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FORCED CONVECTION IN A FLUID LAYER OF TWO IMMISCIBLE FLUIDS OCCUPING A FINITE HEIGHT OVER A NATURALLY PERMEABLE WALL

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The paper examines the influence of a bounding naturally permeable wall on the velocity and temperature distributions in a fluid layer consisting of two immiscible. Newtonian fluids, the flow being caused by a constant pressure fradient. The solution has been obtained subject to a set of appropriate boundary and matching conditions linking various regions, including the B. J. condition at the porous interface. The effect of σ (porosity parameter) and γ (the ratio of heights of upper and lower fluids) on various physical parameters of interest, has been shown graphically or through Tables.

1. Introduction

The importance of theoretical studies pertaining to heat-transfer in a steady flow of immiscible fluids through porous channels does not need much emphasis, as such flows occur in many practical situations (see [1]). A few such studies have lately been attempted by Sacheti et al. [2, 3]. While dealing with coupled fluid flows of immiscible fluids in porous channels, one does often come cross situations where the flow is not necessarily bounded above by a rigid or permeable wall (e.g. see [4]), rather, we have a free surface beneath which flow takes place. In the authors' knowledge, no theoretical attempt has so far been made to look at this problem of practical significance. The present investigation is thus aimed to consider heat-transfer in a fluid layer of two viscous, immiscible fluids which flow over a naturally permeable wall under the application of a constant pressure gradient.

2. Mathematical analysis

Let us consider the steady flow of two immiscible viscous fluids, occupying a finite thickness above a naturally permeable wall of small thickness $H\,(\ll h_2)$. The coordinates are assumed to be x and y, respectively, with the x-axis parallel to the bounding wall and at a height h_2 above it, whilst the positive direction of y is assumed to be directed towards the free surface. The flow is caused by the application of an axial pressure gradient $c\,(= -dp/dx)$.

The upper fluid with viscosity μ_i, density ϱ_1 and thermal conductivity k_1 occupies the region $0 \leqslant y \leqslant h_1$, whereas the lower fluid with viscosity $\mu_2\,(> \mu_1)$, density $\varrho_2\,(> \varrho_1)$ and thermal conductivity k_2 is assumed to occupy the region consisting of $-h_2 \leqslant y \leqslant 0$ and the porous medium [i.e. $-(h_2+H) \leqslant y \leqslant -h_2$]. The governing equations for steady fully developed flow above the permeable wall are usual Navier–Stokes equations and the equation of energy, whilst the flow in the porous medium will be governed by Darcy's law and an appropriate energy equation (cf. [2]). All the physical quantities in the analysis to follow are assumed to be independent of the axial coordinate, x.

The governing equations of motion and energy are

Zone 1 $(0 \leqslant y \leqslant h_1)$:

(2.1) $$\frac{d^2 u_1}{dy^2} = -\frac{c}{\mu_1},$$

(2.2) $$\frac{d^2 T_1}{dy^2} + \frac{\mu_1}{k_1}\left(\frac{du_1}{dy}\right)^2 = 0.$$

Zone 2 $(-h_2 \leqslant y \leqslant 0)$:

(2.3) $$\frac{d^2 u_2}{dy^2} = -\frac{c}{\mu_2},$$

(2.4) $$\frac{d^2 T_2}{dy^2} + \frac{\mu_2}{k_2}\left(\frac{du_2}{dy}\right)^2 = 0,$$

Zone 3 $[-(h_2+H) \leqslant y \leqslant -h_2)]$

(2.5) $$u_3 = \frac{cN}{\mu_2},$$

(2.6) $$\frac{d^2 T_3}{dy^2} + \frac{\mu_2}{k_2}\left(\frac{du_3}{dy}\right)^2 + \frac{\mu_2}{k_2 N} u_3^2 = 0.$$

The boundary and matching conditions are

(2.7) $$y = h_1 : \frac{du_1}{dy} = 0, \quad \frac{dT_1}{dy} = 0;$$

$$y=0:\begin{cases} u_1 = u_2, \quad \mu_1 \dfrac{du_1}{dy} = \mu_2 \dfrac{du_2}{dy}, & (2.8)\\[2ex] T_1 = T_2, \quad k_1 \dfrac{dT_1}{dy} = k_2 \dfrac{dT_2}{dy}, & (2.9)\end{cases}$$

$$y=-h_2:\begin{cases} \dfrac{du_2}{dy} = \dfrac{\alpha}{\sqrt{N}}(u_2-u_3), & (2.10)\\[2ex] T_2 = T_3, \quad \dfrac{dT_2}{dy} = \dfrac{dT_3}{dy}, & (2.11)\end{cases}$$

$$(2.12) \qquad y=-(h_2+H): \quad T_3 = T^{**},$$

where u_1, u_2 and u_3 are velocities in zones 1, 2 and 3 respectively; T_1, T_2 and T_3 are corresponding temperatures; $c\left(=-\dfrac{dp}{dx}\right)$ is the applied pressure gradient, N is the permeability of the porous medium assumed to be low and constant; T^{**} is constant temperature and α is the slip parameter. The boundary condition (2.10) is the well-known Beavers–Joseph slip condition [5]. We have assumed Darcy's law (cf. Eq. (2.5)) to govern the flow inside the porous medium, and Eq. (2.6) is the energy equation governing temperature distribution inside the porous medium.

Now we introduce the nondimensional variables

$$\eta = \frac{y}{h_2}, \quad \bar{u}_1 = \frac{u_1}{V}, \quad \bar{u}_2 = \frac{u_2}{V}, \quad \bar{u}_3 = \frac{u_3}{V}, \quad V = \frac{ch^2}{2\mu_2},$$

$$(2.13) \qquad \theta_1 = \frac{T_1}{T^{**}}, \quad \theta_2 = \frac{T_2}{T^{**}}, \quad \theta_3 = \frac{T_3}{T^{**}}, \quad s = \frac{\mu_2}{\mu_1}, \quad \beta = \frac{k_2}{k_1},$$

$$a = \frac{H}{h_2}, \quad \gamma = \frac{h_1}{h_2}, \quad \sigma = \frac{\sqrt{N}}{h_2}, \quad \mathrm{Br}_1 \ \text{(Brinkman number)} = \frac{\mu_1 V^2}{k_1 T^{**}},$$

where we have used V to non-dimensional various velocities, in the absence of the natural scale.

In view of Eq. (2.13), the set of equations (2.1) through (2.12) will transform to

$$(2.14) \qquad \frac{d^2\bar{u}_1}{d\eta^2} = -2s,$$

$$(2.15) \qquad \frac{d^2\theta_1}{d\eta^2} + \mathrm{Br}_1\left(\frac{d\bar{u}_1}{d\eta}\right)^2 = 0,$$

$$(2.16) \qquad \frac{d^2\bar{u}_2}{d\eta^2} = -2,$$

(2.17) $$\frac{d^2\theta_2}{d\eta^2}+\frac{s}{\beta}\,\mathrm{Br}_1\left(\frac{d\bar{u}_2}{d\eta}\right)^2=0,$$

(2.18) $$u_3=2\sigma^2,$$

(2.19) $$\frac{d^2\theta_3}{d\eta^2}+\frac{s}{\beta}\,\mathrm{Br}_1\left(\frac{d\bar{u}_3}{d\eta}\right)^2+4\sigma^2\,\frac{s}{\beta}\,\mathrm{Br}_1=0.$$

The boundary and matching conditions now modify to

(2.20) $$\eta=\gamma:\quad \frac{d\bar{u}_1}{d\eta}=0,\quad \frac{d\theta_1}{d\eta}=0,$$

(2.21), (2.22) $$\eta=0:\quad \begin{cases}\bar{u}_1=\bar{u}_2, & \dfrac{d\bar{u}_1}{d\eta}=s\,\dfrac{d\bar{u}_2}{d\eta},\\[2mm] \theta_1=\theta_2, & \dfrac{d\theta_1}{d\eta}=\beta\,\dfrac{d\theta_2}{d\eta},\end{cases}$$

(2.23), (2.24) $$\eta=-1:\quad \begin{cases}\dfrac{d\bar{u}_2}{d\eta}=\dfrac{\alpha}{\sigma}\,(\bar{u}_2-\bar{u}_3), & \\[2mm] \theta_2=\theta_3, & \dfrac{d\theta_2}{d\eta}=\dfrac{d\theta_3}{d\eta},\end{cases}$$

(2.25) $$\eta=-(1+a):\quad \theta_3=1.$$

The solutions of the set of Eqs. (2.14) to (2.19) subject to the conditions (2.20) to (2.25) are given by

(2.26) $$\bar{u}_1=-s\eta^2+2\gamma s\eta+B,$$

(2.27) $$\bar{u}_2=-\eta^2+2\gamma\eta+B,$$

(2.28) $$\theta_1=-\mathrm{Br}_1\,s^2\left(\frac{\eta^4}{3}-\frac{2}{3}\,A\eta^3+\frac{A^2}{2}\,\eta^2\right)+M_1\,\eta+(M_2+M_3+M_4)+ +(1+a)\left(\frac{M_1}{\beta}-M_3\right),$$

$$\theta_2=-\frac{s}{\beta}\,\mathrm{Br}_1\left(\frac{\eta^4}{3}-\frac{2}{3}\,A\eta^3+\frac{A^2}{2}\,\eta^2\right)+\frac{M_1}{\beta}\,\eta+(M_2+M_3+M_4)+ +(1+a)\left(\frac{M_1}{\beta}-M_3\right),$$

(2.29)
$$\theta_3 = -2\mathrm{Br}_1 \frac{s}{\beta} \sigma^2 \eta^2 + \left(\frac{M_1}{\beta} - M_3\right)\eta + M_4 + \left(\frac{M_1}{\beta} - M_3\right)(1+a),$$

u_3 being already known by Eq. (2.18).

In the above A, B, M_1, M_2, M_3 and M_4 are given by

$$A = 2\gamma,$$

$$B = \frac{1}{\alpha}(2\sigma + \alpha + 2\gamma\sigma + 2\alpha\gamma + 2\alpha\sigma^2),$$

$$M_1 = \frac{4}{3}\mathrm{Br}_1 \gamma^3 s^2,$$

$$M_2 = \frac{s}{\beta}\mathrm{Br}_1\left(2\gamma^2 + \frac{4}{3}\gamma - \sigma^2 + \frac{1}{3}\right),$$

$$M_3 = 4\frac{s}{\beta}\mathrm{Br}_1\left(\sigma^2 - \frac{1}{3} - \gamma^2 - \gamma\right),$$

$$M_4 = 1 + 2\mathrm{Br}_1 \frac{s}{\beta}\sigma^2(1+a)^2.$$

3. Mass, flux, skin-friction and rate of heat transfer

To obtain mass flux in the region above the permeable wall (i.e., $y \geqslant -h_2$), we follow the procedure described elsewhere [6].

Let M_1 and M_2, respectively, be the mass flow rate in $-h_2 \leqslant y \leqslant h_1$, when the boundary wall is porous and when the boundary is rigid. Then,

$$M_1 = \varrho_1 \int_0^{h_1} u_1\, dy + \varrho_2 \int_{-h_2}^{0} u_2\, dy,$$

while

$$M_2 = \operatorname*{Limit}_{\substack{\alpha\to\infty\\ \sigma\to 0}} M_1.$$

We can now define non-dimensional mass flow rates as

$$M_1^* \equiv \frac{M_1}{\varrho_2 V h_2} = b\left[\frac{2}{3}s\gamma^3 + B\gamma\right] - \left[\frac{1}{3} + \gamma - B\right],$$

and

$$M_2^* \equiv \frac{M_2}{\varrho_2 V h_2} = b\left[\frac{2}{3}s\gamma^3 + 2\gamma^2 + \gamma\right] + \left[\gamma + \frac{2}{3}\right],$$

where

$$b = (\varrho_1/\varrho_2) = 0{,}5.$$

We have computed fractional increase in mass flux as

$$\phi = \frac{M_1^* - M_2^*}{M_2^*}.$$

If τ is the skin-friction at the permeable wall ($\eta = -1$), then a non--dimensional coefficient of skin-friction, τ^*, can be defined as

$$\tau^* = \frac{\tau}{(\mu_2 V/h_2)} = \left.\frac{d\bar{u}_2}{d\eta}\right|_{\eta=-1} = 2 + A.$$

Similarly, one can obtain a non-dimensional expression for the rate of heat transfer q, between the lower fluid and permeable wall as

$$q = \left.\frac{d\theta_2}{d\eta}\right|_{\eta=-1} = 4\mathrm{Br}_1\left[\frac{1}{3} + \gamma^2 + \gamma + \frac{8s^2\gamma^3}{3\beta}\right].$$

Note that both τ^* and q are independent of the porosity parameter, σ.

Table 2 below gives values of the above parameters.

4. Numerical discussion

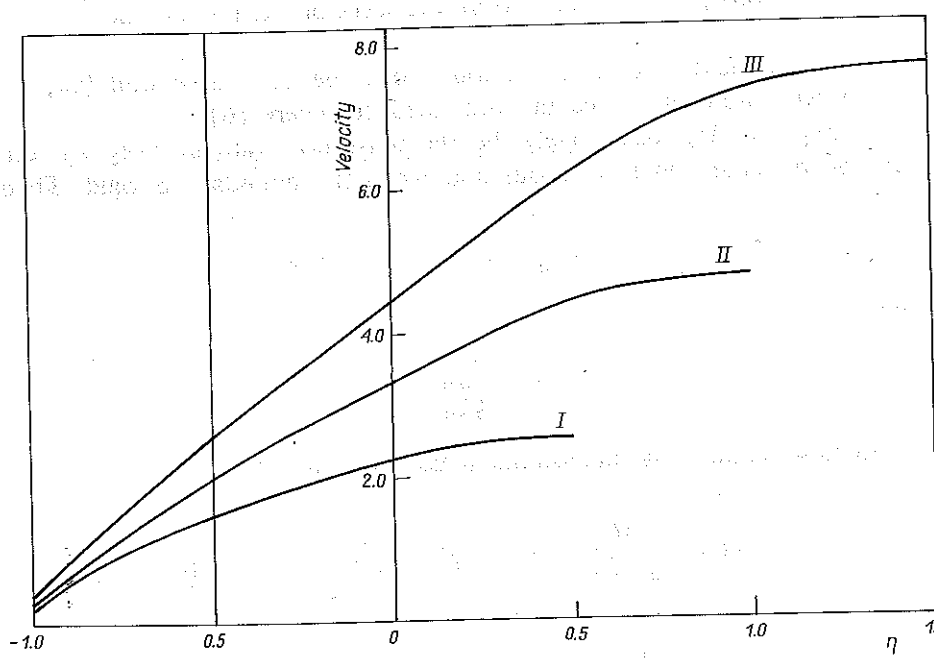

FIG. 1. Velocity distribution in upper fluid ($\eta \geq 0$) and lower fluid ($\eta \leq 0$), $\sigma = 0.1$, $\gamma = 0.5$ (I), 1.0 (II).

Velocity profiles for $\sigma = 0.1$ and $\sigma = 0.3$ have been shown graphically in Figs. 1 and 2, respectively, for fixed values of $s = 1.5$, $\beta = 1.5$, $a = 0.25$, $\mathrm{Br}_1 = 0.025$, $\alpha = 1.4$, and for a range of values of γ. The velocity behaviour

Table 1. Temperature values in different zones.

Temperature	II	$\gamma = 1.0$		$\gamma = 1.5$	
		$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.1$	$\sigma = 0.3$
θ_1	1.50	—*	—	1.576	1.580
	1.40	—	—	1.576	1.580
	1.30	—	—	1.576	1.580
	1.20	—	—	1.576	1.580
	1.10	—	—	1.576	1.580
	1.00	1.230	1.232	1.575	1.579
	0.90	1.230	1.232	1.574	1.578
	0.80	1.230	1.232	1.572	1.576
	0.70	1.230	1.232	1.569	1.573
	0.60	1.230	1.232	1.564	1.568
	0.50	1.229	1.231	1.558	1.562
	0.40	1.228	1.230	1.549	1.553
	0.30	1.226	1.228	1.538	1.542
	0.20	1.223	1.225	1.523	1.527
	0.10	1.218	1.220	1.505	1.510
	0.00	1.212	1.214	1.482	1.486
θ_2	0.00	1.212	1.214	1.482	1.486
	−0.10	1.206	1.208	1.464	1.468
	−0.20	1.200	1.202	1.443	1.447
	−0.30	1.191	1.194	1.420	1.424
	−0.40	1.182	1.184	1.393	1.397
	−0.50	1.170	1.172	1.363	1.367
	−0.60	1.156	1.158	1.328	1.332
	−0.70	1.139	1.141	1.289	1.293
	−0.80	1.120	1.122	1.246	1.250
	−0.90	1.108	1.110	1.197	1.201
	−1.00	1.070	1.072	1.140	1.142
θ_3	−1.00	1.070	1.072	1.140	1.142
	−1.05	1.056	1.056	1.114	1.115
	−1.10	1.042	1.042	1.085	1.086
	−1.15	1.028	1.028	1.056	1.057
	−1.20	1.014	1.014	1.028	1.029
	−1.25	1.000	1.000	1.000	1.000

* The symbol-means 'not applicable'.

seems to be overall similar, except for the increase in its magnitude with increase in permeability of the medium, apparently because of gain in momentum as a result of slip at the permeable surface. Temperature profiles, on the other hand, show little variation with changes in values

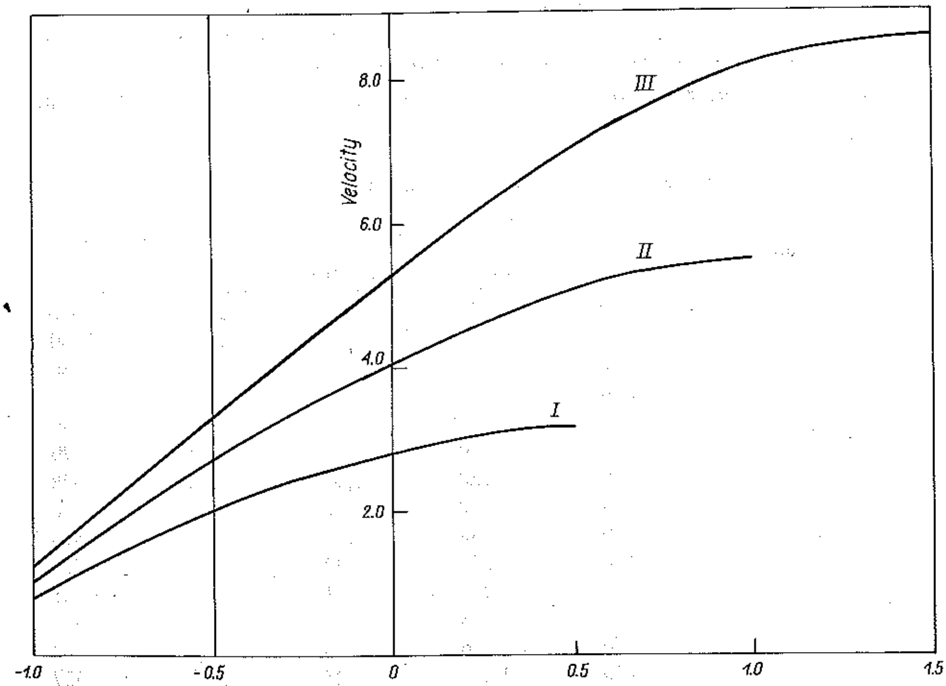

FIG. 2. Velocity distribution in upper fluid ($\eta \geqslant 0$) and lower fluid ($\eta \leqslant 0$), $\sigma = 0.3$.

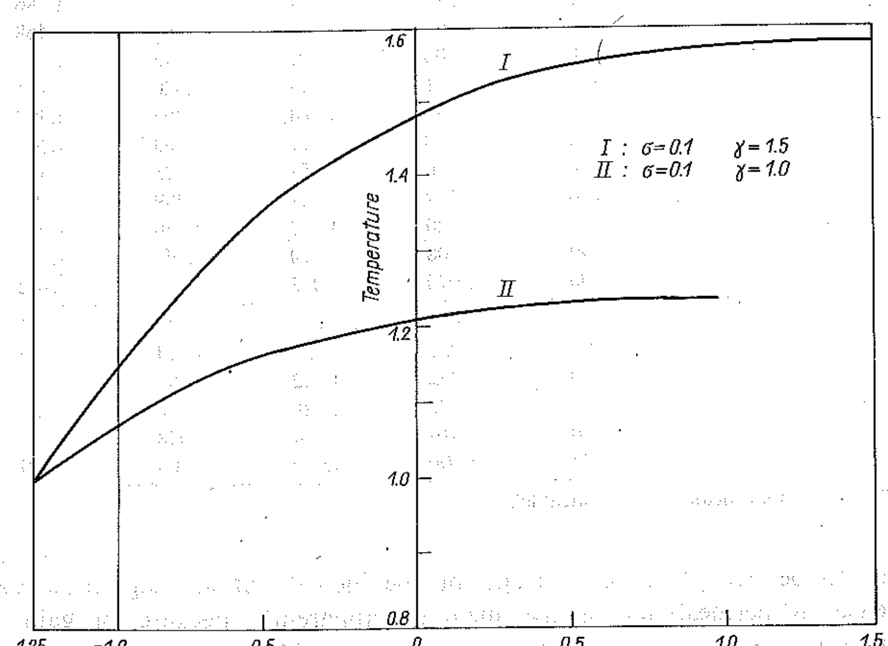

FIG. 3. Temperature distribution: θ_1 ($\eta \geqslant 0$), θ_2 ($1.0 \leqslant \eta \leqslant 0$) and θ_3 ($-1.25 \leqslant \eta \leqslant -1.0$).

Table 2. Values of mass flux, coefficient of skin-fraction and heart transfer.

γ	$[\sigma = 0.1]$	$[\sigma = 0.3]$	τ^*_{lower}	q
0.50	0.155	0.539	3	0.157
1.00	0.398	0.507	4	0.633
1.50	0.080	0.271	5	1.758

of σ (cf. Table 1). However, the variation in ratio of heights of two layers above the permeable wall does have significant bearing on temperature profiles (Fig. 3). A common feature in both velocity and temperature profiles is the observation that near the free surface the profiles tend to be increasingly insensitive to variation in the governing parameters under consideration.

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Streszczenie

WYMUSZONA KONWEKCJA W WARSTWIE O SKOŃCZONEJ GRUBOŚCI SKŁADAJĄCEJ SIĘ Z DWÓCH NIEMIESZAJĄCYCH SIĘ PŁYNÓW I LEŻĄCEJ NA PRZEPUSZCZALNEJ ŚCIANCE

W pracy rozpatrzono wpływ przepuszczalnej ścianki na rozkład prędkości i temperatury w warstwie złożonej z dwu niemieszających się płynów newtonowskich poddanej stałemu gradientowi ciśnienia. Rozwiązanie otrzymano przy uwzględnieniu układu warunków brzegowych i warunków zszycia pomiędzy różnymi obszarami przepływu, jak również na porowatej ściance. Wpływ współczynnika porowatości σ oraz stosunku grubości warstw obu cieczy γ na interesujące nas parametry przepływu zilustrowano graficznie i numerycznie.

Резюме

ВЫНУЖДЕННАЯ КОНВЕКЦИЯ В СЛОЕ КОНЕЧНОЙ ТОЛЩИНЫ СОСТОЯЩЕМ ИЗ ДВУХ НЕСМЕШИВАЮЩИХСЯ ЖИДКОСТЕЙ И ЛЕЖАЩЕМ НА ПРОНИЦАЕМОЙ СТЕНКЕ

В работе рассмотрено влияние проницаемой стенки на распределение скорости и температуры в слое, состоящем из двух несмешивающихся ньютоновских жидкостей, подвергнутых постоянному градиенту давления. Решение получено при учете системы граничных условий и условий сшития между разными областями течения, как тоже на пористой стенке. Влияние коэффициента пористости σ и отношения толщин слоев обоих жидкостей γ на интересующие нас параметры течения иллюстрировано графически и численным образом.

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