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APPLICATIONS OF BURZYŃSKI FAILURE CRITERIA. PART I. ISOTROPIC MATERIALS WITH ASYMMETRY OF ELASTIC RANGE

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The main idea of energy-based hypothesis of material effort proposed by Burzyński is briefly presented and the resulting failure criteria are discussed. Some examples, based on the own studies, which depict applications of these criteria are discussed and visualizations of limit surfaces in the space of principal stresses are presented.

Key words: Burzyński yield condition, yield surface, strength differential effect, energy-based yield criteria.

1. Introduction

The aim of the paper is to present an energy-based approach to failure criteria for materials, which reveal asymmetry in failure characteristics. It means that in the results of tension and compression tests there is observed a difference in the values of elastic, yield or strength limits. The energy-based hypothesis of material effort proposed originally by W. Burzyński is presented [1–3]; and the resulting failure criteria phrased for stress tensor components in an arbitrary Cartesian coordinate system and in particular, with the use of principal stresses, are discussed. As for the new results, our own applications of Burzyński's failure criteria for traditional and new materials are presented.

2. Failure criteria based on Burzyński hypothesis of material effort for isotropic solids

Włodzimierz Burzyński [1] not only summarized the contemporary knowledge about yield criteria but also presented a new idea how to determine the measure of material effort for materials which reveal difference in the failure strength (in particular: the elastic limit) for tension and compression. According to the original Burzyński's hypothesis, *the measure of material effort defining the limit of elastic range is a sum of the density of elastic energy of distortion and a part of density of elastic energy of volume change being a function of the state of stress and particular material properties*. The mathematical formula corresponding to this statement reads:

$$\Phi_f + \eta\Phi_v = K, \qquad \eta = \omega + \frac{\delta}{3p}, \qquad p = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}, \tag{2.1}$$

where

$$\Phi_f = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]$$

means the density of elastic energy of distortion, while:

$$\Phi_v = \frac{1-2\nu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2 = \frac{1-2\nu}{12G(1+\nu)}(\sigma_1 + \sigma_2 + \sigma_3)^2$$

is the density of elastic energy of volume change. The constant K corresponds to the value of the density of elastic energy in a limit state, while ω, δ are material parameters dependent on the contribution of the density of elastic energy of volume change influenced by the mean stress p. By the symbols σ_1, σ_2, σ_3 are meant principal stresses. By introducing the function η Burzyński took into an account the experimentally based observation, that the increase of the mean stress p results in the diminishing contribution of the elastic energy density of volume change Φ_v in the measure of material effort. The above formulation of the measure of material effort is precise for the limit states of linear elasticity, typical for brittle behaviour of materials. When the limit state is related to the loss of material strength preceded by certain plastic strain, then, the measure of material effort (2.1) loses its foundations of linear elasticity, because in this case inelastic states of material may occur. This is the reason why W. Burzyński suggested to treat functions Φ_f and Φ_v in equation (2.1) as general strain functions, and he emphasized this fact by the term "quasi-energies" of strain.

In the discussed measure of material effort (2.1) three material parameters: ω, δ, K are introduced. The final form of failure hypothesis (2.1) reads [1, 2]:

$$\frac{1}{3}\sigma_f^2 + 3\frac{1-2\widetilde{\nu}}{(1+\widetilde{\nu})}\omega p^2 + \frac{1-2\widetilde{\nu}}{(1+\widetilde{\nu})}\delta p = 4GK, \tag{2.2}$$

where $\widetilde{\nu} = \dfrac{k_c k_t}{2k_s^2} - 1$, $K = \dfrac{2k_c k_t}{12G(1+\widetilde{\nu})}$, $\sigma_f^2 = 12G\Phi_f$. The idea of Burzyński's derivation lies in a particular conversion of variables. The triplet (ω, δ, K) is substituted by another one, which results from commonly performed strength tests: elastic (plastic) limit in uniaxial tension – k_t, uniaxial compression – k_c, and torsion – k_s where $(\omega, \delta, K) \to (k_t, k_c, k_s)$ (cf. [1, p. 112]).

Because of mentioned above substitution, (2.2) transforms into the form discussed also in [4]:

$$\frac{k_c k_t}{3k_s^2}\sigma_e^2 + \left(9 - \frac{3k_c k_t}{k_s^2}\right) p^2 + 3(k_c - k_t)p - k_c k_t = 0, \tag{2.3}$$

where $\sigma_e^2 = \frac{1}{2}\sigma_f^2$ is an equivalent stress used in the theory of plasticity. According to the discussion conducted in [1] and [4], the equation (2.3) in the space of principal stresses, depending on the relations among material constants (k_t, k_c, k_s), describes the surfaces: an ellipsoid for $3k_s^2 > k_t k_c$ or a hyperboloid for $3k_s^2 < k_t k_c$, which, however, does not have any practical application. W. Burzyński also noticed that there occur interesting cases if these three material constants are particularly connected, for example if they are bound together as the geometrical average: $\sqrt{3}k_s = \sqrt{k_t k_c}$, then (2.3) takes the form [1]:

$$\sigma_e^2 + 3(k_c - k_t)p - k_c k_t = 0. \tag{2.4}$$

The above equation presents the formula of a paraboloid of revolution in the space of principal stresses. The original hypothesis of W. Burzyński [1] and his comprehensive phenomenological theory of material effort was forgotten and repeatedly "rediscovered" later by several authors, often in parts and without the clarity of the "in depth" analysis and physical foundations of Burzyński's work. Discussion of other works containing the latter equation is presented in [4, 5]. It is worth to mention that the discussed above paraboloid yield condition finds recent applications also in viscoplastic modeling for metal matrix composites [6]. The latter authors, as well as many others, related this condition with the names of R. von Mises and F. Schleicher, although none of these researchers derived the relation (2.4) (cf. [7] for the discussion of a historical background of the studied paraboloid criterion).

3. Recent applications of the Burzyński failure criteria

Defining the strength differential factor $\kappa = k_c/k_t$ allows to determine particular cases of the criterion, for example for $\kappa = 1$, $k_c = k_t = k$ and then $k_s = k/\sqrt{3}$, which suits the condition assumed in the Huber-Mises-Hencky criterion. After suitable transformation (2.3) takes the form expressed by stress tensor components in the system of principal axes:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\left(\frac{k_c k_t}{2k_s^2} - 1\right)(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) + (k_c - k_t)(\sigma_1 + \sigma_2 + \sigma_3) = k_c k_t. \tag{3.1}$$

If $\sigma_2 = 0$, there is obtained a plane state of stress, for which:

$$(3.2) \qquad (\sigma_1^2 + \sigma_3^2) - 2\left(\frac{k_c k_t}{2k_s^2} - 1\right)\sigma_1\sigma_3 + (k_c - k_t)(\sigma_1 + \sigma_3) = k_c k_t.$$

In the space of principal stresses for $\sqrt{3}k_s = \sqrt{k_t k_c}$ the graphical representation of the criterion (2.3) is a paraboloid of revolution with the axis of symmetry given by the axis of hydrostatic compression: $\sigma_1 = \sigma_2 = \sigma_3$. In the plane state of stress for $\sigma_2 = 0$ the graphical representation of the Burzyński hypothesis is an ellipse. The centre of symmetry of such an ellipse is defined by

$$S_e = \left(\frac{k_s^2(k_c - k_t)}{k_c k_t - 4k_s^2}, \frac{k_s^2(k_c - k_t)}{k_c k_t - 4k_s^2}\right)$$

and the axes of symmetry are given by

$$\sigma_3 = \sigma_1, \qquad \sigma_3 = -\sigma_1 + \frac{2k_s^2(k_c - k_t)}{k_c k_t - 4k_s^2}.$$

If $k_c = k_t$ then the centre of the ellipse is given by the beginning of the coordinate system and the Burzyński hypothesis is equal to the Huber hypothesis; in this case, the graphical representation of the yield surface is a cylinder of revolution with the axis of symmetry: $\sigma_1 = \sigma_2 = \sigma_3$.

In [8] the Burzyński material effort hypothesis was specified for some classical experimental data discussed by Theocaris [9] and published in historical papers of Lode [10] as well as by Taylor and Quinney [11]. This paper is devoted to applications of the Burzyński failure criteria for our own experimental data obtained in the recent experimental investigations of mechanical properties of polycarbonate and the results related to the current studies of metal-ceramic composites [8, 12, 13].

The polycarbonate samples were investigated for tension, compression and shear performed with the use of a double shear specimen. The pictures of the sample before and after the shear test are shown in Fig. 1.

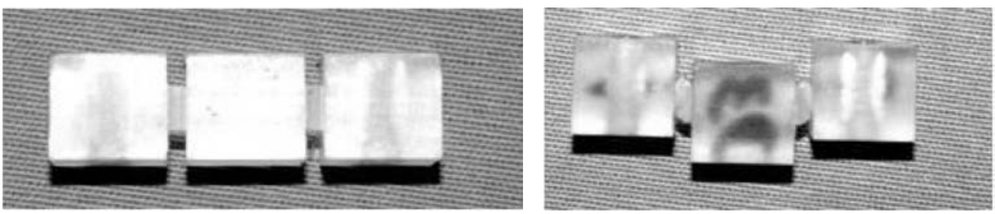
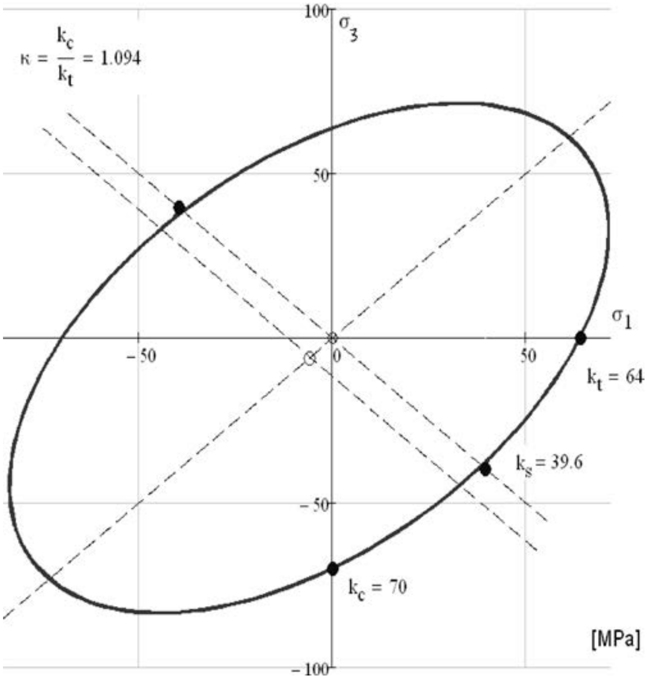


Fig. 1. The sample $12 \times 12 \times 40$ [mm] with the shearing zone $6 \times 6 \times 2$ [mm] prepared for a double shear test before and after deformation.

The numerical analysis of the shear process led to the correction accounting for the geometry of the double shear specimen. As a result, the following data were obtained: $k_c = 70$ MPa, $k_t = 64$ MPa, $k_s = 39.6$ MPa. Application of the formula (3.2) shows that the Burzyński yield criterion fits very well with the experimental data for the investigated polycarbonate. This is depicted in Fig. 2 and Fig. 3, where the graphical representations of Burzyński yield criterion are shown.



The ellipse of the plane state of stress. The experimental data for yield strength in tension, compression and shear.

FIG. 2. Graphical representation of Burzyński yield criterion for the polycarbonate according to our own experimental investigations.

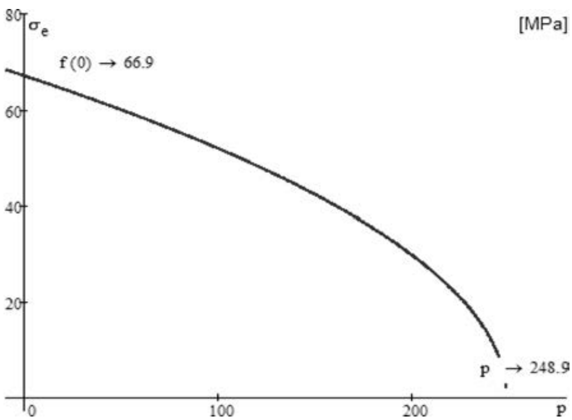
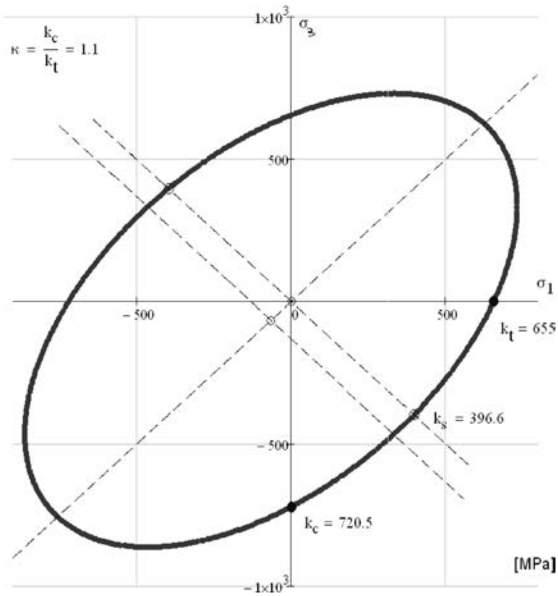


FIG. 3. A half-parabola, being the representation of the Burzyński yield criterion for the polycarbonate in the surface (σ_e, p).

The graphical representation of the limit function for the metal matrix composites (MMC), in particular alumina alloy 6061 reinforced by zircon and corundum particles: $6061{+}2Zr{+}20Al_2O_3$ [12] is presented below, Fig. 4. and Fig. 5.



The ellipse of the plane state of stress. The experimental data are marked with solid points $k_c = 720.5$ MPa, $k_t = 655$ MPa and the limit shear strength $k_s = 39.6$ MPa is marked with an open circle.

FIG. 4. Graphical representation of the Burzyński yield criterion for the MMC composite $6061{+}2Zr{+}20Al_2O_3$.

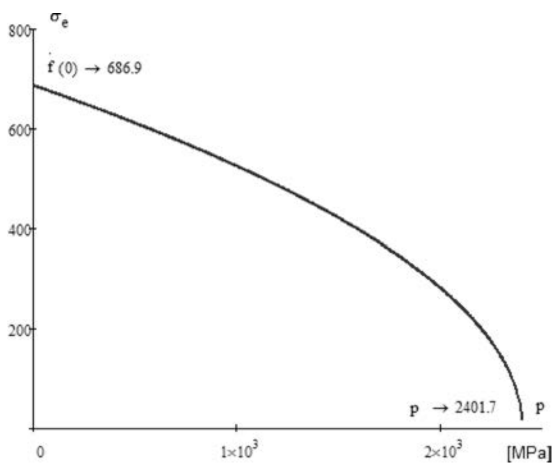


FIG. 5. A half-parabola, being the representation of the Burzyński failure criterion for the MMC composite $6061{+}2Zr{+}20Al_2O_3$ in the coordinates (σ_e, p).

Further experimental tests are necessary to verify the presented above paraboloid failure criterion. At least an independent test delivering information about the strength in shear k_s could be helpful for that. The specified formula for paraboloid failure surface can be applied as plastic potential in calculations of plastic deformation of metallic solids, which reveal the stress differential effect, cf. e.g. [6] or [14]. In such a case, the information of how the ratio $\kappa = k_c/k_t$ changes in strain is necessary. In the numerical simulations of some examples of plastic deformation processes presented in [14] the constant value of κ was assumed. However, the analysis of experimental data of the particle-reinforced metal matrix composite (PRMMC) – Al-47Al_2O_3 in [6] shows that the ratio $\kappa = k_c/k_t$ increases in strain.

Another particle-reinforced metal matrix composite 75%Cr – 25%Al_2O_3 (M) was experimentally investigated [13]. The tests, of compression and tension, were perfomed. The cylindrical specimens of the diameter 12 mm and the height of 10 mm, Fig. 6, were subjected to the compression tests with the use of the strength machine MTS810 of the loading range reaching 250 kN. The corresponding characteristics are given in Table 1.

Fig. 6. Picture of the deformed cylindrical specimen.

Table 1. Material characteristics obtained in the compression test.

Type of the composite	$R_{0.2}$ [MPa]	R_m [MPa]
75%Cr – 25%Al_2O_3(M)	700	920

During the compression of the cylindrical specimen the local failure appeared. The magnified picture (×500) of the surface with the failure sites with the use of scanning microscopy is shown in Fig. 7. The tensile test was performed with the use of the specimens shown in Fig. 8. The plane tensile specimens were cut

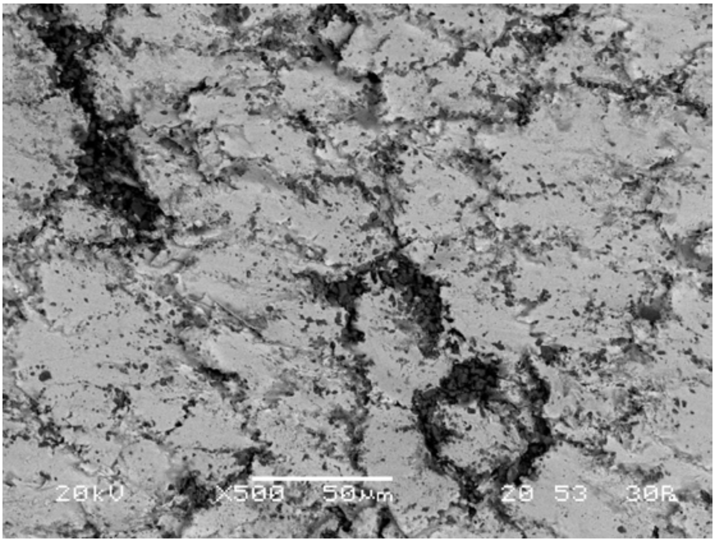


FIG. 7. The picture of the surface of the specimen revealing the sites of failure.

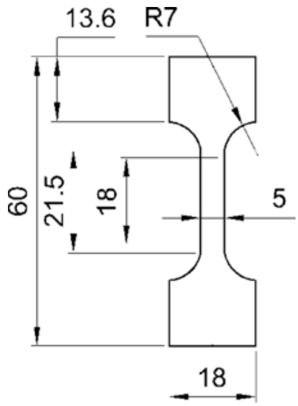


FIG. 8. The shape and dimensions of the ten sile specimen.

out from the roller of the diameter 80 mm and thickness of 5 mm. In Table 2 the measured material parameters are given.

Table 2. Material characteristics obtained in the tensile test.

Type of the composite	$R_{0.2}$ [MPa]	R_m [MPa]
75%Cr – 25%Al_2O_3(M)	23	24

The graphical representation of the limit function for the particle-reinforced metal matrix composite 75%Cr – 25%Al_2O_3 (M) is presented below, Fig. 9 and Fig. 10.

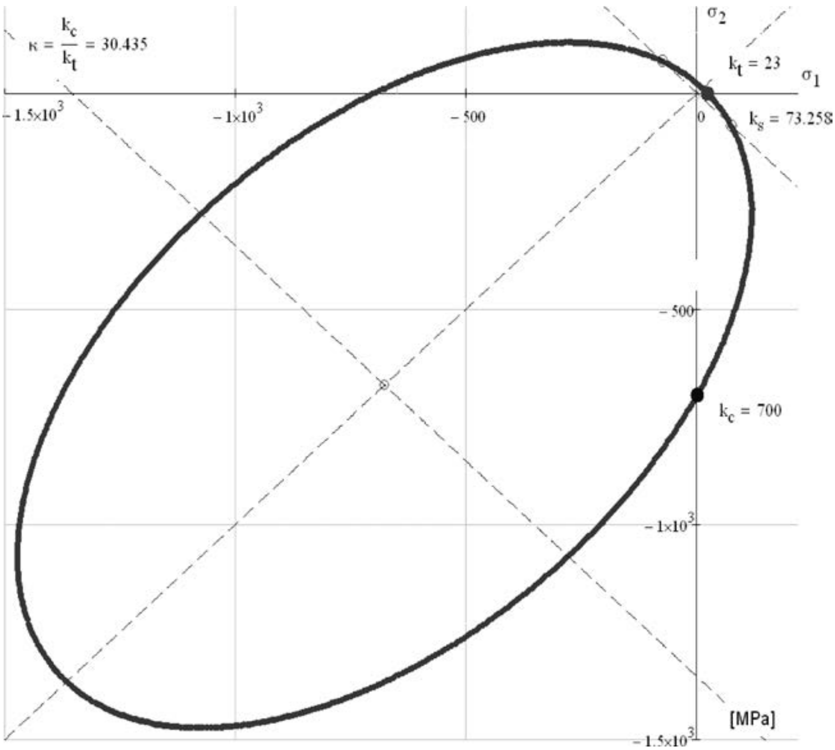


FIG. 9. Graphical representation of the Burzyński yield criterion for the MMC composite $75\%Cr - 25\%Al_2O_3$ (M).

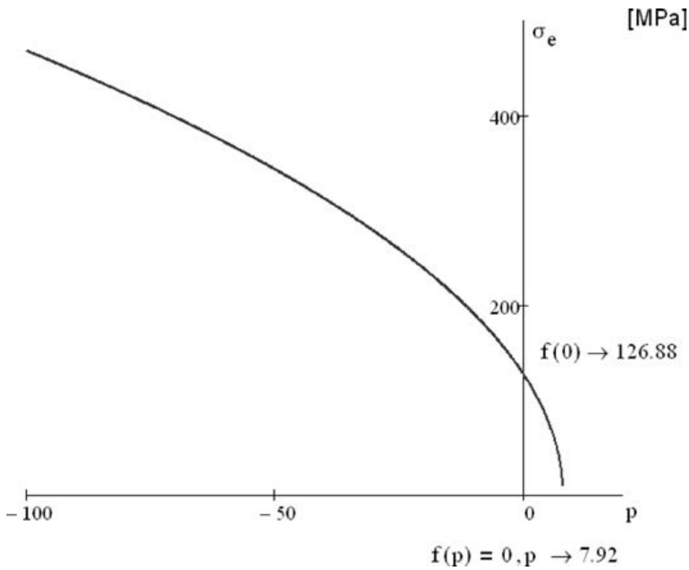


FIG. 10. A half-parabola, being the representation of the Burzyński failure criterion for the MMC composite $75\%Cr - 25\%Al_2O_3$ (M) in the coordinates (σ_e, p).

4. Conclusions

It is worthy of emphasizing that W. Burzyński proposed the hypothesis which was universal in the sense of energy. Therefore, it can be applied not only to isotropic materials, it is also applicable to different kinds of anisotropic solids revealing, in particular, characteristic asymmetry of elastic range. W. Burzyński presented, also for the first time, the energetic approach to determine the failure criteria for a certain class of orthotropic materials [1]. The issue of yielding condition of orthotropic materials, raised by Burzyński, is worth further studies because of its promising possibilities of application for modern materials.

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References

1. W. Burzyński, *Studjum nad Hipotezami Wytężenia*, Akademja Nauk Technicznych, Lwów 1928; also: Włodzimierz Burzyński, "Dzieła Wybrane" part I, Polska Akademia Nauk, PWN Warszawa, 1982, 67–258; see also: *Selected passages from Wlodzimierz Burzyński's doctoral dissertation: "Study on Material Effort Hypotheses" printed in Polish by the Academy of Technical Sciences, Lwów, 1928, 1–192*, Engng. Trans. **57**, 185–215, 2009.
2. W. Burzyński, *Ueber die Anstrengungshypothesen*, Schweizerische Bauzeitung, **94**, 21, 259–262, 1929.
3. W. Burzyński, *Teoretyczne podstawy hipotez wytężenia*, Czasopismo Techniczne, **47**, 1–41, 1929; English translation: *Theoretical foundations of the hypotheses of material effort*, Engng. Trans., **56**, 269–305, 2008.
4. M. Życzkowski, *Discontinuous bifurcation in the case of the Burzynski – Torre yield condition*, Acta Mechanica, **132**, 19–35, 1999.
5. M. Życzkowski, *Combined Loadings in the Theory of Plasticity*, PWN, Warszawa, 1981.
6. H. Zhang, K. T. Ramesh, E. S. C. Chin, *A multi-axial constitutive model for metal matrix composites*, J. Mech. Phys. Solids, **56**, 2972–2983, 2008.
7. R.B. Pęcherski, *Burzyński yield condition vis-à-vis the related studies reported in the literature*, Engineering Transactions, **56**, 47–55, 2008.
8. T. Frąś, R. B. Pęcherski, *Applications of the Burzyński hypothesis of material effort for isotropic solids*, Mechanics & Control, **28**, 2, strony, 2010.
9. P. S. Theocaris, *Failure criteria for isotropic bodies revisited*, Engineering Fracture Mechanics, **51**, 239–264, 1995.

10. W. Lode, *Versuche über den Einfluss der mittleren Hauptspannung auf die Fliessgrenze*, ZAMM, **5**, 1925.

11. G. I. Taylor, H. Quinney, *The plastic distortion of metals*, Phil. Trans. Roy. Soc., **A230**, 323–362, 1931.

12. J. Dutkiewicz, Raport D2-1.1., KomCerMet, 1.07.2009-30.09.2009.

13. Z. L. Kowalewski, Raport D2-2.4., KomCerMet, 1.07.2009-30.09.2009.

14. G. Vadillo, J. Fernandez–Saez, R. B. Pęcherski, *Some applications of Burzyński yield condition in metal plasticity*, Materials & Design, **32**, 628–635, 2011.

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