

SELECTED PASSAGES FROM WŁODZIMIERZ BURZYŃSKI'S
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“STUDY ON MATERIAL EFFORT HYPOTHESES”
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WŁODZIMIERZ BURZYŃSKI (1900–1970)

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NOTES FROM THE SCIENTIFIC EDITOR

It seems that the work of Włodzimierz Burzyński was the most extensive research in the field of failure criteria at that time. We are convinced that it would be very useful for the international scientific community to deliver its translation in the whole. We would like to realize this goal in the future. Adapting however to the recent editorial requirements, which delimit the volume of the published text, we have decided to select some passages, which, according to our opinion, contain the most original and less known or even unknown results. At the beginning, the table of contents is presented. The page numbers remain the same as in the original Lwów edition. The list of references given at the end is a collection of the bibliographic footnotes quoted in the original work. It is also worthwhile mentioning that the biographical note of W. Burzyński was published recently in English by Z.S. Olesiak, *Włodzimierz Stanisław Trzywdar Burzyński*, *Engng. Trans.*, 56, 4, 377–382, 2008.

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INTRODUCTION

(p. 3)

The main subject of the theory of elasticity is to mathematically determine the state of strain or stress in a solid body being under the conditions determined by the action of a system of external forces, the specific shape of the body and its elastic properties. The solution of this question exhausts the role of the elasticity theory and next the theory of strength of materials comes into play.

Its equally important task is to give the dimensions of the considered body with determined exactness, with respect to the states unwanted regarding the body safety on the one hand and the most advantageous economical conditions on the other hand. This problem, very simple in the case of a uniaxial state of stress, becomes so complicated in a general case that from the beginning of the mentioned theories, special attention had to be paid to this question and an intermediate chapter, being at the same time the final part of the theory of elasticity and the introduction to the strength of materials theory, has been introduced. This new passage deals with material effort and different hypotheses related to this notion. The study of these hypotheses is exactly the subject of the present work.

Material effort is of course closely related with the state of strain, or stress, of the considered body. It is then justified to introduce first the basic relations existing in the mentioned states.

[..., p. 25:] III. DEPENDENCE BETWEEN THE STATES OF STRAIN AND STRESS.
ELASTIC ENERGY. NEW RELATIONS.

[..., p. 27:] The sought function Φ (density of elastic strain energy function – ed. note) can be calculated from the formula:

$$(5) \quad 2\Phi = c_{11}\varepsilon_x^2 + 2c_{12}\varepsilon_x\varepsilon_y + 2c_{13}\varepsilon_x\varepsilon_z + 2c_{14}\varepsilon_x\gamma_x + 2c_{15}\varepsilon_x\gamma_y + 2c_{16}\varepsilon_x\gamma_z \\ + c_{22}\varepsilon_y^2 + 2c_{23}\varepsilon_y\varepsilon_z + 2c_{24}\varepsilon_y\gamma_x + 2c_{25}\varepsilon_y\gamma_y + 2c_{26}\varepsilon_y\gamma_z \\ + c_{33}\varepsilon_z^2 + 2c_{34}\varepsilon_z\gamma_x + 2c_{35}\varepsilon_z\gamma_y + 2c_{36}\varepsilon_z\gamma_z \\ + c_{44}\gamma_x^2 + 2c_{45}\gamma_x\gamma_y + 2c_{46}\gamma_x\gamma_z \\ + c_{55}\gamma_y^2 + 2c_{56}\gamma_y\gamma_z \\ + c_{66}\gamma_z^2$$

(the symbols c_{ik} denote elasticity coefficients and the symbols γ_α denote shear strain in the plane with the normal $\alpha = x, y, z$ – ed. note).

[..., p. 27: The above formula] regards solid bodies which are anisotropic in terms of elasticity. However, in case when certain special properties of the body, simplifying its structure, exist – as it happens e.g. in crystals – the elastic constants become related in a particular way and their number becomes lower¹²⁾.

For example, if there exist in the body three perpendicular planes of structural symmetry and the coordinate axes coincide with these three planes, the function simplifies to the following form with 9 elasticity coefficients:

$$(7) \quad 2\Phi = 2c_{11}\varepsilon_x^2 + 2c_{12}\varepsilon_x\varepsilon_y + 2c_{13}\varepsilon_x\varepsilon_z + c_{22}\varepsilon_y^2 + 2c_{23}\varepsilon_y\varepsilon_z \\ + c_{33}\varepsilon_z^2 + c_{44}\gamma_x^2 + c_{55}\gamma_y^2 + c_{66}\gamma_z^2.$$

The mentioned conditions occur with a very good approximation for a timber cube cut out in the particular way.

For materials in which the elastic properties in the three mentioned perpendicular directions are additionally identical, the function Φ simplifies further to the following form:

$$(8) \quad 2\Phi = c_{11} (\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + 2c_{12} (\varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z) + c_{44} (\gamma_x^2 + \gamma_y^2 + \gamma_z^2).$$

Further reduction leads to two elastic constants; [...] this last case is possible for an isotropic body.

[..., p. 30:] As it is known, for components of the state of strain or stress it is allowed to apply arbitrary superposition of two (or more) subcomponents, according to the scheme:

$$\varepsilon = \varepsilon' + \varepsilon'', \quad \frac{1}{2}\gamma = \frac{1}{2}\gamma' + \frac{1}{2}\gamma''$$

or relatively:

$$\sigma = \sigma' + \sigma'', \quad \tau = \tau' + \tau''.$$

[...] Let us pose now the question whether it is possible to do such a decomposition for the function Φ in the sense of the equation $\Phi(\varepsilon, \gamma) = \Phi(\varepsilon', \gamma') + \Phi(\varepsilon'', \gamma'')$. In other words, whether it is possible to apply an arbitrary superposition for the function of elastic energy. The answer in a general case, that is for all $c_{ik} \neq 0$ and for arbitrarily taken ε and γ , must be negative, since Φ is a quadratic function. Nevertheless, such a decomposition may turn out to be possible for a particularly assumed decomposition of ε and γ and for a material with certain specific elastic constants c_{ik} . [...]

To all intents and purposes, there are no physical reasons for the strain energy not to be decomposable into a sum of two other energies, that is into: the energy of volume change and the energy of distortion. [...] This assumption is the essence of the whole reasoning – certainly not quite a theoretical one – and leads to five new relations of the following form:

$$(12) \quad \begin{cases} 3 \text{ relations: } \begin{cases} c_{14} + c_{24} + c_{34} = 0 \\ c_{15} + c_{25} + c_{35} = 0 \\ c_{16} + c_{26} + c_{36} = 0 \end{cases} \\ 2 \text{ relations: } \begin{cases} c_{11} - c_{22} = c_{23} - c_{13} \\ c_{22} - c_{33} = c_{31} - c_{21} \\ c_{33} - c_{11} = c_{12} - c_{32} \end{cases} \end{cases}$$

The number of elastic coefficients would be limited in this case to the number of $21 - 5 = 16$. For a body characterized by the Eq. (7), the number of 9 coefficients would be reduced to 7; and for a model described by the Eq. (8), the number of elastic constants remains the same, i.e. 3.

[..., p. 31:] Let us consider what form takes the function Φ , expressed by the Eq. (5), assuming that the relations (12) are true and that the decomposition of the components [..., of the strain state into a deviatoric and a spherical part] holds. For this purpose, let us replace the coefficients c_{14} , c_{25} and c_{36} , appearing [in (5)] in the terms $2c_{14}\varepsilon_x\gamma_x$, $2c_{25}\varepsilon_y\gamma_y$ and $2c_{36}\varepsilon_z\gamma_z$, with three pairs of other coefficients, resulting from the first three relations [in (12)]. Then, the nine mixed terms with could be expressed in the form:

$$-(c_{24}\gamma_x - c_{15}\gamma_y)(\varepsilon_x - \varepsilon_y) - (c_{35}\gamma_y - c_{26}\gamma_z)(\varepsilon_y - \varepsilon_z) - (c_{16}\gamma_z - c_{34}\gamma_x)(\varepsilon_z - \varepsilon_x).$$

Next, let us change the position of the axes of the coordinate system to a certain characteristic orientation – let us call it the basic one [...] – namely, in such a way to have:

$$(13) \quad \begin{aligned} c_{24}\gamma_x - c_{15}\gamma_y &= 0, \\ c_{35}\gamma_y - c_{26}\gamma_z &= 0, \\ c_{16}\gamma_z - c_{34}\gamma_x &= 0. \end{aligned}$$

In such a case [...], the considered terms will disappear and – leaving the names of the coefficients in the new system unchanged without fear of confusion, or denoting additionally:

$$(14) \quad \begin{aligned} P &= c_{44} + 2c_{45}\frac{c_{24}}{c_{15}}, \\ Q &= c_{55} + 2c_{56}\frac{c_{35}}{c_{26}}, \\ R &= c_{66} + 2c_{64}\frac{c_{16}}{c_{34}} \end{aligned}$$

– the last six terms in (5) will transform into:

$$P\gamma_x^2 + Q\gamma_y^2 + R\gamma_z^2.$$

The constants P , Q , R can be called the reduced elastic moduli of distortion (shear), analogously to the shear modulus G , for the reasons which are to be revealed later. Each of the constants contains four elastic coefficients.

Continuing, it remains now to take care of the rest of the terms in Eq. (5), i.e. the six terms depending solely on the components and the six elastic constants, which also require certain transformation. A glance at the unused until now equations in (12) is sufficient to observe their particular property. By rearranging

them and adding the equations: $c_{12} = c_{21}$, $c_{23} = c_{32}$, $c_{13} = c_{31}$ to both sides, we obtain the system:

$$c_{11} + c_{12} + c_{13} = c_{21} + c_{22} + c_{23},$$

$$c_{21} + c_{22} + c_{23} = c_{31} + c_{32} + c_{33},$$

$$c_{31} + c_{32} + c_{33} = c_{11} + c_{12} + c_{13},$$

or in general:

$$(15) \quad c_{i1} + c_{i2} + c_{i3} = c_{1k} + c_{2k} + c_{3k} = 3B, \quad (i, k = 1, 2, 3).$$

From the last relation and the three initial relations it results that the sum of three normal stresses:

$$(16) \quad 3p = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3 = 3B(\varepsilon_x + \varepsilon_y + \varepsilon_z) = 3Be$$

is noticeably dependent on the sum of three longitudinal strains along the axes x , y , z ; that is: on volume change and one coefficient of elasticity B . Then [the coefficient B] will be further called modulus of elastic volume change.

Lastly, let us substitute:

$$c_{11} = B + \frac{2}{3}(M + N), \quad c_{22} = \frac{2}{3}(N + L), \quad c_{33} = B + (L + M),$$

therefore:

$$c_{12} = B - \frac{2}{3}N, \quad c_{33} = B - \frac{2}{3}L, \quad c_{32} = B - \frac{2}{3}M$$

and let us insert these values into the six discussed terms of the function Φ (7). Then it will turn out that after the rearrangement, they will assume the following form:

$$B(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + \frac{2}{3} [N(\varepsilon_x - \varepsilon_y)^2 + L(\varepsilon_y - \varepsilon_z)^2 + M(\varepsilon_z - \varepsilon_x)^2],$$

where the coefficients

$$(17) \quad L = \frac{2}{3}(B - c_{23}), \quad M = \frac{2}{3}(B - c_{31}), \quad N = \frac{2}{3}(B - c_{12}),$$

can be called the general elastic moduli of distortion.

Finally then, after dividing the Eq. (5) by 2, we obtain:

$$(18) \quad \Phi = \frac{1}{2}B(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + \frac{1}{3} [N(\varepsilon_x - \varepsilon_y)^2 + L(\varepsilon_y - \varepsilon_z)^2 + M(\varepsilon_z - \varepsilon_x)^2] \\ + \frac{1}{2}(P\gamma_x^2 + Q\gamma_y^2 + R\gamma_z^2)$$

as a general normal form of elastic strain energy of an anisotropic solid, in the basic orientation determined by the Eqs. (13). The first part of the function denotes the energy of volume change Φ_v and the remaining one – the energy of distortion Φ_f . The total energy then reads:

$$(19) \quad \Phi = \Phi_v + \Phi_f.$$

[..., p. 33:] The use of principal components simplifies (18) to the following special form of strain energy:

$$(20) \quad \Phi = \frac{1}{2}B(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 + \frac{1}{3}[N(\varepsilon_1 - \varepsilon_2)^2 + L(\varepsilon_2 - \varepsilon_3)^2 + M(\varepsilon_3 - \varepsilon_1)^2].$$

[..., p. 34:] It is not difficult to observe that the whole foregoing reasoning can be easily reversed and applied to the states determined by stress components. The respective relations take the form:

$$(25) \quad \begin{aligned} C_{14} + C_{24} + C_{34} &= 0, \\ C_{15} + C_{25} + C_{35} &= 0, \\ C_{16} + C_{26} + C_{36} &= 0, \\ C_{11} - C_{22} &= C_{23} - C_{13}, \\ C_{22} - C_{33} &= C_{31} - C_{21}, \\ C_{33} - C_{11} &= C_{12} - C_{32}. \end{aligned}$$

The generalized elastic constants are expressed by the equations [...]:

$$(26) \quad \begin{aligned} 3B^* &= C_{i1} + C_{i2} + C_{i3} = C_{1k} + C_{2k} + C_{3k}, \\ L^* &= \frac{3}{2}(B^* - C_{23}), & 4P^* &= C_{44} + 2C_{45} \frac{C_{24}}{C_{15}}, \\ M^* &= \frac{3}{2}(B^* - C_{31}), & 4Q^* &= C_{55} + 2C_{56} \frac{C_{35}}{C_{26}}, \\ N^* &= \frac{3}{2}(B^* - C_{12}), & 4R^* &= C_{66} + 2C_{64} \frac{C_{16}}{C_{34}}, \end{aligned}$$

where the additional relation reads:

$$(27) \quad \begin{aligned} e = \varepsilon_x + \varepsilon_y + \varepsilon_z = e_1 + e_2 + e_3 &= 3B^*(\sigma_1 + \sigma_2 + \sigma_3) \\ &= 3B^*(\sigma_x + \sigma_y + \sigma_z) = 9B^*p. \end{aligned}$$

The formulae for elastic energy will take the [following] forms – a general one in the basic system:

$$(28) \quad \Phi = \frac{1}{2}B^*(\sigma_x + \sigma_y + \sigma_z)^2 + \frac{1}{3} [N^*(\sigma_x - \sigma_y)^2 + L^*(\sigma_y - \sigma_z)^2 + M^*(\sigma_z - \sigma_x)^2] \\ + 2(P^*\tau_x^2 + Q^*\tau_y^2 + R^*\tau_z^2)$$

(the symbols τ_α denote the shear stress in the lane with the normal $\alpha = x, y, z$ – ed. note) and a particular one in the principal system:

$$(29) \quad \Phi = \frac{1}{2}B^*(\sigma_1 + \sigma_2 + \sigma_3)^2 \\ + \frac{1}{3} [N^*(\sigma_1 - \sigma_2)^2 + L^*(\sigma_2 - \sigma_3)^2 + M^*(\sigma_3 - \sigma_1)^2].$$

[..., p. 38:] As the conclusion of this chapter there will be given a group of formulae [...], assuming certain special states. These include: the case of uniaxial tension or relative compression and the case of simple torsion [...]. The first one is characterized by the components:

$$\sigma_x = \sigma_0, \quad \sigma_y = \sigma_z = 0, \quad \tau_x = \tau_y = \tau_z = 0$$

and the second one by:

$$\sigma_x = \tau_0, \quad \sigma_y = 0, \quad \sigma_z = -\tau_0, \quad \tau_x = \tau_y = \tau_z = 0,$$

or

$$\sigma_x = \sigma_y = \sigma_z = 0, \quad \tau_x = 0, \quad \tau_y = \tau_0, \quad \tau_z = 0.$$

From the respective relations of the present chapter we obtain for the first case:

$$(49) \quad \Phi_f = \frac{1}{6G}\sigma_0^2, \quad \Phi = \frac{1}{2E}\sigma_0^2, \quad \varepsilon_0 = \frac{1}{E}\sigma_0$$

and similarly for the second case:

$$(50) \quad \Phi_f = \frac{1}{2G}\tau_0^2, \quad \Phi = \frac{1}{2G}\tau_0^2, \quad \gamma_0 = \frac{1}{G}\tau_0.$$

[..., p. 39:] IV. MATERIAL EFFORT

[..., p. 40:] Generally, under the notion *material effort* we understand a physical state of a body, comprehended in the sense of elasticity or plasticity or material strength, generated by a system of stresses, and related with them strains, in the body. This brief qualitative definition will become – I suppose –

completely clear after looking through the discussion in this and next chapters. [...] Generally then, the new notion *material effort* depends on the manner in which external forces act, on the body shape and on individual properties of the body. These notions deserve a few words of explanation.

Under the manner in which external forces act, one should understand not only the distribution of loading but also its variability in time. The recent state of the strength theories does not allow to consider this important factor in calculations, except for a few particular cases.

The body shape is one of the reasons for the dependence of the stress state components on the position of the considered point in the body. It is clear that the uniformity or non-uniformity of the state of stress strongly influences the quality of the physical state of the whole body¹⁵). One has the impression that authors of various hypotheses overlooked this fact; however, the ways of conducting experiments contradict that. Furthermore, it is not known whether the local grouping of stress components leading to the limiting numerical value of material effort accounts for unwanted changes exclusively in this particular point of the body, or influences the physical behaviour of the whole body in general. Similarly, it is unknown whether the experimental observation of the existence of certain planes of unwanted states (planes of shear, *etc.*) is a proof that the corresponding to this planes components [of the state of stress] are the reasons for creation of internal disorders.

These remarks fall out if a uniform state of stress is ascertained in the whole body. For this reason, the results obtained in following chapters should be limited to the case of a *uniform* state of stress or, otherwise, they should be limited exclusively to a point.

[..., p. 41:] Regarding the structure, two kinds of solid bodies are distinguished: crystalline and amorphous ones, depending whether the particles of the body are distributed in space regularly or irregularly. The majority of technical materials (metals) are continuous macroscopic conglomerates of both types of structure. For this reason such bodies behave as isotropic ones, since the anisotropy of particular crystals cannot be shown individually at the macroscopic level. Such bodies are called quasi-isotropic. However, secondary circumstances can trigger, even in such a conglomerate, some remarkable differences in the material behaviour along certain directions – e.g. the influence of rolling [...] *etc.* and the respective differences should be accounted for. There are no such attempts in a general sense; the hypotheses discussed in following chapters assume isotropy of materials without any explanation.

Elastic properties of a body are determined by the so-called elastic moduli or elastic constants, which were discussed in the previous chapter. For a large group of materials these coefficients are constant, so they ascertain that the generalized Hooke's law remains valid. A series of recent precise experi-

ments show irrefutably that the range of solids undergoing the Hooke's law is pretty large (e.g. concrete)¹⁶⁾. The limit of validity of Hooke's law is called *the limit of proportionality*. If additionally, the strains induced within such limits are ideally – at least in the technical meaning – elastic, we obtain a model of a body, to which all equations from the foregoing chapter are applicable. However, these equations are valid only up to the elasticity limit. By the existence of both limits, they ascertain, in general, proximity of these limits. Therefore, confusing both the terms in the vast technical literature does not implicate too serious mistakes. *The proportionality limit* plays the role of a mathematical condition rather than a physical one – whereas it is opposite for *the limit of elasticity*.

Bodies which do not have the limit of proportionality show more or less distinct limits of elasticity; thus the relations of the foregoing chapter have the character of the first approximation only.

[..., p. 42:] Beyond the elastic range, the elasticity coefficients should be considered to be variable or generally, they should not be used in the meaning they were referred to until now. Instead, they should be replaced by certain constants specific not only for the body but also for the considered stress process itself.

To such ranges belongs, first of all, the range of plastic strains, which begins from the so-called *limit of plasticity* [...].

[..., p. 43:] A few words should be said also about the third process connected with a particular body. To the phenomena accompanying permanent strains is related the third stage – belonging, undisputably, to the strength of materials theory – that is the range of material cracking, ending with *the limit of strength* (in a technical sense). Conditions of failure are usually very complicated and, up till now, also not too much theoretically explained²⁰⁾. Uniformity or rather, on the opposite, non-uniformity of the state of stress, which is – as a matter of fact – difficult to be analysed in connection with the shape of a body, plays a considerable role in this region. Disregarding the surface energy^{15,21)} can be a reason of serious errors, even in preliminary calculations. Apart from that, it is not known whether the specific for given materials constants reflect satisfactorily the essence of the phenomenon of failure, as it is assumed by some authors. Qualitative diversity in different strength processes persuaded researchers [...] to divide failure surfaces into two categories, i.e. *the surfaces of shear and tear*²²⁾.

Under the stress properties we understand the behaviour of a body in certain special states; these properties reveal themselves as numerical values of stress in the above-described limit ranges. We know a whole series of such states and, because of obvious benefits and applications in the following chapters, let us set them schematically by means of normal principal stresses, under the assumption $\sigma_1 > \sigma_2 > \sigma_3$, as follows:

- I. Uniaxial tension: $\sigma_1 = k_r, \sigma_2 = 0, \sigma_3 = 0$;
- II. Uniaxial compression: $\sigma_1 = 0, \sigma_2 = 0, \sigma_3 = -k_c$;
- III. Simple torsion (shear): $\sigma_1 = k_s, \sigma_2 = 0, \sigma_3 = -k_s$.

Only few metals are characterized by the equality $k_r = k_c = k$; in principle, these constants are different, namely $k_c > k_r$. The discussed states are accounted as the simplest – the fundamental ones – for the study of material effort.

[..., p. 44:] To the similar, simplest states of stress should be added also the following ones, though more complex indeed, yet in the present state of our knowledge on material effort they can not be omitted. These are in sequence:

- IV. Biaxial uniform tension: $\sigma_1 = k_{rr}, \sigma_2 = k_{rr}, \sigma_3 = 0$;
- V. Biaxial uniform compression: $\sigma_1 = 0, \sigma_2 = -k_{cc}, \sigma_3 = -k_{cc}$;
- VI. Triaxial uniform tension: $\sigma_1 = k_{rrr}, \sigma_2 = k_{rrr}, \sigma_3 = k_{rrr}$;
- VII. Triaxial uniform compression: $\sigma_1 = -k_{ccc}, \sigma_2 = -k_{ccc}, \sigma_3 = -k_{ccc}$.

Any other experimental states can be of course put between the ones given above. The aforementioned values k refer to the states lying on the elasticity limit (or proportionality limit), the limit of plasticity and the strength limit. [...]

[..., p. 48:] V. ANALYTICAL AND GRAPHICAL METHODS OF PRESENTATION
OF MATERIAL EFFORT. CLASSIFICATION OF HYPOTHESES.

[..., p. 51:] In the present work, the following classification is assumed as the best illustration of the contents of the [material effort] hypotheses²⁹⁾ and at the same time, as it partly corresponds to chronological relations of these theories.

- A. *The hypotheses of limit stresses.*
- B. *The hypotheses of limit strains.*
- C. *The hypotheses of limit energies. [...]*

[..., p. 96:] VIII. THE HYPOTHESES OF LIMIT ENERGIES.
THE AUTHOR'S HYPOTHESIS.

C1. THE HYPOTHESIS OF CONSTANT LIMIT ENERGY OF STRAIN

The mentioned in the title hypothesis is known since the times of BELTRAMI⁵⁴⁾, who for the first time suggested the use of strain energy for calculation of material effort. Independently of BELTRAMI⁵⁴⁾, HUBER⁵⁶⁾ stated an identical theory; already then, however, emphasizing certain additional thought resulting in the change of the contents of (C1) into (C2), which will be discussed in the

next paragraph. Finally, in recent time HAIGH³²⁾, an Englishman, repeated the suggestion of Beltrami, not having known – like Huber – Beltrami’s publication.

[..., p. 97: Formula of the hypothesis reads:]

$$(C1_0) \quad \boxed{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + 2(1+\mu)(\tau_x^2 + \tau_y^2 + \tau_z^2) = k^2},$$

where: $k = k_c = k_r$.

With the use of the principal components [of stress], the hypothesis takes a shorter form [...]:

$$(C1) \quad \boxed{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = k^2}.$$

[..., p. 99:] The above equation represents a rotationally symmetric ellipsoid of the axis oriented at equal angles to the axes of the system $\sigma_1, \sigma_2, \sigma_3$, with lengths of the half-axes:

$$b_1 = b_3 = \frac{k}{\sqrt{1+\mu}},$$

$$b_2 = \frac{k}{\sqrt{1-2\mu}}.$$

[..., p. 100:] C2. THE HYPOTHESIS OF CONSTANT LIMIT ENERGY
OF VOLUME CHANGE AND DISTORTION

As it was mentioned before, independently of Beltrami, Huber brought forward a similar proposition. He used his hypothesis for limit states of strength supposing however, that the theory would be valid also for elastic ranges. Basing on certain facts related experimentally to exceeding the strength limit, he observed that in the case of the states with three negative normal components [of stress], one should consider rather the energy of distortion Φ_f than the total Φ as a measure of material effort.

About his final, mathematically precisely stated position [on this matter] we learn from the letter to FÖPPL⁸⁾ and the following statement contained there: “Material effort is measured by the sum of these parts of density of strain energy, which result from the distortion and increase of volume”. The measure of material effort is then $\Phi = \Phi_v + \Phi_f$ if the above assumption is fulfilled, i.e. when $e > 0$ or $\sigma_x + \sigma_y + \sigma_z > 0$; in the opposite case, i.e. when $e < 0$ or $\sigma_x + \sigma_y + \sigma_z < 0$, the assessment of material effort is given by Φ_f exclusively. In this way, a discontinuous hypothesis is created; the states I, IV, and VI belong to the first group of phenomena, while the states II, V and VII belong to the latter one; the state III is proved in both ranges.

This last state fits best to express the Huber hypothesis; comparing then respectively the complete or partial formula (37) with the pertinent ones (49) and (50), from the Chapter III we obtain:

$$\frac{1}{2(1+\mu)}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\mu}{1+\mu}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + (\tau_x^2 + \tau_y^2 + \tau_z^2) = k_s^2$$

(C2₀) for: $\sigma_x + \sigma_y + \sigma_z \geq 0$, furthermore:

$$\frac{1}{3}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x) + (\tau_x^2 + \tau_y^2 + \tau_z^2) = k_s^2$$

for: $\sigma_x + \sigma_y + \sigma_z \leq 0$,

as a mathematical formula of Huber's hypothesis in a general case. The particular form [for principal stress components] reads of course:

$$\frac{1}{2(1+\mu)}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{\mu}{1+\mu}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = k_s^2$$

for: $\sigma_1 + \sigma_2 + \sigma_3 \geq 0$, and

(C2)

$$\frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1) = k_s^2,$$

for: $\sigma_1 + \sigma_2 + \sigma_3 \leq 0$.

[..., p. 103:] C3. THE HYPOTHESIS OF LIMIT ENERGY OF DISTORTION

The decomposition of elastic energy into two characteristic parts, applied for the first time for the assessment of material effort by Huber, has earned in the process of time a well-deserved experimental and theoretical confirmation and created the foundation of unusually fine and mathematically simple hypothesis (C3).

According to this new theory, the measure of material effort is exclusively the energy of distortion Φ_f . The hypothesis was for the first time proposed, it seems, by MISES⁵⁹). Having drawn the attention to the fact that the spatial picture of the hypothesis (A3) [related with the criterion of Tresca] in the orthogonal system of axes [of the principal shear stresses] $\tau_I, \tau_{II}, \tau_{III}$ shows a cube of the edge k , Mises expressed a conviction, that this rather should be the sphere:

$$\tau_I^2 + \tau_{II}^2 + \tau_{III}^2 = \frac{k^2}{2}.$$

The second author who raised the mathematical formula (C3) to the rank of the fundamental equation of the theory of plasticity was HENCKY^{19),60)}.

[..., p. 104:] The mathematical form of the hypothesis (C3) [...] in a general case reads:

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x + 3(\tau_x^2 + \tau_y^2 + \tau_z^2) = k^2$$

or in particular [for principal stresses]:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 = k^2.$$

[..., p. 106:] C4. THE HYPOTHESIS OF VARIABLE LIMIT ENERGY OF STRAIN

In such a way one could name the hypothesis which was – as it appears – presented during one of Mises's lectures in 1925 and published by SCHLEICHER⁵⁰⁾ in 1925/1926.

According to Schleicher's theory, the "equivalent" stress, expressed by the left-hand side of (C1) – let us denote it shortly: $\sigma_{vf} = \sqrt{2E\bar{\Phi}}$ – is in the limit state a variable value depending on the state of stress, that is on:

$$p = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}.$$

In other words, [the equation] $\sigma_{vf} = f(p)$ is a mathematical form of the hypothesis of variable limit energy.

Schleicher relates his theory to elastic and plastic states and recommends to seek for the shape of the function f experimentally, similarly as it was advised by Mohr in the case of the shape of envelope.

[..., p. 107] It is possible to approximate the experimental curve, according to Schleicher, by means of

$$(C4^*) \quad \text{a parabola} \quad \sigma_{vf}^2 = s^2 - 3mp,$$

$$(C4^{**}) \quad \text{or a line} \quad \sigma_{vf} = t - 3mp.$$

In the first case it is: $s^2 = k_c k_r$, $m = k_c - k_r$, and in the second one:

$$t = \frac{2k_c k_r}{k_c + k_r}$$

and the already known

$$n = \frac{k_c - k_r}{k_c + k_r};$$

in other words, the hypothesis is dependent on two parameters k_c and k_r . [...]

Expressing p and σ_{vf} by means of the components of stress, we get (from (C4*) and (C4**)) the following relations:

$$(C4') \quad \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu(\sigma_x\sigma_y + \sigma_z\sigma_y + \sigma_x\sigma_z) + 2(1 + \mu)(\tau_x^2 + \tau_y^2 + \tau_z^2) \\ + (k_c - k_r)(\sigma_x + \sigma_y + \sigma_z) = k_c k_r$$

and

$$(C4'') \quad \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu''(\sigma_x\sigma_y + \sigma_z\sigma_y + \sigma_x\sigma_z) + 2(1 + \mu'')(\tau_x^2 + \tau_y^2 + \tau_z^2) \\ + (k_c - k_r)(\sigma_x + \sigma_y + \sigma_z) = k_c k_r,$$

or with use of principal stresses:

$$(C4_1) \quad \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3) \\ + (k_c - k_r)(\sigma_1 + \sigma_2 + \sigma_3) = k_c k_r$$

as well as:

$$(C4_2) \quad \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu''(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3) \\ + (k_c - k_r)(\sigma_1 + \sigma_2 + \sigma_3) = k_c k_r,$$

whereas:

$$\mu'' = \frac{\mu + n^2}{1 - n^2} = \frac{\mu(k_c + k_r)^2 + (k_c - k_r)^2}{4k_c k_r}.$$

For $k_c = k_r$, the Schleicher hypothesis expressed either by (C4') and (C4'') or by (C4₁) and (C4₂), transforms in the hypothesis of Beltrami.

[..., p. 111:] C5. THE HYPOTHESIS OF VARIABLE LIMIT ENERGY
OF VOLUME CHANGE AND DISTORTION

The review of enormous theoretical material, which was presented in the previous chapters, together with an equally extensive set of experiments, allows judging discerningly the merits and drawbacks of the discussed hypotheses. This assessment leads to the rejection of the theories A and B and compels to accept the theories C, which are more consistent mathematically and therefore more flexible for experiments.

Individual properties of the studied bodies suggest that basing the theories on one or two experimental data does not in general render faithfully the phenomenon of material effort and demands to introduce more parameters into account, as it was suggested by Schleicher. Controlling the phenomena by the modulus of elasticity causes many problems and the only rarely returns reliable services.

For this reason I have tried to state a hypothesis as general as (C4) which would be, however, free of these additions which seemed for me inadequate in the study of material effort. The starting point is the attitude similar to what Huber stated in (C2); however, much more general and continuous. It is the following conviction: *The measure of local material effort in elastic and plastic ranges is the sum of density of quasi-energy of distortion and a certain part – dependent on the state of stress and individual properties of a body – of the density of the pseudo-energy of volume change.*

By adding “quasi” – or “pseudo” – we try to emphasize that the analytic expressions used in continuation, quoted in the third chapter, do not mean – for a certain group of bodies or relatively in certain experimental fields – elastic energy in the sense discussed in this chapter.

The mathematical formula for the hypothesis is the equation:

$$\Phi_f + \eta\Phi_v = K.$$

Expansions of the functions Φ_f and Φ_v are very well known to us. Determination of constant K does not present difficulties; it is the value of the left-hand side of the equation, determined for one of the basic states, the simplest ones, that is: I, II or III. The remaining to be discussed η is – as it results from the assumption – a function of individual material properties as parameters and of the state of stress as an independent variable. The individual properties should be expressed also by the moduli of the simplest states. To the latter one should apply several magnitudes created from the components of state of stress; because of the proved minor significance of the component τ , one should express the independent variable of the function η by the component σ . From possible expressions, due to the mathematical character of the energies Φ_f and Φ_v , there suggests itself the invariant which does not privilege any of the three components, namely

$$p = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}.$$

In general then, we assume that: $\eta \equiv \eta(p)$. Considering series of correct experiments seems to suggest generally that the influence of Φ_v decreases with the algebraic increase of the mean stress p ; this leads to a very well applicable function:

$$\eta = \omega + \frac{\delta}{3p}.$$

The written [above] type [of function] does not always stand in ideal agreement with experimental facts, but increasing of the number of the introduced parameters K , ω , δ leads to a very complicated hypothesis, so this was abandoned and the sometimes unavoidable shortcomings [of the expression] were compensated in continuation in a more – as it will appear – appropriate manner.

On this occasion, it should be remarked that the role of experiment is not to determine directly the constants K , ω , δ , as it would seem at the first moment, but three other data which will be discussed now.

Anticipating what will follow, let us put here:

$$\frac{1-2\mu}{1+\mu} \omega = \frac{1-2\nu}{1+\nu}, \quad 12GK = \frac{2k_c k_r}{1+\nu}, \quad \frac{1-2\mu}{1+\mu} \delta = \frac{2(k_c - k_r)}{1+\nu}.$$

Moreover, let us substitute for shortening: $12G\Phi_f = \sigma_f^2$. And [now] let us insert the complete set of the mentioned transformations into the main equation. After a simple transformation we obtain:

$$\frac{1+\nu}{3} \sigma_f^2 + 3(1-2\nu)p^2 + 3(k_c - k_r)p - k_c k_r = 0.$$

The introduction of the parameters k_r and k_c into the last equation is justified, since it is easy to demonstrate that it is identically fulfilled for the states I and II. By assuming, additionally, the state III, we obtain the relation:

$$\nu = \frac{k_c k_r}{2k_s^2} - 1.$$

From the last reasoning it follows that the hypothesis (C5) is a theory based on the three constants: k_r , k_c , k_s or relatively: k_r , k_c , ν . Let us hold the last group for later consideration because of vital mathematical benefits which will appear in the course of time. The coefficient ν – as it will also appear – very strongly determines individual properties of an examined body in the range of its brittle or – opposite – plastic behaviour. It could be advantageous to call it the “*plasticity coefficient*”, because it turns out that for tough and brittle materials there is: $\nu < \frac{1}{2}$, for tough and plastic materials there is: $\nu = \frac{1}{2}$, and for soft (plastic) bodies: $\nu > \frac{1}{2}$. There is no way to the state the limits within which ν ranges; the possible excess over $\frac{1}{2}$ grows not so high, the matter of decreasing the value presents itself similarly. There arises a supposition that the interval where ν ranges fits between 0 and 1. With the course of the discussion it will turn out that in the main we need to do this kind of assumption out of necessity. After the above remarks, there arises the question whether the coefficient ν , or another approximate one, could be – by instance – used for determining the extent of [a magnitude] quite close to plasticity, i.e. hardness, with mathematical description of which theoretical researchers have been bothering for so many years.

Returning, however, to the recently written equation, let us transform it into the following formula:

$$(C5') \quad \frac{1 + \nu}{3} \sigma_f^2 + 3(1 - 2\nu)(p + \sigma')^2 = k'^2$$

where:

$$\sigma' = \frac{k_c - k_r}{2(1 - 2\nu)},$$

$$k'^2 = k_c k_r + \frac{3}{4} \frac{(k_c - k_r)^2}{1 - 2\nu} = \frac{3k_s^2(k_c + k_r)^2 - 4k_c^2 k_r^2}{4(3k_s^2 - k_c k_r)} = -k_1^2.$$

In the system of axes (p, σ_f) , the equation (C5') represents – similarly to the system of the axes (p, σ_{vf}) by Schleicher – curves of the second degree, the type of which should be now considered.

Of course there occur to mind the [three] cases: $\nu > \frac{1}{2}$; $\nu = \frac{1}{2}$; $\nu < \frac{1}{2}$. Due to the dependent on that algebraic value of k'^2 or k_1^2 , one detail should be emphasized here. Namely, from some later discussion it will follow that within the sphere of experimental facts there should be: $k_s \geq \frac{2}{\sqrt{3}} \frac{k_c k_r}{k_c + k_r}$. The lower limit of this inequality seems to be quite convincing, since it is enough to assume: $k_c = k_r = k$ to obtain: $k_s = \frac{k}{\sqrt{3}}$, that is the relation well known to us from (C3) and currently strongly emphasized in a series of publications. While for: $k_r \neq k_c$ $\left[\varkappa = \frac{k_c}{k_r} \neq 1 \right]$ this inequality would indicate that for technically possible materials in the group of $\nu > \frac{1}{2}$, there has to be $\nu < 3.5$ ($\varkappa \cong 8$). However, one can be assured that ν will not reach such a value, because: as k_c increases in comparison to k_r , at the same time k_s begins to significantly more strongly exceed the given [above] limit $\frac{2}{\sqrt{3}} \frac{k_c k_r}{k_c + k_r}$, which results in the fact that ν considerably lowers its limiting value. In any case, a bound on k_s is followed by a bound on ν ; in the especially important case $k_r = k_c$, that is: $\varkappa = 1$, we obtain – as it was mentioned – $k_s \geq \frac{k}{\sqrt{3}}$ and consequently: $\nu \leq \frac{1}{2}$. After such bounds on the magnitude k_s , we can start the promised discussion.

And so, in the case when $\nu < \frac{1}{2}$, which means $k_s > \sqrt{\frac{k_c k_r}{3}}$, there is: $1 - 2\nu > 0$ and moreover $k'^2 > 0$, and the equation (C5') represents in the mentioned system [of coordinate axes] an ellipse, or relatively a circle, whose centres lie on the negative direction of p (Fig. 64) – or in the special case: $k_r = k_c$ i.e. $\varkappa = 1$ – they coincide with the origin of the coordinate system (Fig. 65).

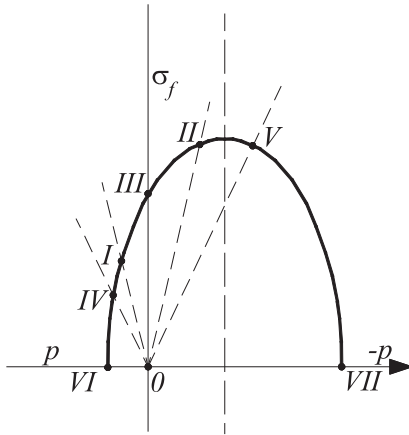


Fig. 64.

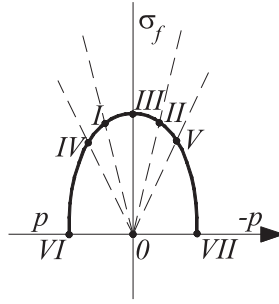


Fig. 65.

In the case of $\nu = \frac{1}{2}$, the equation (C5') turns into a parabola of the second degree for $\chi > 1$ (Fig. 66), or into two lines parallel to the axis p for $\chi = 1$ (Fig. 67).

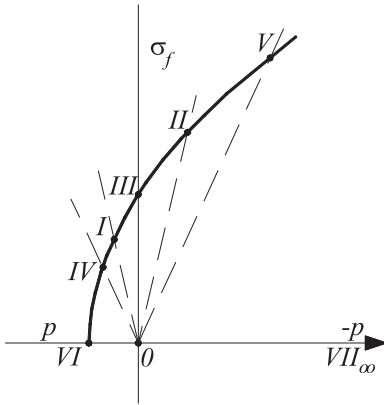


Fig. 66.

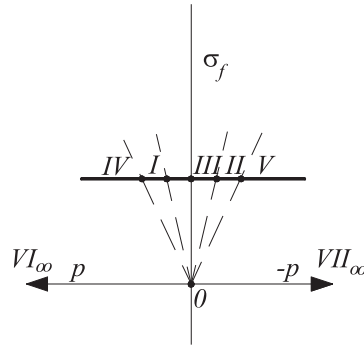


Fig. 67.

In the case of $\nu > \frac{1}{2}$, that is $\frac{2}{\sqrt{3}} \frac{k_c k_r}{k_c + k_r} < k_s < \sqrt{\frac{k_c k_r}{3}}$, there is [both $1 - 2\nu > 0$ as well as $k_1'^2 > 0$ (which means also that $k_1^2 > 0$) and the equation (C5') represents a hyperbola, whose one branch only, of course, comes into play (Fig. 68). In the case when: $k_s = \frac{2}{\sqrt{3}} \frac{k_c k_r}{k_c + k_r}$ the hyperbola degenerates into two crossing lines [only one of lines is depicted due to the symmetry] (Fig. 69). Because of the already mentioned bound on the lower limit of k_s , the case of a hyperbola rotated by the angle $\frac{\pi}{2}$ from the formerly discussed position is excluded.

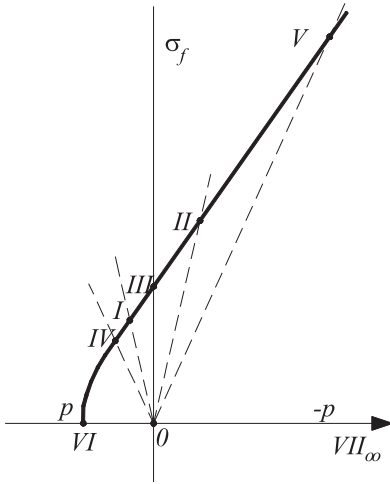


Fig. 68.

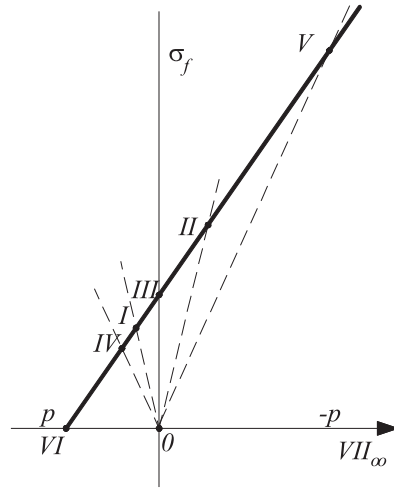


Fig. 69.

In the enclosed schemes are shown only halves of the considered curves relevant for $\sigma_f > 0$. Besides, there are added lines for the states I and II, III, IV and V, VI, and VII, similarly as it was performed in the discussed Schleicher's theory. The equations for these lines read: $\sigma_f = \pm 3\sqrt{2}p$, $p = 0$, $\sigma_f = \pm \frac{3}{2}\sqrt{2}p$ and finally, for the last two ones: $\sigma_f = 0$. Positions of intersections of those lines with the referred curves characterize very well the category of the investigated material.

The given discussion, together with the set graphs allows – under the assumption of the truthfulness of the theory (C5) – judging certain phenomena, and especially it graphically explains changes of the limit value of quasi-energy of distortion in the critical range. Uniformity of this study demands, however, to expand the formula (C5') into the types used in the present work. With this aim let us expand σ_f according to (C3₀), with p – as above; then we obtain directly:

$$(C5_0) \quad \boxed{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + 2(1 + \nu)(\tau_x^2 + \tau_y^2 + \tau_z^2) + (k_c - k_r)(\sigma_x + \sigma_y + \sigma_z) = k_c k_r}$$

or in a simpler form:

$$(C5) \quad \boxed{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) + (k_c - k_r)(\sigma_1 + \sigma_2 + \sigma_3) = k_c k_r.}$$

At first sight, the new hypothesis – apart from the change of the notation μ or μ'' into ν – does not differ from Schleicher's theory and therefore from the equations (C4') and (C4'') or relatively (C4₁) and (C4₂); however, exactly this

subtle difference in notation constitutes the fundamental superiority of (C5) over (C4). Since while μ or μ'' are elastic constants in direct or relatively – let us say – reduced meaning, ν does not have – with advantage to the hypothesis – this property at all.

It could seem that the discussed superiority is ostensible as Schleicher's theory employs only two constants in the quoted equations and the hypothesis (C5) uses three of them. However, it should be reminded that Schleicher doubts the possibility of sufficient representation of the material effort phenomenon by two parameters and, as I mentioned, by examining some experiments, he assumed four of those parameters using independently both equations quoted in the section (C5) for one research series. Finally, regardless of the number of these coefficients, the hypothesis (C4) cannot free itself from the disturbing influence of the constant μ , the lack of which is particularly advantageous in (C5).

That there is some distinguishing generality in employing [the plasticity coefficient] ν into the range of the theory (C5), can be proved by the following facts. For $\nu < \frac{1}{2}$ the hypothesis can transform in a special case into Schleicher's theory; namely, if there is: $\nu = \mu$ or $\nu = \mu''$. Similarly for $\nu = \mu$ and $\varkappa = 1$, the theory (C5) transforms directly into Beltrami's hypothesis (C1), or partly into Huber's theory (C2). In the case of $\nu = \frac{1}{2}$ we create a new eventuality: namely for $\varkappa = 1$ the theory (C5) becomes identical to (C3). For $\nu > \frac{1}{2}$ and in both previous cases, the hypothesis contains a whole series of eventualities, which are not considered in other theories.

[. . . , p. 127:] There arises the question if and how the current formula takes into account the influences of – often inevitable – anisotropy of material. Comparison of the expressions for Φ_v and Φ_f for isotropic and anisotropic bodies in the Eqs. (28) and (31), or relatively (29) and (32), in the Chapter III indicates a distinct difference only in the expressions for Φ_f . Therefore one should suppose that also in the discussed hypothesis, this elusive anisotropy must become visible through an analogous change.

The use of the word “elusive” is deeply grounded. Indisputably, creation of hypotheses of material effort for anisotropic bodies is the distant future. Although, even today it can be supposed that the measure of material effort of some bodies, indicating certain simplified properties in three directions, can be pretty well [expressed by] the energy Φ_f – as it was ascertained in the theory (C3) regarding certain isotropic materials. However, the currently discussed task consists in catching the influences of slight anisotropy, [being] difficult to state in terms of quantity but to some extent visible in terms of quality.

For this purpose, the best suitable will be certainly the general [form of the] function Φ_f (Chapter III). However, [it should be] appropriately simplified,

since introducing it in the complete form with six elasticity constants would give as a result a hypothesis with eight constants. So, in the first place, not being interested in introducing approximate relations between groups L^* , M^* , N^* and P^* , Q^* , R^* at the cost of losing the energy-based character of the functions, let us rather at the start give up on expressing the theory in the basic system – as it was called in the Chapter III – and let us refer it from now on to the system of principal directions. In this manner we obtain a hypothesis of five constants instead of three, as it was until now.

However, even this number could turn out to be too large for the approximate assessment of the symptoms of anisotropy and – even though such a kind of increase would introduce into the account two new mutually supplementing parameters k_{rr} and k_{cc} – one should rather give up on this symmetry and try to continue the reduction of the number of constants down to four. Successful solution of this question presents itself obvious after the provided till now direct reasoning. Let us assume, beforehand, that the general type of the equation linking the variables p and σ_f – presented in the beginning of this section – will not receive any external change after the present remarks.

This equation is obtained analogically as previously. Namely, let us substitute into the main equation: $\Phi_f + \eta\Phi_v = K$ the complete expressions for Φ_f , Φ_v from the formula (29) in the Sec. III. Let us multiply both sides of the equation by $\frac{3M^*}{L^*N^*}$ and put for reduction the replacements:

$$\frac{1 - 2\nu^*}{1 + \nu^*} = \frac{3B^*M^*}{2L^*N^*}\omega, \quad \frac{3(k_c - k_r)}{1 + \nu^*} = \frac{3B^*M^*}{2L^*N^*}\delta, \quad \frac{3k_c k_r}{1 + \nu^*} = \frac{3KM^*}{L^*N^*}$$

and furthermore:

$$\frac{M^{*2}}{L^*N^*} = 2\lambda.$$

By multiplying both sides of the equation transformed in such a way by $\frac{1 + \nu^*}{3}$, we will obtain:

$$\frac{1 + \nu^*}{3}\sigma_f^{*2} + 3(1 - 2\nu^*)p^2 + 3(k_c - k_r)p - k_c k_r = 0,$$

where:

$$\sigma_f^{*2} = \frac{M^*}{N^*}(\sigma_2 - \sigma_3)^2 + 2\lambda(\sigma_3 - \sigma_1)^2 + \frac{M^*}{N^*}(\sigma_1 - \sigma_2)^2 \quad \text{and} \quad p = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

are variables of the function written above. Instead of the variable p it would be more rational to use in this case a slightly different one, namely:

$$p^* = \frac{\lambda\sigma_1 + (1 - \lambda)\sigma_2 + \lambda\sigma_3}{1 + \lambda}.$$

The reasons for such a remark, significantly changing the energy-based sense of the hypothesis, will appear in the Chapter X.

The obtained equation ostensibly does not differ in its structure from the previous one. [Nevertheless,] the essential difference lies, first of all, in the plasticity coefficient ν^* , whose numerical value can be now quite seriously modified by the anisotropy. Besides, the difference can possibly be in p^* but mainly in σ_f^{*2} , which is currently remarkably different from σ_f^2 involving – at least for this moment – three parameters. These last ones – needless to say – are not treated as representations of the ratio of elasticity constants but as coefficients particularly connected with the experimental essence of material effort. It seems, apparently, that σ_f^{*2} involves three such parameters, but assuming such a special structure of the equation entails certain consequences. We find about them by assuming States I and II; there occur from that the following results:

$$\frac{M^*}{L^*} = \frac{M^*}{N^*} = 2(1 - \lambda),$$

in which case, finally, [the following]:

$$\sigma_f^{*2} = 2(1 - \lambda)(\sigma_2 - \sigma_3)^2 + 2\lambda(\sigma_3 - \sigma_1)^2 + 2(1 - \lambda)(\sigma_1 - \sigma_2)^2$$

is a function of one parameter λ only, and the whole hypothesis will now belong to the category of theories of the four constants k_r , k_c , ν , and λ , or other four if convenience would demand to introduce them.

For $\lambda = \frac{1}{2}$ there is $\sigma_f^{*2} = \sigma_f^2$ and $p^* = p$ and the hypothesis as a whole transforms into the previous, comprehensively discussed one. If one assumes that accidental influences of anisotropy are quite strongly limited, it seems reasonable to expect that the interval within which λ varies is quite modest, and so that it ranges e.g. from 0 to 1. The significance of the parameter λ will come out from the assumption of the State III for the previously written equation; namely, after the auxiliary substitution:

$$\varphi = \sqrt{\frac{2(1 + \lambda)}{3}}$$

we will obtain the relation:

$$\nu^* = \frac{1}{\varphi^2} \frac{k_c k_r}{2k_s^2} - 1,$$

very strongly reminding the previous formula expressing ν .

The “coefficient of anisotropy” φ modifies then quite significantly the “plasticity coefficient” ν to the value ν^* . The last one then will not be contained within the limits from 0 to 1, but within a little more extended ones. If we assume the conditions $0 \leq \lambda \leq 1$ and $0 \leq \nu \leq 1$, the interval of changes of ν^* will

be described by the inequality $-\frac{1}{4} \leq \nu \leq 2$; similarly, the coefficient φ will be limited within the interval $\sqrt{\frac{2}{3}} \leq \varphi \leq \frac{2}{\sqrt{3}}$. However – similarly as it was previously considered – going up or down from the value $\nu^* = \frac{1}{2}$ will be distinctively reflected in the contents of the theory.

The mutual dependence of the discussed coefficients is described by the expression:

$$\nu = \frac{2\lambda(1 + \nu^*) - (1 - 2\nu^*)}{3}.$$

The difference: $\delta^* = \nu^* - \nu = \frac{1 + \nu^*}{3}(1 - 2\lambda)$ can be $\delta^* > 0$ or $\delta^* = 0$ or $\delta^* < 0$, depending on $\lambda > \frac{1}{2}$ or $\lambda = \frac{1}{2}$ or $\lambda < \frac{1}{2}$. Now, the use of the parameter δ^* instead of the parameter $\lambda = \frac{1 + \nu^* - 3\delta^*}{2(1 + \nu^*)}$ can turn out to be more advantageous. For the assumptions made, the parameter δ^* is described by the interval: $-\frac{1 + \nu^*}{3} \leq \delta^* \leq \frac{1 + \nu^*}{3}$.

Nevertheless, first let us notice also what follows: the previously written equation can be transformed, analogously to the initial reasoning, into the following form:

$$(C5)'^* \quad \frac{1 + \nu^*}{3} \sigma_f^{*2} + 3(1 - 2\nu^*)(p + \sigma'^*)^2 = k'^{*2},$$

where:

$$\sigma'^* = \frac{k_c - k_r}{2(1 + \nu^*)},$$

$$k'^{*2} = k_c k_r + \frac{3}{4} \frac{(k_c - k_r)^2}{1 - 2\nu^*} = \frac{3\varphi^2 k_s^2 (k_c + k_r)^2 - 4k_c^2 k_r^2}{4(3\varphi^2 k_s^2 - k_c k_r)} = -k_1'^{*2}.$$

In the system (p, σ_f^*) or (p^*, σ_f^*) the Eq. (C5)'* represents figures analogous to the ones given before – of course with certain subtle differences, the presence of which is obvious for $\lambda \neq \frac{1}{2}$ that is $\varphi \neq 1$ or $\nu^* \neq \nu$ i.e. $\delta^* \neq 0$. These differences mean that everywhere instead of k_s we will write φk_s , and instead of ν we will insert ν^* and finally, we will replace σ_f for σ_f^* .

The present discussion has only a sketchy character; for this reason we omit discussion of these new details. Let us notice, however, that the current and continued mathematical argument is in the present conditions valid only with the assumption of inequality $\sigma_1 > \sigma_2 > \sigma_3$ or relatively $\sigma_1 < \sigma_2 < \sigma_3$, without which we managed in the previous part of the section (C5).

By expanding the last equation we come to the fundamental formula of the author's hypothesis for quasi-isotropic bodies, as below:

$$(C5)^* \quad \boxed{\sigma_1^2 + (1 + 2\delta^*)\sigma_2^2 + \sigma_3^2 - 2(\nu^* + \delta^*) \left(\sigma_2\sigma_3 + \frac{\nu^* - \delta^*}{\nu^* + \delta^*} \sigma_3\sigma_1 + \sigma_1\sigma_2 \right) + (k_c - k_r)(\sigma_1 + \sigma_2 + \sigma_3) = k_c k_r.}$$

The equation (C5)* represents – with omission of certain slight changes which would result from the introduction of p^* – the final form of the improved hypothesis, and so we should devote next a few comments to it. With the assumption that $\nu^* = \nu$, which means $\delta^* = 0$, [the formula] (C5)* transforms, of course, into (C5), i.e. into the form involving – depending on the values of ν and \varkappa – various special cases, [including] among others all hypotheses of the group C, which were already extensively commented. [...]

As for the special cases of the formula (C5)*, these arise, before all, in the case of $\nu^* = \frac{1}{2}$; then the hypothesis transforms into the equation:

$$(1 - \lambda)(\sigma_2 - \sigma_3)^2 + \lambda(\sigma_3 - \sigma_1)^2 + (1 - \lambda)(\sigma_1 - \sigma_2)^2 + (k_c - k_r)(\sigma_1 + \sigma_2 + \sigma_3) = k_c k_r.$$

The assumption $\lambda = 0$ leads now to one special form, unknown to us until now. The assumption $\lambda = \frac{1}{2}$ gives one of the forms of (C5) already discussed for $\nu = \frac{1}{2}$. Finally, putting $\lambda = 1$, we obtain an equation which for plane states (i.e. for $\sigma_2 = 0$) becomes identical to the corresponding one in Mohr's [theory] (A5). If for an arbitrary λ we assume $k_c = k_r = k$, we will obtain the correct theory (C3), namely:

$$(1 - \lambda)(\sigma_2 - \sigma_3)^2 + \lambda(\sigma_3 - \sigma_1)^2 + (1 - \lambda)(\sigma_1 - \sigma_2)^2 = k^2.$$

The simplicity of the last equation, hiding in itself the theories (C3) and (A3), deserves special emphasizes and attention; let us devote some time to it at the end of this section.

Coming back to the general form (C5)*, let us try to show it graphically. For this purpose, let us – similarly to previous considerations – ascertain that the discussed equation can be transformed to the form:

$$(C5')^* \quad \sigma_1'^{*2} + (1 + 2\delta^*)\sigma_2'^{*2} + \sigma_3'^{*2} - 2(\nu^* + \delta^*) \cdot \left(\sigma_2'^* \sigma_3'^* + \frac{\nu^* - \delta^*}{\nu^* + \delta^*} \sigma_3'^* \sigma_1'^* + \sigma_1'^* \sigma_2'^* \right) = k'^{*2} = -k_1'^{*2},$$

where: $\sigma_i'^* = \sigma_i + \sigma'^*$ and the meanings of the expressions $\sigma_i'^*$ and $k_i'^*2 = -k_1'^*2$ remain unchanged. The equation (C5)* is valid for arbitrary \varkappa and $\nu^* \neq \frac{1}{2}$. In the case of $\nu^* = \frac{1}{2}$ and $\varkappa > 1$, the transformation leads to the function:

$$(C5'')^* \quad \sigma_1''*2 + (1 + 2\delta^*)\sigma_2''*2 + \sigma_3''*2 - 2(1 + 2\delta^*)\sigma_2''*\sigma_3''* - 2(1 - 2\delta^*)\sigma_3''*\sigma_1''* \\ - (1 + 2\delta^*)\sigma_1''*\sigma_2''* + (k_c - k_r)(\sigma_1''* + \sigma_2''* + \sigma_3''*) = 0,$$

where: $\sigma_i''* = \sigma_i + \sigma''*$ and $\sigma''* = -\frac{k_c k_r}{3(k_c - k_r)}$.

Finally, for $\nu^* = \frac{1}{2}$ and $\varkappa = 1$ the hypothesis will be expressed by the equation just written above, which – because of the currently reduced relation: $\lambda = \frac{1}{2} - \delta^*$ – will assume after rearrangement the form:

$$(C5''')^* \quad \sigma_1^2 + (1 + 2\delta^*)\sigma_2^2 + \sigma_3^2 - 2(1 + 2\delta^*)\sigma_2\sigma_3 \\ - 2(1 - 2\delta^*)\sigma_3\sigma_1 - (1 + 2\delta^*)\sigma_1\sigma_3 = k^2.$$

Introduction of the parameter a into the equations (C5')*, (C5'')* and (C5''')* leads to the types similar to (C5₁), (C5₂), (C5'₁) and (C5''₂). Their discussion leads to appropriate determination of the intervals in which Mohr's circles have, or relatively do not have, envelopes and for the first ones leads to the shapes of the envelopes, picture of which is slightly different from the previous graphs. For this reason we omit the respective illustration devoting more attention to Haigh's limit surfaces.

The last one, in the case of $\nu^* < \frac{1}{2}$, is a triaxial ellipsoid with the lengths of the half-axes:

$$b_1^* = \frac{k'^*}{\sqrt{1 + \nu^* + 3\delta^*}}, \quad b_2^* = \frac{k'^*}{\sqrt{1 - 2\nu^*}}, \quad b_3^* = \frac{k'^*}{\sqrt{1 + \nu^* - \delta^*}},$$

[the ellipsoid] is shifted to the centre: $\sigma_1 = \sigma_2 = \sigma_3 = -\sigma'^*$. In the case of $\nu^* = \frac{1}{2}$, $\varkappa > 1$, we obtain an elliptical paraboloid with a vertex in the point: $\sigma_1 = \sigma_2 = \sigma_3 = -\sigma''*$ and the parameters:

$$q_1^* = \frac{k_c - k_r}{1 + 2\delta^*} \frac{1}{\sqrt{3}}, \quad q_3^* = \frac{k_c - k_r}{3 - 2\delta^*} \sqrt{3}.$$

Under the conditions: $\nu^* = \frac{1}{2}$ and $\varkappa = 1$ the critical surface is an elliptical cylinder with the semi-axes:

$$b_1^* = \frac{k_1^*}{\sqrt{1 + \nu^* + 3\delta^*}}, \quad b_2^* = \frac{k_1^*}{\sqrt{2\nu^* - 1}}, \quad b_3^* = \frac{k_1^*}{\sqrt{1 + \nu^* - \delta^*}}.$$

This cylinder can degenerate into two parallel planes for $\delta^* = -\frac{1}{2}$ or relatively $\lambda = 1$ (or even transform into a hyperboloid cylinder for $\lambda > 1$). Finally, the assumption $\nu^* > \frac{1}{2}$ hides in itself a two-shell triaxial hyperboloid with the centre: $\sigma_1 = \sigma_2 = \sigma_3 = -\sigma'^*$ and the semi-axes:

$$b_1^* = \frac{k_1^*}{\sqrt{\nu^* + 1 + 3\delta^*}}, \quad b_2^* = \frac{k_1^*}{\sqrt{2\nu^* - 1}}, \quad b_3^* = \frac{k_1^*}{\sqrt{\nu^* + 1 - \delta^*}}.$$

Here belongs also the special case determined by the assumption:

$$k_s = \frac{1}{\varphi} \frac{2}{\sqrt{3}} \frac{k_c k_r}{k_c + k_r},$$

leading to an elliptical cone as the searched surface.

The contour of a plane state is shown regardless of $\nu^* < \frac{1}{2}$, $\nu^* = \frac{1}{2}$, $\nu^* > \frac{1}{2}$ by the equation:

$$\sigma_{1^*}^2 + \sigma_{3^*}^2 - 2(\nu^* - \delta^*)\sigma_{1^*}\sigma_{3^*} = k_*^2,$$

where:

$$\sigma_{i^*} = \sigma_i + \sigma_*, \quad \sigma_* = \sigma'^* \frac{1 - 2\nu^*}{1 - 2\nu^* + \delta^*} = \frac{kc - kr}{2(1 - \nu^* + \delta^*)},$$

$$k_*^2 = k'^* - \sigma'^* 2 \frac{(1 + \nu^* + 3\delta^*)(1 - 2\nu^*)}{1 - \nu^* + \delta^*} = kc kr + \frac{kc - kr}{2(1 - \nu^* + \delta^*)}.$$

Taking into account the initial assumption: $\nu^* - \delta^* = \nu$ and resulting from this the following: $1 - \nu^* + \delta^* = 1 - \nu$, we recognize in the last equation the contour known to us from the basic hypothesis (C5). It is an ellipse with the semi-axes:

$$\frac{k^*}{\sqrt{1 + \nu^* - \delta^*}} \quad \text{and} \quad \frac{k^*}{\sqrt{1 + \nu^* + \delta^*}},$$

properly translated and rotated or relatively two parallel lines.

[..., p. 136:] IX. OVERVIEW OF EXPERIMENTAL DATA

[..., p. 160:] It is possible to show that the function [defining a measure of material effort, which is] created from the components of the state of stress and possessing an assumed characteristic property, can only be the expression build from the differences between those [stress] components, that is in general:

$$f_1(\sigma_2 - \sigma_3) + f_2(\sigma_3 - \sigma_1) + f_3(\sigma_1 - \sigma_2) = K.$$

If we keep the restriction to a homogeneous form of the second degree, we will obtain from the above the equation:

$$L^*(\sigma_2 - \sigma_3)^2 + M^*(\sigma_2 - \sigma_3)^2 + N^*(\sigma_2 - \sigma_3)^2 = 3\Phi_f,$$

i.e. the formula already known to us from the Chapters III and VIII.

Finally, the demand upon the invariance of this form [with respect to arbitrary rotation] leads to the equation:

$$(\sigma_2 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 = 2k^2,$$

i.e. directly to the hypothesis (C3); additionally, for $L^* = N^* = 0$, we obtain the hypothesis (A3). [...]

[..., p. 188:] *Lwów, in December 1927.*

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