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DETERMINATION OF GENERALIZED INERTIAL FORCES IN RELATIVE MOTION OF MECHANICAL SYSTEMS OF A RAILWAY-VEHICLE TYPE

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Comparing with what is known from the literary sources, this paper presents a simpler method of determining the inertial forces which result from the relative motion for the mechanical systems consisting of many rigid bodies with holonomic and nonholonomic constraints. The method is based on a theorem, the proof of which will be included in this work. Although the presented method has been worked out primarily to meet the need of modelling railway vehicle vibrations, for which the assumptions of the theorem are easily satisfiable, it can also be applied to any mechanical system satisfying the mentioned assumptions.

1. Introduction

Nowadays it is common to apply nonlinear mathematical models in mechanics and this is due to the growing capabilities of carrying out numerical analyses of such models. The same happens in railway vehicle dynamics. Railway vehicle vibrations can be treated as relative motion in relation to the transportation coordinate systems (i.e., moving at a constant speed along an ideal track, that is without any geometrical irregularities and purely rigid). In the case when the motion of a vehicle takes place either along a circular railway track or a transition curve, the above-mentioned coordinate system are noninertial with respect to the system which is rigidly connected with the earth and which, in turn, can be treated as the inertial one. This is why in the equations of vibrational motion of a vehicle that moves along a curved track there should appear the so-called imaginary (this term is also known and we shall use it here interchangeably with the term "inertial") forces [2, 4] which result from relativity of motion. In [5] it has been shown that the maximal values of the components defining the imaginary forces are of the same order as the maximal values of other components with nonlinearities of a kinematic type. These last components appear in the equations of motion as a result of differentiating the kinetic energy of the system according to one of the chosen formalisms of building the equations of motion [1, 4] – the formalism should be taken from Analytical Mechanics. Thus it would be an inconsistency to ignore the inertial (imaginary) forces in the equations of relative motion where nonlinearities of a kinematic type are included.

It appears, however, that the determination of forces of interest to us may be difficult in the case of a system consisting of even a few rigid bodies, the motion of which is restricted by constraints. The difficulties grow along with the increase in the dimension of the system and the number of constraints.

2. Generalized inertial forces

In the case of a single free rigid body in relative motion, both the scalar equations and inertial forces can be determined basing on the vectorial equations of relative motion [2].

$$
\begin{gathered}
m\mathbf{p}_{wc} = \mathbf{P} - m\boldsymbol{\omega}\times\mathbf{v} - m\boldsymbol{\varepsilon}\times\mathbf{r}_0' - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_0') - 2m\boldsymbol{\omega}\times\mathbf{v}_c',\\
J^*\dot{\boldsymbol{\omega}}' + \boldsymbol{\omega}'\times J^*\boldsymbol{\omega}' = \mathbf{M}_C - J^*\boldsymbol{\varepsilon} - \boldsymbol{\omega}\times J^*\boldsymbol{\omega} - 2\boldsymbol{\omega}'\times\left(J^* - \frac{1}{2}\vartheta^* E\right)\boldsymbol{\omega},
\end{gathered}
\tag{2.1}
$$

where the dot "$\cdot$" designates (here and throughout the paper) the time differentiation, m is the body mass, J is the body inertia tensor, $\mathbf{p}_{wc}$ is the vector of the body mass center acceleration in relative motion, $\mathbf{P}$ is the main vector of the external forces, $\mathbf{M}_C$ is the main vector of the moments of force in relation to the body mass center, $\mathbf{v}$ is the transportation linear velocity vector, $\boldsymbol{\omega}$ is the transportation angular velocity vector, $\mathbf{r}_0'$ is the radius-vector of the body mass center in a noninertial transportation system, $\boldsymbol{\omega}'$ is the body angular velocity in relative motion, E is the unit tensor and ϑ is the first invariant of the tensor J.

The first of the equations (2.1) describes the motion of the body mass center C while the second one describes the spherical motion round this center. While building the scalar equations on the basis of the second of the equations (2.1), it is advisable to express it in the system rigidly connected with the body — the system of the principal and central axes of inertia of the body would be the best one since then the components of tensor J are constant. However, from the formal point of view, it is not obligatory. To stress the fact that we shall prefer expressing this equation in the system of the principal central axes of inertia, the tensor J has been distinguished by the index "*".

In the case of the systems with constraints, we have to apply one of the formalisms derived from Analytical Mechanics in order to build the equations of motion. This, as we know, results from the fact that it is impossible to consider the equations of constraints directly in the vectorial equations of a rigid body in relative motion (these equations are a direct adaptations of Newton's equations for this case). As it has been shown in [2], generalized forces of inertia should be added to the remaining generalized forces without regard to both the applied formalism of building the equations of motion and to whether the assumed coordinates are quasi-coordinates [2, 4] or generalized coordinates [1, 2] (Lagrange's coordinates [3]). And thus, for the commonest Lagrange's formalism [1, 3, 4], the equations of relative motion have the following form [2]:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\sigma}\right)-\frac{\partial T}{\partial q_\sigma}=Q_{Z\sigma}+Q_{B\sigma}, \qquad \sigma=1,\ldots,k, \tag{2.2}$$

where T is the kinetic energy of the system in relative motion, q_σ is the generalized coordinate, $Q_{Z\sigma}$ are generalized external forces, $Q_{B\sigma}$ are generalized inertial forces and k is the number of degrees of freedom of the system [1, 3, 4].

In [2] it has also been shown that the generalized imaginary forces Q_B, which should be adjoined to generalized forces Q_Z while building the differential equations of relative motion for a free rigid body (its position is defined by l independent generalized coordinates q_λ where $\lambda = 1, \ldots, l = 6$), can be expressed in the following way[(1)]:

$$\begin{aligned} Q_{B\lambda} = &-m\left(\frac{d'\mathbf{v}}{dt}+\boldsymbol{\omega}\times\mathbf{v}\right)\frac{\partial \mathbf{r}'_0}{\partial q_\lambda}-m[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}'_0)]\frac{\partial \mathbf{r}'_0}{\partial q_\lambda}-\\ &-\boldsymbol{\omega}\times J\boldsymbol{\omega}\frac{\partial \boldsymbol{\omega}'}{\partial \dot{q}_\lambda}-m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}'_0\frac{\partial \mathbf{r}'_0}{\partial q_\lambda}-J\frac{d\boldsymbol{\omega}}{dt}\frac{\partial \boldsymbol{\omega}'}{\partial \dot{q}_\lambda}-\\ &-2m\boldsymbol{\omega}\times\frac{d'\mathbf{r}'_0}{dt}\frac{\partial \mathbf{r}'_0}{\partial q_\lambda}-2\boldsymbol{\omega}'\times\left(J-\frac{1}{2}E\vartheta\right)\boldsymbol{\omega}\frac{\partial \boldsymbol{\omega}'}{\partial \dot{q}_\lambda}. \end{aligned} \tag{2.3}$$

The meanings of the designations appearing in Eqs. (2.3) are the same as of those appearing in Eqs. (2.1). Only the symbol d'/dt requires an explanation. It is the designation of the local derivative in the transportation coordinate system. Equation (2.3) has been obtained in [2] for a free rigid body, starting from the expression that defines generalized forces of inertia for the system of m particles. Following the transition from the system of m particles to a rigid body, it can be easily noticed that there would be no qualitative differences if, instead of passing to a single rigid body, we pass to the system of $p < m$ of such bodies. We have done it, and thus the equation equivalent to Eq. (2.3) receives the form

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(1) The following definitions [2] have been assumed for the components appearing at the right-hand side of Eq. (2.3): the first component – generalized inertial forces of translatory motion; the sum of the second and the third one – generalized centrifugal forces of inertia; the sum of the fourth and the fifth – generalized rotational forces of inertia; the sum of the sixth and the seventh – generalized gyroscopic forces.

$$(2.4)\qquad Q_{B\lambda n}=\sum_{n=1}^{p}\Big\{-m_n\Big(\frac{d'\mathbf{v}_n}{dt}+\boldsymbol{\omega}_n\times\mathbf{v}_n\Big)\frac{\partial\mathbf{r}'_{0n}}{\partial q_{\lambda n}}-$$
$$-m_n[\boldsymbol{\omega}_n\times(\boldsymbol{\omega}_n\times\mathbf{r}'_{0n})]\frac{\partial\mathbf{r}'_{0n}}{\partial q_{\lambda n}}-\boldsymbol{\omega}_n\times J\boldsymbol{\omega}_n\frac{\partial\boldsymbol{\omega}'_n}{\partial\dot{q}_{\lambda n}}-m_n\frac{d\boldsymbol{\omega}_n}{dt}\times\mathbf{r}'_{0n}\frac{\partial\mathbf{r}'_{0n}}{\partial q_{\lambda n}}-J_n\frac{d\boldsymbol{\omega}_n}{dt}\frac{\partial\boldsymbol{\omega}'_n}{\partial\dot{q}_{\lambda n}}-$$
$$-2m_n\boldsymbol{\omega}_n\times\frac{d'\mathbf{r}'_{0n}}{dt}\frac{\partial\mathbf{r}'_{0n}}{\partial q_{\lambda n}}-2\boldsymbol{\omega}'_n\times\Big(J_n-\frac{1}{2}E\vartheta_n\Big)\boldsymbol{\omega}_n\frac{\partial\boldsymbol{\omega}'_n}{\partial\dot{q}_{\lambda n}}\Big\}.$$

With that, the position of such a system is defined by the coordinates $q_{\lambda n}$ to the number of $l\cdot p=6p$. We shall also define the index $\alpha=\lambda+6(n-1)=1,\ldots,6p$ that is a substitute for λn. For a single rigid body, Eq. (2.4) passes, of course, to Eq. (2.3).

If we want Eqs. (2.3) and (2.4) to be used to determine generalized forces of inertia, the coordinates q_λ and $q_{\lambda n}$ must fulfill certain conditions. These have, of course, been given in Eqs. (2.2). Namely, they must be assumed for each of the free bodies in such a way as to let three of them define the position of the body mass center, and the other three define the body spherical motion round the mass center (to express it more precisely, the angular motions of the coordinate system rigidly connected with the body in relation to the transportation system). This restriction imposed upon coordinates is easy to be guessed by comparing Eq. (2.3) with the inertial forces in Eqs. (2.1) — these forces are expressed by the terms appearing in Eqs. (2.1) appropriately after P and M_C. It is obvious that the same results will be obtained on the basis of Eqs. (2.1) and (2.3) on condition that we assume the same coordinates for both cases.

Let us agree that from now on, to distinguish the inertial forces obtained for a free rigid body on the basis of Eq. (2.3), we shall designate them with the index "′". And so

$$Q_{B1}=Q'_{B1},\quad Q_{B2}=Q'_{B2},\quad Q_{B3}=Q'_{B3},\quad Q_{B4}=Q'_{B4},\quad Q_{B5}=Q'_{B5},\quad Q_{B6}=Q'_{B6}.$$

If we restrict the motion of the system of free rigid bodies with holonomic constraints to the number of $w\leqslant 6p$ of the form

$$(2.5)\qquad a_{\mu1}(q_1\ldots q_k)\dot{q}_1+a_{\mu2}(q_1\ldots q_k)\dot{q}_2+\ldots+a_{\mu\alpha}(q_1\ldots q_k)\dot{q}_\alpha=0,$$

where $k=(6p-w)$ is the number of degrees of freedom of a holonomic system [1, 3, 4], then Eq. (2.4), as the one that defines generalized imaginary forces, is still in force. However, it is necessary, while determining Q_B, to consider Eqs. (2.5) in such a way as to make (2.4) dependent on independent coordinates q_σ ($\sigma=1,\ldots,k$).

Equation (2.4) can be criticized considering its practical application to determine imaginary forces for mechanical systems consisting of several bodies with imposed constraints. And so, there is a flaw in it — it requires arduous determining of k expressions of $Q_{B\sigma}$ of a much more complicated form in the case of the systems with constraints than the form of Q'_{B1}, Q'_{B2}, Q'_{B3}, Q'_{B4}, Q'_{B5},

Q'_{B6} expressions which are obtained on the basis of Eq. (2.1) or Eq. (2.3). At the same time the cost of labour increases significantly along with the increase in the number of degrees of freedom of the system and in the number of equations of holonomic constraints. Determination of $Q_{B\lambda n}$ expressions is labour-consuming, mainly due to the following activities and factors: determination of the multiple vector and scalar products of the vectors on the basis of their components, determination of these components and the fact that the components of the tensor J are not, in general, constant. Thus, similarly to Eq. (2.1), it is convenient to express the components of inertia tensors J_n in the systems of the principal and central axes of inertia of an appropriate body. But then, the components of the vectors that appear next to J_n must be expressed in these mentioned systems. Later on, it requires a projection of the resultant vectors, expressed in the systems of the principal central axes, into directions matching the assumed generalized coordinates. If, for instance, we considered a system of 21 degrees of freedom, 21 equations of holonomic constraints and consisting of 7 rigid bodies (a model of an eight-wheel car has similar dimension), then we would have to perform operations on about 600 different components of vectors and tensors appearing in Eq. (2.4).

The above-mentioned difficulties connected with the practical application of the relationship (2.4), led us make an attempt to formulate a less complicated method of determining the forces of inertia. It was advisable that the method would make use of the expressions that define imaginary forces for a free rigid body. It was also important to make the method usable for any railway vehicle, using at the same time the specific properties of mechanical systems of such a type in order to simplify the method.

These properties allow reducing our considerations to the case when the transportation for each of p rigid bodies is common (i.e., the systems in relation to which the relative motion of each body is considered are motionless in relation to one another) and when the coordinates that define the position of each body fulfill a certain condition. We shall demand of the coordinates (three linear and three angular for each body) to be defined analogically. The linear coordinates must define the position of the mass center for each of the bodies, and the angular ones should define the mutual position of two systems: the system of the principal central axes of inertia and the transportation system (e.g., these could be Euler's angles [2, 3, 4] for all the bodies). Having the coordinates defined in such a way, the imaginary forces for the system of p free rigid bodies are defined by $6p$ equations of Eq. (2.3) type, which are similar for each of the bodies and differentiated only by the index n. At the same time this similarity refers also to the form of the equations obtained after introducing the components of both vectors and tensors into them. The above sentences reduce themselves to the fact that for the conditions as described above, if we have the imaginary forces written for one free body, then on this basis we can write them for each of the remaining $(p-1)$ free bodies, giving all quantities an index matching each body. In the case of a system of p rigid bodies with holonomic constraints, a statement analogous to the previous one, basing directly on Eq. (2.4), is impossible. And thus the following theorem has been formulated and proved.

3. The theorem and the proof

THEOREM. *If generalized coordinates that define the position of the system of p rigid (free) bodies are determined analogically for each of the bodies and the transportation is common for them (in the sense understood in the above paragraph) and if we restrict the motion of the system with the help of $w \leqslant 6p$ holonomic constraints, then in order to determine generalized imaginary forces that would match the accepted independent generalized coordinates, it is enough to know generalized imaginary forces for one of the p bodies that is treated as the free one. With that, the imaginary forces for the system of p bodies with imposed holonomic constraints are the linear functions of the imaginary forces that refer to the system of the free rigid bodies.*

Proof. Let us first consider in Eq. (2.4) the dependence [1, 2, 3, 4] which is obligatory for the generalized coordinates

$$\frac{\partial \mathbf{r}'_{0n}}{\partial q_{\lambda n}} = \frac{\partial \mathbf{v}'_{cn}}{\partial \dot{q}_{\lambda n}}. \tag{3.1}$$

Then let us consider any system of p free rigid bodies. And so we will have $6p$ equations of the (2.3) type which are similar in hexads for each of the bodies and different in the index n which distinguishes the body (this statement refers to the forms of these equations before the components of vectors and tensors are introduced into them — compare it with a similar statement which appears 4 sentences before the text on the theorem). Then we can write

$$\mathbf{v}'_{cn} = \sum_{\lambda=1}^{3} \mathbf{e}_{\lambda n} \dot{q}_{\lambda n} \quad \text{and} \quad \boldsymbol{\omega}'_n = \sum_{\lambda=4}^{6} \mathbf{e}_{\lambda n} \dot{q}_{\lambda n}, \tag{3.2}$$

where $\dot{q}_{\lambda n}$ for $\lambda = 1, 2, 3$ are the linear generalized velocities and $\dot{q}_{\lambda n}$ for $\lambda = 4, 5, 6$ are the angular ones. The vectors $\mathbf{e}_{\lambda n}$ are the versors of the axes which are defined by the directions of action of the generalized velocities. For Eqs. (3.1) and (3.2) it will be

$$\frac{\partial \mathbf{v}'_{cn}}{\partial \dot{q}_{\lambda n}} = \mathbf{e}_{\lambda n}, \quad \lambda = 1, 2, 3 \quad \text{and} \quad \frac{\partial \mathbf{v}'_{cn}}{\partial \dot{q}_{\lambda n}} = \mathbf{0}, \quad \lambda = 4, 5, 6, \tag{3.3}$$

$$\frac{\partial \boldsymbol{\omega}'_n}{\partial \dot{q}_{\lambda n}} = \mathbf{0}, \quad \lambda = 1, 2, 3 \quad \text{and} \quad \frac{\partial \boldsymbol{\omega}'_n}{\partial \dot{q}_{\lambda n}} = \mathbf{e}_{\lambda n}, \quad \lambda = 4, 5, 6. \tag{3.4}$$

In the case of Eqs. (3.3) and (3.4), the values of $Q_{\beta\lambda n}$ forces are the sums of the scalar products of the versors $\mathbf{e}_{\lambda n}$ and of vectors appearing in front of them (in the formula (2.4)). Since $|\mathbf{e}_{\lambda n}| = 1$, then from the definition of the scalar products of vectors it results that the forces $Q_{B\lambda n}$ are simply the lengths of projections of the vectors which appear in front of the versors $\mathbf{e}_{\lambda n}$ into the directions of these versors.

Now we shall introduce the equations of holonomic constraints to the number of $w \leqslant 6p$ into the system. We shall also demand of Eqs. (2.5) to assume the following form:

$$\dot{q}_d = A_{d1}(q_1 \dots q_k)\dot{q}_1 + A_{d2}(q_1 \dots q_k)\dot{q}_2 + \dots + A_{dk}(q_1 \dots q_k)\dot{q}_k, \tag{3.5}$$

where $q_1 \dots q_k$ form a set of independent generalized coordinates [1, 4] and q_d are dependent coordinates where $d = (6p - w + 1), \dots, 6p$. Equation (3.5), due to using a duplex indexing, can be written in the following form:

$$\dot{q}_{\tau s} = \sum_{t=1}^{p} \sum_{\mu=1}^{6} A_{\tau s \mu t} \dot{q}_{\mu t}, \tag{3.6}$$

where the coefficient $A_{\tau s \mu t}$ is the function of only the independent coordinates $q_{\mu t}$; $\dot{q}_{\tau s}$ is the dependent generalized velocity ($\dot{q}_{\tau s} \neq \dot{q}_{\mu t}$ so if $\tau = \mu$, then $s \neq t$ or if $s = t$, then $\tau \neq \mu$); $\tau, \mu = 1 \dots 6$; $s, t = 1 \dots p$; the number of coordinates $q_{\tau s}$ is equal to w, and the number of coordinates $q_{\mu t}$ is equal to k. Of course, still if $\tau, \mu = 1, 2, 3$, then the generalized velocity is linear and if $\tau, \mu = 4, 5, 6$, then it is angular. If we introduce Eq. (3.6) into Eqs. (3.2), then a change in the form of (3.2) will take place for the expressions for which $n = s$ and

$$\mathbf{v}'_{cs} = \sum_{\lambda \neq \tau} \mathbf{e}_{\lambda s} \dot{q}_{\lambda s} + \sum_{\tau} \mathbf{e}_{\tau s} \Big(\sum_{t=1}^{p} \sum_{\mu=1}^{6} A_{\tau s \mu t} \dot{q}_{\mu t} \Big), \qquad \lambda, \tau = 1, 2, 3, \tag{3.7}$$

$$\boldsymbol{\omega}'_s = \sum_{\lambda \neq \tau} \mathbf{e}_{\lambda s} \dot{q}_{\lambda s} + \sum_{\tau} \mathbf{e}_{\tau s} \Big(\sum_{t=1}^{p} \sum_{\mu=1}^{6} A_{\tau s \mu t} \dot{q}_{\mu t} \Big), \qquad \lambda, \tau = 4, 5, 6. \tag{3.8}$$

Of course, in general, the number of the versors $\mathbf{e}_{\lambda s}$ and $\mathbf{e}_{\tau s}$ for the expressions (3.7) and (3.8) equals three for each of them. If we differentiate these expressions, then in the following cases we shall obtain

$$\frac{\partial \mathbf{v}'_{cs}}{\partial \dot{q}_{\tau s}} = \mathbf{0}, \qquad \text{for} \quad \lambda, \tau = 1, 2, 3,$$

$$\frac{\partial \mathbf{v}'_{cs}}{\partial \dot{q}_{\mu t}} = \sum_{\tau} \mathbf{e}_{\tau s} A_{\tau s \mu t} + \mathbf{e}_{\mu t}, \qquad \text{for} \quad \lambda, \tau = 1, 2, 3 \quad \text{and} \quad t = s,\ \mu = 1, 2, 3,\ \mu \neq \tau, \tag{3.9}$$

$$\frac{\partial \mathbf{v}'_{cs}}{\partial \dot{q}_{\mu t}} = \sum_{\tau} \mathbf{e}_{\tau s} A_{\tau s \mu t}, \qquad \text{for} \quad \lambda, \tau = 1, 2, 3 \quad \text{and} \quad t = s,\ \mu = 4, 5, 6, \quad \text{or } \lambda, \tau = 1, 2, 3 \quad \text{and} \quad t \neq s,\ \mu = 1, \dots, 6.$$

and

$$\frac{\partial \boldsymbol{\omega}'_s}{\partial \dot{q}_{\tau s}} = \mathbf{0}, \qquad \text{for } \lambda, \tau = 4, 5, 6,$$

(3.10)
$$\frac{\partial \boldsymbol{\omega}'_s}{\partial \dot{q}_{\mu t}} = \sum_{\tau} \mathbf{e}_{\tau s} A_{\tau s \mu t} + \mathbf{e}_{\mu t}, \qquad \text{for } \lambda, \tau = 4, 5, 6 \text{ and } t = s,\ \mu = 4, 5, 6,\ \mu \neq \tau,$$

$$\frac{\partial \boldsymbol{\omega}'_s}{\partial \dot{q}_{\mu t}} = \sum_{\tau} \mathbf{e}_{\tau s} A_{\tau s \mu t}, \qquad \text{for } \lambda, \tau = 4, 5, 6 \text{ and } t = s,\ \mu = 1, 2, 3 \text{ or } \lambda, \tau = 4, 5, 6 \text{ and } t \neq s,\ \mu = 1, \ldots, 6.$$

Now let us pay attention to the fact that the equations of constraints given by Eqs. (3.5) and (3.6) depend, in general, on k generalized velocities. It may, however, happen that none of w equations of constraints will depend on one or a greater number of velocities $\dot{q}_{\mu t}$. This means that the appropriate coefficients $A_{\tau s \mu t} = 0$. In such a situation both the second and the third of the expressions (3.9) and (3.10) will have the right-hand sides equal appropriately to $\mathbf{e}_{\mu t}$ and $\mathbf{0}$.

Now we shall assume that $n = t$ and we shall see what form Eqs. (3.2) will take after differentiating them by $\dot{q}_{\tau s}$ and $\dot{q}_{\mu t}$. We shall consider, one after another, such cases as those for Eqs. (3.9) and (3.10). We shall obtain

$$(3.11)_1 \qquad \frac{\partial \mathbf{v}'_{ct}}{\partial \dot{q}_{\tau s}} = \mathbf{0}, \qquad \text{for } \lambda, \tau = 1, 2, 3$$

the second case and the first part of the third one are included in the expression (3.9) since there $t = s$

$(3.11)_{2,3}$
$$\frac{\partial \mathbf{v}'_{ct}}{\partial \dot{q}_{\mu t}} = \mathbf{e}_{\mu t}, \qquad \text{for } \lambda, \tau, \mu = 1, 2, 3,\ t \neq s,$$
$$\frac{\partial \mathbf{v}'_{ct}}{\partial \dot{q}_{\mu t}} = \mathbf{0}, \qquad \text{for } \lambda, \tau = 1, 2, 3,\ t \neq s,\ \mu = 4, 5, 6,$$

which is consistent with Eqs. (3.3) and

$$(3.12)_1 \qquad \frac{\partial \boldsymbol{\omega}'_t}{\partial \dot{q}_{\tau s}} = \mathbf{0}, \qquad \text{for } \lambda, \tau = 4, 5, 6,$$

the second case and the first part of the third one are included in Eqs. (3.10) since there $t = s$

$$(3.12)_{2,3} \qquad \frac{\partial \boldsymbol{\omega}'_t}{\partial \dot{q}_{\mu t}} = \mathbf{0}, \qquad \text{for } \lambda, \tau = 4, 5, 6, \ t \neq s, \ \mu = 1, 2, 3,$$

$$\frac{\partial \boldsymbol{\omega}'_t}{\partial \dot{q}_{\mu t}} = \mathbf{e}_{\mu t}, \qquad \text{for } \lambda, \tau, \mu = 4, 5, 6, \ t \neq s,$$

which is consistent with Eqs. (3.4).

Considering now Eqs. (3.9) up to Eqs. (3.12) in Eq. (2.4) for the successively regarded cases, we shall notice that

$$(3.13) \qquad Q_{B\tau s} = 0, \qquad \text{(the first case)},$$

$$Q_{B\mu t} = \sum_s \sum_\tau A_{\tau s \mu t} \cdot Q'_{B\tau s} + Q'_{B\mu t}, \qquad \text{(the second and the third case)},$$

where "′", as before, distinguishes the free body while $Q'_{B\tau s}$ and $Q'_{B\mu t}$ are equal to generalized inertial forces which match the coordinates $q_{\tau s}$ and $q_{\mu t}$ for the free bodies. The summation after s and τ means that, as a result of it, we shall have to take, in general, w components of $A_{\tau s \mu t} \cdot Q'_{B\tau s}$ type. However, we should remember that it may happen that $A_{\tau s \mu t} = 0$, and that the number of components may be smaller than w. In particular, they can be equal to zero for all the pairs of indices τs. Then $Q_{B\mu t} = Q'_{B\mu t}$. In the expression (3.13), we have shown that generalized imaginary foces for the system of p rigid bodies, which is restricted by holonomic constraints, can always be presented in the form of linear functions of the imaginary forces obtained for the system of the same p bodies which are treated as free. The expression (3.13) closes the proof since the forces $Q'_{B\tau s} Q'_{B\mu t}$ can be built by appropriate indexing on the basis of the forces Q'_{B1}, Q'_{B2}, Q'_{B3}, Q'_{B4}, Q'_{B5}, Q'_{B6} which are valid for a single free rigid body (see some closing sentences preceding the text on the theorem). The above theorem and the proof can be found only in the dissertation [5] since they were worked out in the process of writing it.

4. Practical rules for applying the theorem and the case of nonholonomic constraints

To make the practical application of the theorem less complicated, we give the following rules which define the sequence of operations indispensable in determining imaginary forces for a holonomic system when the forces for a single free rigid body are already known (if the assumptions of the theorem are, of course, satisfied).

1. After determining the inertial forces for a single rigid body, we should choose indices for each of the bodies of the system and then, with their help, we should designate coordinates, velocities, accelerations and mass quantities that appear in the forces $Q_{B\lambda}$, where $\lambda = 1 \ldots 6$, given by the formula (2.3).

2. We should write the equations of holonomic constraints for the accepted independent generalized coordinates in a special way. This should be done in such a way that only the generalized velocities, which do not belong to the set of independent generalized velocities, would appear at the left-hand side (just as it is in Eq. (3.6)).

3. We should determine generalized imaginary forces. The forces which correspond to the coordinates that do not belong to the set of independent coordinates should be assumed to be equal to zero. The forces which correspond to the independent coordinates that do not appear in the equations of constraints should be assumed to be equal to the appropriate forces for a free body and each of the quantities mentioned in 1) should be designated with an appropriate index for each body. The forces which correspond to the independent coordinates appearing in the equations of constraints should be calculated as a sum. It is the sum of an appropriate force for a free body (with an appropriate index) and of certain expressions. The latter are of a form equal to the product of the coefficient which appears in the equation of constraints, satisfying 2), in front of the coordinate under consideration and the force which corresponds, for a free body, to the coordinate that in the equation of constraints appears at the left-hand side (all quantities with appropriate indices). To make it clearer, see Eq. (3.13).

4. Using the equations of holonomic constraints in the form (3.6), i.e., in the form that satisfies 2), we should write generalized inertial forces obtained by means of 3) as the functions of independent velocities and coordinates.

5. Later on, while building the equations of motion, the forces obtained on the basis of 1) to 4) should be added, e.g., according to the rule (2.2), to the right-hand sides of the equations. One should remember here that while building the left-hand sides of Eqs. (2.2), it is obligatory to use the same indexing for each of the bodies, as it happens while determining the inertial forces.

In the case of a nonholonomic system where the number of equations of constraints equals b and the number of degrees of freedom $l = (k-b)$ where k is the number of coordinates which describe the position of the system, we should do what follows: we determine k generalized imaginary forces as if it were a holonomic system or a free one if there are no holonomic constraints. Then we treat these forces in the same way we treat generalized external forces by carrying out operations which are required for them in the accepted formalism of building the equations of a nonholonomic system.

5. Concluding remarks

The above theorem can make it much more easier to determine generalized forces of inertia for mechanical systems consisting of many bodies and having a great number of holonomic constraints. When one starts using it, it can be fully appreciated. In [5] it has been used practically twice. One case was modelling vibrations of a wheel set considering the constraints in the track-wheelset system. Then it was applied while modelling vibrations of an eight-wheel freight car. The first of the models had 4 degrees of freedom (2 equations of holonomic constraints) in generalized coordinates; the second one had 21 degrees of freedom (21 equations of holonomic constraints) and it was a model in quasi-coordinates.

It seems that the presented method of determining the inertial forces in relative motion is a certain noticeable step forward in polishing this problem up for large mechanical systems consisting of rigid bodies.

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Streszczenie

WYZNACZANIE UOGÓLNIONYCH SIŁ BEZWŁADNOŚCI W RUCHU WZGLĘDNYM UKŁADÓW MECHANICZNYCH TYPU POJAZDU SZYNOWEGO

Przedstawiono prostszą, niż znane z literatury, metodę wyznaczania sił bezwładności wynikających z ruchu względnego dla układów mechanicznych złożonych z wielu brył sztywnych z więzami holonomicznymi i nieholonomicznymi. Metoda oparta jest na przedstawionym w pracy twierdzeniu i jego dowodzie. Metodę początkowo opracowano do modelowania drgań pojazdów szynowych (dla których założenia twierdzenia dają się łatwo spełnić), można ją również stosować dla dowolnych układów mechanicznych o tych samych założeniach.

Резюме

ОПРЕДЕЛЕНИЕ ОБОБЩЕННЫХ СИЛ ИНЕРЦИИ В ОТНОСИТЕЛЬНОМ ДВИЖЕНИИ МЕХАНИЧЕСКИХ СИСТЕМ ТИПА РЕЛЬСОВОГО ТРАНСПОРТНОГО СРЕДСТВА

По сравнению с тем, что известно из литературы, настоящая статья представляет более простый метод определения сил инерции, вытекающих из относительного движения,

для механических систем, состоящих из многих жестких тел с голономическими и неголономическими связями. Метод базирует на теореме, доказательство которой содержится в работе. Несмотря на то, что представленный метод разработан первоначально для потребностей моделирования колебаний рельсовых транспортных средств, для которых предположения теоремы легко удовлетворяются, то однако метод может быть применен для произвольных механических систем, удовлетворяющих упомянутым предположениям.

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