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AN ENERGY BASED MULTIAXIAL FATIGUE CRITERION

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The aim of this investigation was to develop a method of prediction of the multiaxial fatigue life. A form of the cyclic strain energy density is proposed as a damage parameter for multiaxial fatigue failure. The strain energy density includes both elastic and plastic strain energy densities, and it can be used to describe both the low- and high-cycle fatigue. The criterion proposed has an invariant property and is hydrostatic pressure sensitive. Predictions of the proposed criterion are compared with the experimental results of biaxial fatigue tests for the A-516 Gr. 70 carbon low-alloy steel. The comparison has shown good agreement.

1. INTRODUCTION

Many components are exposed to varying degrees of multiaxial loading. Fatigue under such loadings is generally referred to as multiaxial fatigue. Costly failures of this type required more reliable methods for the prediction of fatigue life. Such a prediction should based on general and basic principles and significant constants can be obtained from common tests.

Many studies have been made to propose adequate criteria for correlating the test results. The experimental and theoretical works related multiaxial fatigue were recently reviwed by KREMPL [1] and BROWN and MILLER [2]. Most of these criteria are stress or strain based. In these approaches the interrelation between the stresses and strains is overlooked. Therefore, the idea of relating fatigue life to the plastic strain energy density has been proposed [3, 4]. However, the plastic strain energy density is pressure insensitive.

The aim of this investigation is to develop a method for the prediction of the multiaxial fatigue life. A cyclic strain energy density concept is introduced for multiaxial states of stress. The approach based on the plastic strain energy density dissipated during cyclic loading and the elastic strain energy density. The proposed criterion has an invariant property, and is hydrostatic pressure sensitive and a function of strain ratio. Materials parameters can be determined mainly based on uniaxial test data.

The experimental data of biaxial fatigue for A-516 Gr. 70 steel are presented to show the correlation between the strain energy density and number of cycles to failure for different strain ratio.

2. MULTIAXIAL FATIGUE CRITERION

During cyclic loading energy is dissipated due to plastic deformation. A part of this energy is converted into heat, and the other part is rendered irrecoverable at every cycle in plastic strain energy density, ΔW^{p}. The plastic strain energy per cycle is nearly constant during the life cycle, under strain controlled conditions [5, 6]. However, when the strain range $\Delta\varepsilon^{p}_{ij} \to 0$, also corresponding $\Delta W^{p} \to 0$. It is obvious that in this case, the elastic strain energy density controls the fatigue life. Therefore the idea of relating the number of cycles to failure with strain energy density includes both elastic and plastic strain energy densities has been proposed. Consequently, to obtain the failure criterion in multiaxial fatigue, an estimation of the plastic and elastic strain energy density per cycle is required.

The elastic strain energy density for an element subjected to stress and strain history, σ_{ij} and ε_{ij}, is

$$W^{el} = \frac{1}{4\mu}\left(\sigma_{ij}\sigma_{ij} - \frac{\lambda}{2\mu+3\lambda}(\sigma_{kk})^2\right), \qquad i,j = 1,2,3, \tag{2.1}$$

where μ and λ are Lame's constants.

In the case of cyclic loading, when $\sigma^{(2)}_{ij}$ and $\sigma^{(1)}_{ij}$ are the maximum and minimum values of a cyclic stress components, cyclic elastic tensile strain energy density ΔW^{el} can be expressed as

$$\Delta W^{el} = \frac{1}{4\mu}\left(\sigma^{(2)}_{ij}\sigma^{(2)}_{ij} - \frac{\lambda}{2\mu+3\lambda}(\sigma_{ll})^2\right). \tag{2.2}$$

In general, stress components $\sigma^{(2)}_{ij}$ can be split into a mean and alternating stress, σ^{m}_{ij} and σ^{a}_{ij}, i.e.

$$\sigma^{(2)}_{ij} = \sigma^{m}_{ij} + \sigma^{a}_{ij}. \tag{2.3}$$

The mean and alternating stresses are defined by

$$\sigma^{m}_{ij} = \frac{1}{2}(\sigma^{(2)}_{ij} + \sigma^{(1)}_{ij}); \tag{2.4}$$

$$\sigma^{a}_{ij} = \frac{1}{2}(\sigma^{(2)}_{ij} - \sigma^{(1)}_{ij}). \tag{2.5}$$

Substituting Eqs. (2.4), (2.5) in relation (2.2) we get

$$\Delta W^{el} = \frac{1}{4\mu}\left[\sigma^{a}_{ij}\sigma^{a}_{ij} + 2\sigma^{a}_{ij}\sigma^{m}_{ij} + \sigma^{m}_{ij}\sigma^{m}_{ij} - \frac{\lambda}{2\mu+3\lambda}(\sigma_{ll})^2\right]. \tag{2.6}$$

Equation (2.6) can be rewritten as

$$\Delta W^{el} = \Delta W^{d} + \Delta W^{v}, \tag{2.7}$$

where the first term is interpreted as the strain energy density due to distortion of an infinitesimal element, and the second term is the strain energy density due to volume change. These relations can not be used directly in the investigations of multiaxial fatigue problem. They are insensitive to the superposed pressure, and therefore not adequate for fatigue phenomena.

According to extensive series of tests made by Morrison et al. [7, 8] the fatigue fracture is a hydrostatic pressure-sensitive process. They have found that with increasing hydrostatic compression the fatigue life increases, and hydrostatic tension reduces it. They concluded that hydrostatic part of the stress tensor has an influence on the phenomena under investigation.

We propose the following relation

$$\Delta W^e = [1+H(\varrho)]\Delta W^{el}, \tag{2.8}$$

where ϱ is the strain ratio, and H is Heaviside function. The strain ratio is defined as follows

$$\varrho = \varepsilon_2/\varepsilon_1, \tag{2.9}$$

where ε_1, ε_2 are principal strains. It is seen that for uniaxial case, $\varrho < 0$ and we obtain $\Delta W^e = \Delta W^{el}$, and for the most damaging biaxial strain $\varrho > 0$ and thus $\Delta W^e = 2\Delta W^{el}$. The plastic energy dissipated per unit volume during a given loading cycle for an element subjected to cyclically varying stress and strain history, σ_{ij} and ε_{ij}^P, is

$$\Delta W^P = \oint \sigma_{ij} d\varepsilon_{ij}^P. \tag{2.10}$$

In the analysis the attention is limited to incompressible isotropic material behaviour. It is perhaps worthwhile to mention that a material which initially may exhibit anisotropy, in certain cases may become less anisotropic upon reaching a steady-state cyclic response [6].

Because of incompressibility of plastic deformation (i.e. $d\varepsilon_{kk} = 0$), the foregoing relation can also be expressed as follows

$$\Delta W^P = \oint S_{ij} d\varepsilon_{ij}^P, \tag{2.11}$$

where $S_{ij} = \sigma_{ij} - (\delta_{ij}\delta_{kk})/3$ is the deviatoric stress tensor.

In the case of multiaxial states of stress and proportional stressing, the plastic strain, according to the deformation theory, is given by

$$\varepsilon_{ij}^P = \frac{3}{2}\left(\frac{1}{E_S} - \frac{1}{E}\right) S_{ij}, \tag{2.12}$$

where $E_S = \bar{\sigma}/\bar{\varepsilon}$ is the secant modulus of the stress-strain curve.

According to this theory, $\bar{\sigma}$ and $\bar{\varepsilon}$ are defined as follows

$$\bar{\sigma} = \left(\frac{3}{2} S_{ij} S_{ij}\right)^{1/2}; \tag{2.13}$$

$$\bar{\varepsilon} = \left(\frac{2}{3} \varepsilon_{ij} \varepsilon_{ij}\right)^{1/2}. \tag{2.14}$$

The incremental form of the plastic strain (2.12) is

$$d\varepsilon_{ij}^{P} = \frac{3}{2}\left(\frac{1}{E_S} - \frac{1}{E}\right) dS_{ij} + \frac{3}{2}\left(\frac{1}{E_t} - \frac{1}{E_S}\right) S_{ij} \frac{d\bar{\sigma}}{\bar{\sigma}}, \tag{2.15}$$

where $E_t = d\bar{\sigma}/d\bar{\varepsilon}$ is the tangent modulus of the stress-strain curve.

It has been demonstrated that under multiaxial symmetric or almost symmetric cyclic loading, a closed steady loop is obtained after a transition period. Our consideration will be limited to the steady shape of the hysteresis loop. Connecting the tips of steady loops drawn on the $\bar{\sigma} - \bar{\varepsilon}$ diagram we obtain an effective stress-strain curve. It is generally assumed that the effective stress-strain curve for multiaxial proportional loading takes the form analogous to the uniaxial cyclic stress-strain relationship. Therefore, in coordinates $\overline{\Delta\sigma}' - \overline{\Delta\varepsilon}'$ (see Fig. 1), it can be expressed as

$$\frac{\overline{\Delta\varepsilon}'}{2} = \frac{\overline{\Delta\varepsilon}^{e'}}{2} + \frac{\overline{\Delta\varepsilon}^{P'}}{2} = \frac{\overline{\Delta\sigma}'}{2E} + \left(\frac{\overline{\Delta\sigma}'}{2K'}\right)^{1/n'}, \tag{2.16}$$

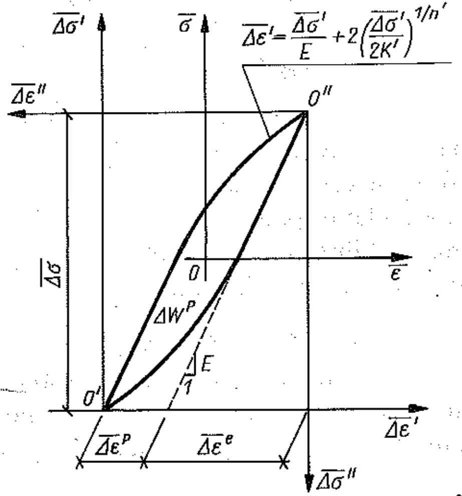

FIG. 1. Hysteresis loop.

where $\overline{\Delta\varepsilon}^{P'}$ is the effective plastic strain range, $\overline{\Delta\sigma}'$ is the effective stress range, E is Young's modulus, and K' and n' are the material constants.

The values of $\overline{\Delta\varepsilon}^{P'}$, $\overline{\Delta\sigma}'$ are defined as follows:

$$\begin{aligned} \overline{\Delta\varepsilon}^{P'} &= \left[\frac{2}{3}(\Delta\varepsilon_{ij}^{P})'(\Delta\varepsilon_{ij}^{P})'\right]^{1/2}, \\ (\Delta\varepsilon_{ij}^{P})' &= (\varepsilon_{ij}^{P(2)})' - (\varepsilon_{ij}^{P(1)})'; \end{aligned} \tag{2.17}$$

$$\overline{\Delta\sigma'} = \left[\frac{3}{2}(\Delta S_{ij})'(\Delta S_{ij})'\right]^{1/2}, \quad (\Delta S_{ij})' = (S_{ij}^{(2)})' - (S_{ij}^{(1)})', \tag{2.18}$$

where subscripts (2) and (1) denote again the maximum and minimum values.

Therefore, by introducing Eq. (2.15) into (2.11) we obtain

$$\Delta W^P = \oint \bar{\sigma}\left(\frac{1}{E_t} - \frac{1}{E}\right) d\bar{\sigma} = \oint \bar{\sigma}\, d\bar{\varepsilon}^P. \tag{2.19}$$

With reference to Fig. 1 and coordinates $\overline{\Delta\sigma'} - \overline{\Delta\varepsilon'}$, the plastic strain energy density for Masing material behaviour, ΔW^P, can be calculated from Eq. (2.19) by noting [5]

$$\Delta W^P = \int_0^{\Delta\sigma'} \overline{\Delta\sigma'}\, d(\overline{\Delta\varepsilon^{P'}}) - \int_0^{\Delta\sigma'} \overline{\Delta\varepsilon^{P''}}\, d(\overline{\Delta\sigma''}) = \overline{\Delta\sigma'}\,\overline{\Delta\varepsilon'} - 2\int_0^{\sigma} \overline{\Delta\varepsilon^{P'}}\, d(\overline{\Delta\sigma'}) = \frac{1-n'}{1+n'}\overline{\Delta\sigma'}\,\overline{\Delta\varepsilon^{P'}}. \tag{2.20}$$

Hence for the given values of

$$\overline{\Delta\sigma'} = \overline{\Delta\sigma} \equiv \left[\frac{3}{2}(S_{ij}^{(2)} - S_{ij}^{(1)})(S_{ij}^{(2)} - S_{ij}^{(1)})\right]^{1/2}, \tag{2.21}$$

and

$$\overline{\Delta\varepsilon^{P'}} = \overline{\Delta\varepsilon^P} \equiv \left[\frac{2}{3}(\varepsilon^{P(2)}_{ij} - \varepsilon^{P(1)}_{ij})(\varepsilon^{P(2)}_{ij} - \varepsilon^{P(1)}_{ij})\right]^{1/2}, \tag{2.22}$$

we obtain

$$\Delta W^P = \frac{1-n'}{1+n'}\overline{\Delta\sigma}\,\overline{\Delta\varepsilon^P}. \tag{2.23}$$

Combining Eqs. (2.6), (2.8) and (2.23) we get the strain energy density as

$$\Delta W^t = \Delta W^e + \Delta W^P = \frac{1}{4\mu}\left[1 + H(\varrho)\right]\left[\sigma^a_{ij}\sigma^a_{ij} + 2\sigma^a_{ij}\sigma^m_{ij} + \sigma^m_{ij}\sigma^m_{ij} - \frac{\lambda}{2\mu + 3\lambda}(\sigma_{ll})^2\right] + \frac{1-n'}{1+n'}\overline{\Delta\sigma}\,\overline{\Delta\varepsilon^P}. \tag{2.24}$$

A failure criterion for multiaxial fatigue failure can now be proposed relating ΔW^t to the number of cycles to failure, N_f, and strain ratio, ϱ,

$$\Delta W^t = f(N_f, \varrho). \tag{2.25}$$

As a particular form of equation (2.25) we suggest the power law type

$$\Delta W^t = \mathscr{K}(\varrho) N_f^{\alpha} + C. \tag{2.26}$$

In the above equation $\mathscr{K}(\varrho)$ is a functiom of strain ratios. In the first approximation these relation can be expressed as a linear function of strains ratios, i.e.

$$\mathscr{K}(\varrho) = \mathscr{K}_0 + \mathscr{K}_1 \cdot \varrho, \tag{2.27}$$

where $\mathcal{K}_0$ and $\mathcal{K}_1$ are material constants to be evaluated. By equating the relations (2.24) and (2.26) we can get the following multiaxial fatigue failure criterion

$$\left\{\frac{1}{4\mu}\left[1+H(\rho)\right]\left[\sigma_{ij}^{a}\sigma_{ij}^{a}+2\sigma_{ij}^{a}\sigma_{ij}^{m}+\sigma_{ij}^{m}\sigma_{ij}^{m}-\frac{\lambda}{2\mu+3\lambda}(\sigma_{ll})^2\right]+ \right.$$
$$\left. +\frac{1-n'}{1+n'}\overline{\Delta\sigma\,\Delta\varepsilon^{P}}\right\}=\mathcal{K}(\varrho)N_f^{\alpha}+C. \tag{2.28}$$

At the high cycle fatigue (when $N_f\to\infty$), the term within the second curly bracket vanish ($\overline{\Delta\varepsilon^P}\approx 0$), and we get the condition

$$[1+H(\varrho)]\,\Delta W^{el}=C. \tag{2.29}$$

The magnitude of C can be obtained from the uniaxial conditions

$$C=[1+H(\varrho)]\,C_u=\frac{1}{2E}[1+H(\varrho)]\,\sigma^2_{\text{endurance}}. \tag{2.30}$$

That is, constant C_u in Eq. (2.30) is the elastic strain energy density at the material endurance limit.

Similarly, at the low cycle fatigue regimes the terms described as elastic energies are small in comparison with the other terms, and if they are neglected, we obtain

$$\frac{1-n'}{1+n'}\overline{\Delta\sigma\,\Delta\varepsilon^{P}}=\mathcal{K}N_f^{\alpha}. \tag{2.31}$$

We have thus demonstrated that the fatigue failure criterion (2.28) is a general one which applies to the entire spectrum of the failure lives.

Noting that $\bar{\sigma}=(3J_2)^{1/2}$ and $\sigma_{ll}=I_1$, the equation (2.31) can be presented in general form as

$$G(I_1,\,J_2)=L(N_f,\,\varrho). \tag{2.32}$$

It should be noted that the criterion (2.28) is hydrostatic pressure-sensitive and has an invariant property, i.e. it is a frame-indifference criterion. Additionally, the criterion (2.28) includes the influence of mean stress on fatigue fracture.

For symmetrical loading ($\sigma_{ij}^{m}=0$) the equation (2.28) is reduced to the following form:

$$\frac{1+H(\varrho)}{4\mu}\left[\sigma_{ij}^{a}\sigma_{ij}^{a}-\frac{\lambda}{2\mu+3\lambda}(\sigma_{ll})^2\right]+\frac{1-n'}{1+n'}\overline{\Delta\sigma\,\Delta\varepsilon^{P}}=\mathcal{K}N_f^{\alpha}+C. \tag{2.33}$$

For uniaxial loading, criterion (2.33) can be expressed as [9, 10]

$$\frac{1-n'}{1+n'}\Delta\sigma\,\Delta\varepsilon^{P}+\frac{\Delta\sigma^2}{8E}=[\mathcal{K}_0+\mathcal{K}_1(-\gamma)]\,N_f^{\alpha}+C_u. \tag{2.34}$$

3. Comparison with experimental results

To examine the applicability of the proposed criterion we must first determine the material parameters n', k, σ_{end}, E, γ; their values can be obtained from uniaxial tests or found in material handbooks. To determine the last two paraeters appearing in Eq. (2.28), $\mathscr{K}_0$ and $\mathscr{K}_1$, the data for two strain ratios ϱ are necessary, in that one can be uniaxial case. The results are compared with experimental data from fully reversed strain-controlled tests under biaxial stress condition reported by Lefebvre *et al.* [11]. The tests were conducted on thin-walled tubes made of A-516 Gr. 70 low-alloy carbon steel, subjected to a combination of cyclic axial loading and external-internal pressure. Specimens were cyclically loaded in the axial direction, while pressure were applied to the inside and outside, alternatively during each half cycle. During the test the strain ratio was kept constant.

From uniaxial tests we obtain $\alpha = -0.51$ and, using the results concerning the strain ration $\varrho = 1$, we have determined $\mathscr{K}_0 = 180$ MJ/m^3 and $\mathscr{K}_1 = -142$ MJ/m^3.

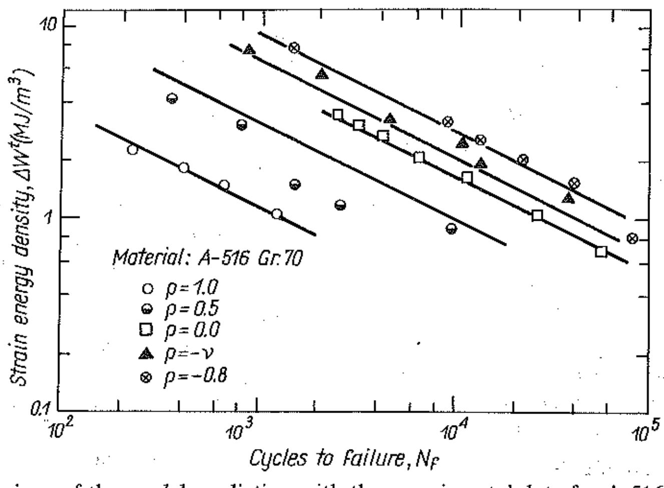

Fig. 2. Comparison of the model prediction with the experimental data for A-516 Gr. 70 steel.

The predictions of equation (2.28) are compared with experimental data in Fig. 2 for different strain ratios; it is seen that the criterion exhibits the same general tendency as the experimental data. Correlation with the experimental data is rather good except for strain ratio $\varrho = 0.5$. The dispersion is probably due to additional factors such as the effect of bending strain etc. Lefebvre [11] suggested the need for additional tests. Generally, there exists a good correlation between the proposed criterion and test results.

4. Conclusions

A new criterion for multiaxial fatigue is developed based on the strain energy density. According to this criterion, the dissipated plastic strain energy density and elastic energy density can be used to describe fatigue life. The criterion has an invariant property and is hydrostatic stress-sensitive. The criterion can be used for life prediction in both the low- and high-cycle regimes. The majority of material parameters can be determined from the uniaxial test data.

This concept has been used to correlate the biaxial strain fatigue data for the A-516 Gr. 70 carbon low–alloy steel. The comparison of the investigated model and experimental data has shown good agreement.

The proposed criterion seems to yield a promising approach for fatigue analysis of materials under multiaxial loading.

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Streszczenie

ENERGETYCZNE KRYTERIUM ZMĘCZENIA W WIELOOSIOWYM STANIE ODKSZTAŁCENIA

Celem rozważań jest opracowanie metody pozwalającej przewidywać trwałość zmęczeniową przy wieloosiowych stanach odkształcenia. Zaproponowano pewną postać gęstości energii odkształceń cyklicznych jako parametr uszkodzenia w wieloosiowym zniszczeniu zmęczeniowym. Do gęstości energii włącza się zarówno energię odkształcenia sprężystego jak i plastycznego i stosuje się ją do oceny zmęczenia nisko- i wysokocyklowego. Zaproponowane kryterium ma pewne własności niezmienniczości oraz uwzględnia efekt ciśnienia hydrostatycznego. Wyniki przewidywane na podstawie tego kryterium porównano z danymi doświadczalnymi otrzymanymi z dwuosiowego testu zmęczeniowego wykonanego na próbkach niskostopowej stali A-516 Gr. 70. Porównanie wykazuje dobrą zgodność wyników.

Резюме

ЭНЕРГЕТИЧЕСКИЙ КРИТЕРИЙ УСТАЛОСТИ В МНОГООСЕВОМ ДЕФОРМАЦИОННОМ СОСТОЯНИИ

Целью рассуждений является разработка метода позволяющего предсказывать усталостную прочность при многоосевых деформационных состояниях. Предложен некоторый вид плотности энергии циклических деформаций как параметра повреждения в многоосевом усталостном разрушении. К плотности энергии включается так энергия упругой деформации, как и пластической деформации и применяется она для оценки низко- и высокоциклической усталостей. Предлагаемый критерий имеет некоторые свойства инвариантности, а также учитывает эффект гидростатического давления. Результаты, предсказываемые на основе этого критерия, сравнены с экспериментальными данными, полученными из двухосевого усталостного теста, проведенного на образцах низколегированной углеродистой стали A-516 Gr. 70. Сравнение показывает хорошее совпадение результатов.

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