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GENERALIZED STRAIN AND STRESS MEASURES: CRITICAL SURVEY AND NEW RESULTS

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Four basic principles: objectivity, isotropy, consistency and regularity are proposed to restrict the concepts of generalized strain and (more originally) of generalized stress. These principles are used to derive two general representations of the corresponding strain and stress functions. Based on a material definition of conjugacy, each candidate strain is then placed in one-to-one correspondence with a certain conjugate stress and vice versa. Besides the classical strain-stress pairs already current in the literature, an interesting family of new strains and conjugate stresses is disclosed in the process. The main contributions of this paper, however, are to demonstrate the superiority of a particular class of strain and stress measures, herein called "congruent", and to reveal the coexistence of different definitions of conjugacy, which is a source of confusion.

1. Introduction

In nonlinear mechanics, the form assumed by the stress-strain *law* adopted to model the intrinsic response of a material depends on the stress-strain *pair* selected to formulate this constitutive law. Indeed, if the stress measure conjugate to a given strain mesasure is unique once a specific definition of conjugacy is adopted, the choice of such a stress-strain pair is by no means unique, even if certain pairs are generally considered as favorites.

The usual tendency is to use the simplest stress-strain pair (namely the Green strain – second Piola-Kirchhoff stress) and to transfer all the complexity of the material response to the stress-strain law, which is a sound approach. A legitimate question however is whether a more elaborate choice of stress-strain pair could perhaps simplify the form of certain stress-strain laws, preferably of the most common ones. Of course, any attempt to simplify the modeling of materials by complicating the description of deformations and stresses may rightly appear vacuous to many people. Nevertheless, the prospect of extending the range of application of the classical constitutive theories (such as linear elasticity or classical plasticity to quote two typical examples) from infinitesimal deformations to moderate ones, without any modification of their parts, may represent a sufficient incentive for others.

Several pairs of conjugate stress-strain measures have long been identified in nonlinear mechanics [1, 2]. A brief history of strain and stress, providing the key to the names by which they are referred to in this paper, is given in Appendix A. Among them, the strains which are here attributed to Green $\mathbf{E}^G$ and Karni $\mathbf{E}^K$, together with their conjugate stresses denoted $\mathbf{S}^G$ and $\mathbf{S}^K$ to show the correspondence, play a fundamental role. This is mainly because they are *simple* (especially the Green-second-Piola-Kirchhoff pair $\mathbf{E}^G - \mathbf{S}^G$), but also because they may be viewed as an upper and lower *bound* for other candidate stress-strain pairs $\mathbf{E} - \mathbf{S}$. Somewhere in between these bounds, the logarithm or natural strain $\mathbb{G}$ (often attributed to Hencky) together with its almost conjugate stress, the rotated true stress $\mathbb{T}$ (apparently due to Noll) represent an attractive compromise, a sort of pivot for all other candidate measures. Several penetrating studies have already been published on the subject, e.g. [3, 4, 5, 6, 7, 8, 9, 10, 11], with a special mention for the erudite accounts found in [1, 2, 12] and the masterful synthesis [13]. Dominant among the findings is the fundamental definition of a *generalized strain* measure with its *conjugate stress* [13]. Slightly less essential is the underlying consequence that all admissible stress-strain pairs are *equivalent*, a fact already recognized in [1], but mainly from a kinematical standpoint. If the fundamental definition provides a clear framework for the study of specific measures, its direct consequence shows the theoretical emptiness of such inquiries. Only initial ignorance of these results combined with the fascinating appeal of the natural strain and rotated stress measures can explain persistence with further studies in this field, beginning with the present one and including [14, 8, 10, 58, 60, 62, 64] among the most recent ones.

In the present article, the concept of generalized strain is developed and, more originally, the concept of generalized stress is defined. Both concepts are governed by the same set of four basic principles, namely objectivity, isotropy, consistency and regularity. Two general tensorial representations of the generalized strain and stress functions based on these principles are derived separately. Relying on the material definition of conjugacy introduced by [13], each candidate strain is then placed in one-to-one-correspondence with a conjugate stress and vice versa. Incidentally, the coexistence of alternative definitions of conjugacy is detected and the differences are pointed out to avoid apparent contradictions. The class of generalized stresses congruent to the second Piola–Kirchhoff stress $\mathbf{S}^G$, together with their conjugate strains, are shown to present definite advantages over all other candidate pairs.

Of course, all the classical stress-strain pairs already mentioned in the literature are included in this general framework, which opens up several other directions of investigation. In particular, an interesting family of generalized strain and conjugate stress measures, which seems to have gone unnoticed until now, (except for a few hints in the rubber elasticity literature of the forties [15, 16, 17, 18]) is disclosed in the process. In essence, the strain family consists of a *convex combination* of the material forms of (any) two basic strains such as the classical Green and Karni strains. For instance their arithmetical mean, herein called the *Mooney strain* $\mathbf{E}^M$, is the most promising member

$$\mathbf{E}^M = \frac{1}{2}\mathbf{E}^G + \frac{1}{2}\mathbf{E}^K = \frac{1}{4}(\mathbf{C} - \mathbf{C}^{-1}),$$

where $\mathbf{C}$ is the Green deformation tensor defined in terms of the deformation gradient $\mathbf{F}$ by $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. The corresponding *strain rate* is shown to be related to the *rotated strain rate* $\mathbf{D} = \mathbf{R}^T d\mathbf{R}$ (material counterpart of the usual spatial rate of deformation d), taken as reference, by the direct (explicit) formula

$$\dot{\mathbf{E}}^M = \frac{1}{2}\mathbf{UDU} + \frac{1}{2}\mathbf{U}^{-1}\mathbf{D}\mathbf{U}^{-1} = f(\mathbf{D}).$$

In the above, $\mathbf{R}$ and $\mathbf{U}$ represent the rotation and stretch tensors issuing from the polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{RU}$. In this particular case, the *conjugate stress* $\mathbf{S}^M$ is found to be related to the *rotated stress* $\mathbf{T} = \mathbf{R}^T(J\mathbf{t})\mathbf{R}$ (material counterpart of the true stress of Cauchy $\mathbf{t}$ scaled by the Jacobian $J = \det \mathbf{F}$), conjugate to $\mathbf{D}$, by the inverse (implicit) formula

$$\mathbf{T} = \frac{1}{2}\mathbf{U}\mathbf{S}^M\mathbf{U} + \frac{1}{2}\mathbf{U}^{-1}\mathbf{S}^M\mathbf{U}^{-1} = f(\mathbf{S}^M).$$

The Mooney strain $\mathbf{E}^M$ and its conjugate stress $\mathbf{S}^M$ prove to be good approximations of the logarithm strain $\mathbf{G}$ and corresponding rotated stress $\mathbf{T}$. However, because they do not result from a congruent transformation of the

Green strain $\mathbf{E}^G$ and second Piola–Kirchhoff stress $\mathbf{S}^G$ respectively, they suffer from a number of shortcomings which compromise their utilization.

This article is divided into two parts. The first part deals with the concept of strain or kinematics. The second part focusses on the concept of stress or dynamics. (An estimate of the incidence of the choice of a specific stress-strain pair on the eventual form of a stress-strain law closes part two). In each part, both the "tensorial forms" and the "spectral forms" are discussed to give additional insight. However, this presentation is limited to the "material forms" of the different measures (referred to the undeformed configuration), their "spatial forms" (referred to the deformed configuration) merely being alluded to, so certain difficulties and an excessive proliferation of symbols and names can be avoided. The adjectives "material" and "spatial" are extensively used in this paper to distinguish entities referred to the undeformed configuration from their analogues referred to the deformed configuration. This terminology conforms to a meaningless but deep-rooted tradition (since both descriptions are in fact material in a strict sense). A popular alternative is the "Lagrangean"–"Eulerian" pair but it is historically inaccurate. Our preference would go to "initial"–"actual". For simplicity also, both deformed and undeformed configurations are implicitly referred to the same rectangular coordinate system. Finally a rather descriptive style in line with our engineering backgrounds characterizes the article, which is meant to be tutorial at the cost of some repetition.

2. Generalized strain

2.1. Elementary introduction

Consider a bar of initial length L to be deformed along its axis into a final length l, as shown in Fig. 1. A basic quantity for studying the deformation of this bar is the ratio of its deformed length over its original length, $\lambda = l/L$, called the *stretch ratio.* It is a positive nondimensional number which represents a relative (rather than absolute) elongation when greater than unity, and a contraction if smaller. As such it is a legitimate scale of deformation but it is not a *measure* in the strict sense, since it does not vanish with the deformation. The most natural alternative which vanishes for $\lambda = 1$ is the Cauchy strain defined as the change in length over the original length:

(2.1) $$E^C = \frac{l-L}{L} = \lambda - 1 = \varepsilon \qquad \text{(Cauchy)}.$$

This linear measure in λ has acquired a reference status since the entire theory of small deformations has been developed using it as a basis.

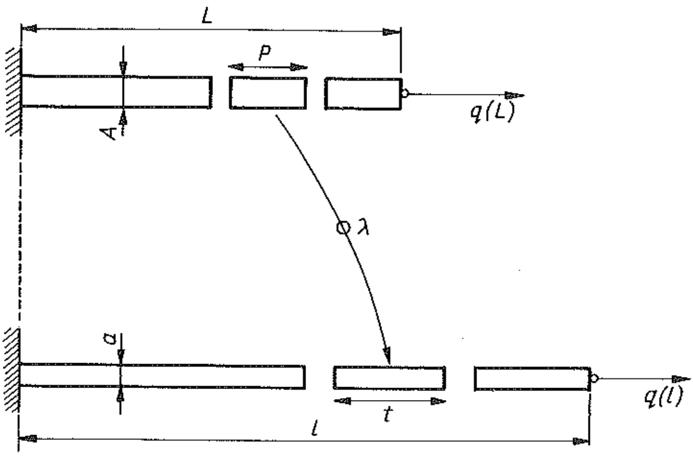

FIG. 1.

For various reasons, which would be too long and premature to explain in this section, several other measures have been introduced over the years, the most classical of which are listed below (e.g. [3]):

(2.2)
$$\begin{aligned}
E^G &= \frac{1}{2}\frac{l^2-L^2}{L^2} = \frac{1}{2}(\lambda^2-1) &&= \varepsilon + \frac{\varepsilon^2}{2} &&\text{(Green)},\\
E^B &= \frac{l-L}{L} = \lambda - 1 &&= \varepsilon &&\text{(Biot)},\\
G &= \operatorname{Log}\frac{l}{L} = \operatorname{Log}\lambda &&= \varepsilon - \frac{\varepsilon^2}{2} + \ldots &&\text{(natural)},\\
E^H &= \frac{l-L}{l} = 1 - \frac{1}{\lambda} &&= \varepsilon - \varepsilon^2 + \ldots &&\text{(Hill)},\\
E^K &= \frac{1}{2}\frac{l^2-L^2}{l^2} = \frac{1}{2}\left(1-\frac{1}{\lambda^2}\right) &&= \varepsilon - \frac{3}{2}\varepsilon^2 + \ldots &&\text{(Karni)}.
\end{aligned}$$

The Biot strain is distinguished from the Cauchy strain because the fact that they appear to be identical is a mere coincidence in the particular case of pure elongation under consideration. All the above measures have been cleverly integrated by DOYLE and ERICKSEN [19] and also by SETH [4] into a one-parameter family in the form

$$E^{(m)} = \frac{1}{m}(\lambda^m - 1) = \varepsilon + \frac{m-1}{2}\varepsilon^2 + \ldots \qquad \text{(Seth).} \tag{2.3}$$

The Green, Biot, Hill and Karni strains are clearly recovered for the *integer* values of the parameter $m = +2, +1, -1$ and -2, respectively, whereas the natural strain is found to correspond to the pivot value $m = 0$ by a limiting process to overcome the indeterminacy.

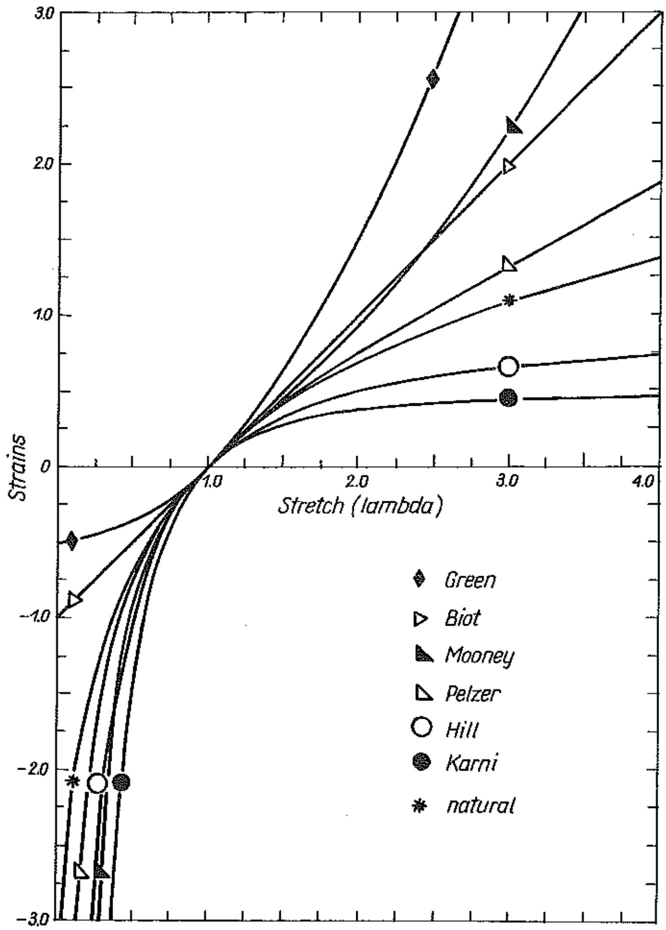

FIG. 2.

The resemblances and differences between these five measures are best appreciated by looking at their graphs in Fig. 2.

First, it is checked that all measures *vanish* in the reference configuration, i.e. at $\lambda = 1$. It is also observed that all curves are *tangent* to each other at this reference point. This conformity indicates that all measures coincide with the small strain ε of Cauchy for infinitesimal deformations around the reference configuration (as confirmed by the expansions in terms of ε included in Eqs. (2.2)). Finally, it is noted that all measures are monotone increasing functions of the positive stretch ratio.

Besides these convergences around unity, the different measures significantly diverge away from it. Their erratic asymptotic behaviour, as the deformation ratio shrinks to zero or extends to infinity, is the most intriguing discrepancy at first glance. It takes some time to convince oneself that all these measures are in fact equivalent and that any *invertible* function $E(\lambda)$ is a perfectly legitimate strain measure provided it complies with the requirements

$$(2.4) \quad E = E(\lambda), \quad E(1) = 0, \quad E'(1) = 1, \quad E'(\lambda) > 0, \quad \text{or} \quad \exists E^{-1}.$$

The idea of the proof lies in the fact that any such generalized strain $E = E(\lambda)$ may be related to any other invertible strain, say $G = \operatorname{Log}\lambda$ thus $\lambda = \operatorname{Exp} G$, by the composition $E = E(\operatorname{Exp} G)$ and *vice versa*.

In spite of this equivalence, one strain measure may present definite practical advantages over another one, beginning with its simplicity, but also including such a feature as a certain progressiveness in its behaviour at large strains. In this latter respect, the Green and Karni strains appear in Fig. 2 as an upper and a lower bound that one "would not like" to exceed. On the contrary, the graph of the natural strain is "pleasing" which reflects the progressiveness inherent in its definition and two other attractive properties:

$$\begin{aligned}
\operatorname{Log}\lambda &= \int_1^{\lambda} \frac{d\mu}{\mu}, && \text{"progressivity";}\\
\operatorname{Log}\lambda\mu &= \operatorname{Log}\lambda + \operatorname{Log}\mu, && \text{"additivity";}\\
\operatorname{Log}\frac{1}{\lambda} &= -\operatorname{Log}\lambda, && \text{"symmetry".}
\end{aligned}$$

However, the logarithm is not a very easy function to compute and it seems worthwhile to look for a good approximation.

A comparative examination of the classical measures (2.2) soon reveals several intermediate alternates, named here after the rubber elasticians [15, 16, 20, 18], later found to give definite hints of these expressions:

$$
(2.5) \quad
\begin{aligned}
E^P &= \frac{1}{2}\frac{l^2 - L^2}{lL} = \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) &&= \varepsilon - \frac{\varepsilon^2}{2} + \dots && \text{(Pelzer)},\\
E^M &= \frac{1}{4}\frac{l^4 - L^4}{l^2L^2} = \frac{1}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right) &&= \varepsilon - \frac{\varepsilon^2}{2} + \dots && \text{(Mooney)},\\
E^W &= \frac{1}{3}\frac{l^3 - L^3}{l^2L} = \frac{1}{3}\left(\lambda - \frac{1}{\lambda^2}\right) &&= \varepsilon - \varepsilon^2 + \dots && \text{(Wall)},\\
E^R &= \frac{1}{3}\frac{l^3 - L^3}{lL^2} = \frac{1}{3}\left(\lambda^2 - \frac{1}{\lambda}\right) &&= \varepsilon + 0 + \dots && \text{(Rivlin)}.
\end{aligned}
$$

REMARK. Consult Appendix A for additional details concerning the historical origin of these strains.

A second look at Fig. 2 confirms some of the expectations placed in these four measures, especially with regard to their capacity to approximate the natural scale. In fact, application of the trapezoidal rule to the integral definition of the logarithm gives an analytical justification for the former:

$$G = \operatorname{Log}\lambda = \int_1^\lambda \frac{d\mu}{\mu} \simeq \frac{1}{2}(\lambda - 1)\left(1 + \frac{1}{\lambda}\right) = \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right) = E^P.$$

A closer comparison of the new measures (2.5) with the classical ones (2.2) suggests yet another interpretation. The Pelzer and Mooney strains are identified with the *arithmetical mean* of the Biot–Hill and Green–Karni pairs, respectively. Similarly, the Wall and Rivlin strains are recognized as weighted averages of the Biot–Karni and Green–Hill pairs:

$$
(2.6) \quad
\begin{aligned}
E^P &= \frac{1}{2}E^B + \frac{1}{2}E^H,\\
E^M &= \frac{1}{2}E^G + \frac{1}{2}E^K,\\
E^W &= \frac{1}{3}E^B + \frac{2}{3}E^K,\\
E^R &= \frac{2}{3}E^G + \frac{1}{3}E^H,
\end{aligned}
$$

REMARK. Some of these relationships were known by [21]. The Pelzer strain may also be interpreted as the geometric mean of the Green and Karni strains

$$E^P = \sqrt{E^G E^K} = \frac{1}{2}\sqrt{(\lambda^2 - 1)(1 - \lambda^{-2})} = \frac{1}{2}\sqrt{\lambda^2 - 2 + \lambda^{-2}} = \frac{1}{2}(\lambda - \lambda^{-1}).$$

From a classical inequality between the arithmetic mean and the geometric mean, it follows that $E^M \geq E^P$ for $\lambda \geq 1$, and *vice versa* for $\lambda < 1$.

Starting from the Seth family (2.3), the eight strain measures involved in Eqs. (2.6) are conveniently collected in a two-parameter family, referred to herein as the "*rubber*" family, in the form

$$
(2.7)\qquad \begin{aligned} E &= \frac{p}{p-q}E^{(p)} + \frac{q}{q-p}E^{(q)} \qquad \text{(rubber)},\\ &= \frac{1}{p-q}(\lambda^p - \lambda^q) = \varepsilon + \frac{p+q-1}{2}\varepsilon^2 + ... \\ & -2 \leq q \leq 0 \leq p \leq +2, \end{aligned}
$$

where $E^{(p)}$ and $E^{(q)}$ are any two *basic* members of the Seth family (2.3) (i.e. corresponding to integer values of opposite signs of the parameters p and q).

Because any member of the rubber family (2.7) is a *convex* combination of two basic strains, it automatically satisfies the admissibility requirements Eq. (2.4):

$$
\begin{aligned} E(1) &= \frac{p}{p-q}E^{(p)}(1) + \frac{q}{q-p}E^{(q)}(1) = 0,\\ E'(1) &= \frac{p}{p-q}E^{(p)\prime}(1) + \frac{q}{q-p}E^{(q)\prime}(1) = \frac{p-q}{p-q} = 1,\\ E''(1) &= \frac{p}{p-q}E^{(p)\prime\prime}(1) + \frac{q}{q-p}E^{(q)\prime\prime}(1) = p+q-1. \end{aligned}
$$

For further reference, a general expression for the strain rate $\dot{E}$ is derived in terms of the basic stretch rate $\dot{\lambda}$, by differentiating the definition (2.4) with respect to time

$$
(2.8)\qquad \dot{E} = E'(\lambda)\dot{\lambda}.
$$

Applying this chain rule to the generic formulas of the Seth family (2.3) and the rubber family (2.7) leads to

$$
(2.9)\qquad \begin{aligned} &\dot{E}^{(m)} = \lambda^{m-1}\dot{\lambda},\\ &\dot{E} = \frac{p}{p-q}\dot{E}^{(p)} + \frac{q}{q-p}\dot{E}^{(q)} = \frac{p\lambda^{p-1} - q\lambda^{q-1}}{p-q}\dot{\lambda}. \end{aligned}
$$

The eight members of the rubber family are summarized in Table 1 together with their rates. The only "symmetric" members of the family (excluding the peculiar rigid body case E^0) are the Pelzer and Mooney strains located on the diagonal ($p = -q$) of Table 1. As such they are favorite elements of the family with a computational preference for the Mooney strain to appear in three dimensions.

Table 1. The eight members of the "rubber" strain family (Eqs.(2.7)) and their rates (Eq.(2.9)).

p \ q	0	−1	−2
0	$E^0 = 0$ $\dot{E}^0 = 0$	$E^H = 1 - \frac{1}{\lambda}$ $\dot{E}^H = \frac{1}{\lambda^2}\dot{\lambda}$	$E^K = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)$ $\dot{E}^K = \frac{1}{\lambda^3}\dot{\lambda}$
1	$\dot{E}^B = \alpha - 1$ $\dot{E}^B = 1\dot{\lambda}$	$E^P = \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right)$ $\dot{E}^P = \frac{1}{2}\left(1 + \frac{1}{\lambda^2}\right)\dot{\lambda}$	$E^W = \frac{1}{3}\left(\lambda - \frac{1}{\lambda^2}\right)$ $\dot{E} = \frac{1}{2}\left(1 + \frac{2}{\lambda^3}\right)\dot{\lambda}$
2	$E^G = \frac{1}{2}(\lambda^2 - 1)$ $\dot{E}^G = \lambda\dot{\lambda}$	$E^R = \frac{1}{3}\left(\lambda^2 - \frac{1}{\lambda}\right)$ $\dot{E}^R = \frac{1}{3}\left(2\lambda + \frac{1}{\lambda^2}\right)\dot{\lambda}$	$E^M = \frac{1}{4}\left(\lambda^2 - \frac{1}{\lambda^2}\right)$ $\dot{M}^R = \frac{1}{2}\left(\lambda + \frac{1}{\lambda^3}\right)\dot{\lambda}$

2.2. Basic geometry of deformation

Let the deformed configuration of a continuous body be defined in terms of its reference configuration by the *placement*

$$\mathbf{x} = \mathbf{x}(\mathbf{X}). \tag{2.10}$$

The point $\mathbf{X}$ locates the original position of a typical material particle in space and the image $\mathbf{x}$ its new position after deformation as shown in Fig. 3. Both sets of points are implicitly referred to the same rectangular coordinate

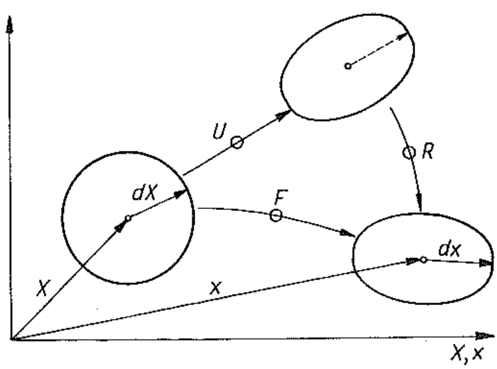

FIG. 3.

system fixed in space. Their components X_I and x_i, $I, i = 1, 2, 3$, on the common basis vectors $\mathbf{E}_I = \mathbf{e}_i$, are called the "material" and "spatial" coordinates of the particle, respectively.

An infinitesimal oriented fiber $d\mathbf{x}$ in this body is deformed according to the differential relation

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X}, \qquad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}. \tag{2.11}$$

The derivative $\mathbf{F}$ is called the *deformation gradient.* By definition, this mixed spatial-material tensor constitutes the fundamental quantity for the analysis of local deformations, with respect to which all other candidate measures will ultimately have to be referred to. The regularity of $\mathbf{F}$ ($J = \det \mathbf{F} > 0$) expresses the cohesion of matter during a deformation.

An infinitesimal oriented surface $d\mathbf{S}$ in the shape of a parallelogram delimited by two distinct fibers $d\mathbf{Y}$ and $d\mathbf{Z}$, is deformed according to the less obvious relation, e.g. [22],

$$d\mathbf{s} = J\mathbf{F}^{-T}\, d\mathbf{S}, \tag{2.12}$$

where $d\mathbf{S} = d\mathbf{Y} \times d\mathbf{Z}$ is a vector normal to the undeformed parallelogram with magnitude equal to its area and $d\mathbf{s} = d\mathbf{y} \times d\mathbf{z}$ is its counterpart in the deformed configuration.

The change in volume of a parallelepiped delimited by three fibers follows at once from Eqs. (2.11) and (2.12),

$$dv = J\, dV. \tag{2.13}$$

REMARK. Briefly, $\mathbf{F}^T d\mathbf{s} = \mathbf{F}^T(d\mathbf{y} \times d\mathbf{z}) = \mathbf{F}^T(\mathbf{F}d\mathbf{Y} \times \mathbf{F}d\mathbf{Z}) = J d\mathbf{Y} \times d\mathbf{Z} = J d\mathbf{S}$, and $dv = d\mathbf{x} \cdot d\mathbf{s} = J d\mathbf{X} \cdot \mathbf{F}^T\mathbf{F}^{-T} d\mathbf{S} = J d\mathbf{X} \cdot d\mathbf{S} = J dV$. For a uniform bar of initial volume $V = AL$ homogenously stretched to a final volume $v = al$, the differential formulas (2.11) to (2.13) degenerate into trivial finite identities $l = (l/L)L = \lambda L$, $a = v/l = (v/V)(V/l) = J\lambda^{-1}A$ and $v = (v/V)V = JV$, respectively.

The squared length of a deformed fiber can be computed in terms of its undeformed counterpart using the scalar product

$$d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X}^T \cdot \mathbf{F}^T\mathbf{F}\, d\mathbf{X}, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{2.14}$$

The symmetric positive definite material tensor $\mathbf{C}$ naturally produced in the process is called the Green on material *deformation* tensor.

The squared area of an infinitesimal parallelogram is evaluated in the same way

$$ds \cdot ds = J^2\, dS \cdot \mathbf{F}^{-1}\mathbf{F}^{-T}\, dS, \qquad \mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-T}. \tag{2.15}$$

The inverse of the material strain tensor $\mathbf{C}$, which is itself material, is called the Piola deformation tensor.

REMARK. For simplicity, we confine ourselves to the material or Lagrangean description of deformations, referring all tensors to the undeformed configuration. A parallel (but not equivalent) spatial or Eulerian presentation, referred to the deformed configuration, would depart from here by introducing the Cauchy or spatial deformation tensor $\mathbf{c} = \mathbf{F}\mathbf{F}^T$ and its inverse $\mathbf{c}^{-1}$ due to Finger.

Paraphrasing [23], formula (2.15) shows that the tensor $J^2\mathbf{C}^{-1}$ measures changes of infinitesimal areas in precisely the same way as $\mathbf{C}$ measures changes of infinitesimal lengths.

The square root of the deformation tensor is called the *stretch* tensor

$$\mathbf{U} = \sqrt{\mathbf{C}}. \tag{2.16}$$

By applying the representation theorem of isotropic functions of the next section to the square root, a tensorial expression of the stretch tensor $\mathbf{U}$ in terms of the deformation tensor $\mathbf{C}$ of the form $\mathbf{U} = x\mathbf{C} + y\mathbf{I} + z\mathbf{C}^{-1}$ may theoretically be obtained [24]. However, the expressions of the coefficient x, y, z in terms of the principal invariants I_c, II_c, III_c of $\mathbf{C}$ (or equivalently of the principal stretches λ_a) are so complicated [25, 59] that they are difficult to use in practice. Consequently, the stretch tensor is best defined and interpreted in spectral form as follows (e.g. [13]). Because the deformation tensor $\mathbf{C}$ is symmetric and positive definite, it has three real positive principal values λ_a^2 along three orthogonal (or at least "orthogonalizable") principal directions $\mathbf{N}_a$ (normalized to unity) concomitantly defined by

$$\begin{aligned} \mathbf{C}\mathbf{N}_a &= \lambda_a^2 \mathbf{N}_a, \qquad & a = 1,2,3\ , \\ \mathbf{N}_a \cdot \mathbf{N}_b &= \delta_{ab}, \qquad & b = 1,2,3\ . \end{aligned} \tag{2.17}$$

The three dyadic self-products $\mathbf{N}_a \otimes \mathbf{N}_a$ of the principal vectors form a principal basis for the material tensor derived from $\mathbf{C}$. In particular the deformation tensor itself and its inverse can be alternately written in spectral form

$$\begin{aligned} \mathbf{C} &= \lambda_a^2 \mathbf{N}_a \otimes \mathbf{N}_a, \\ \mathbf{C}^{-1} &= \lambda_a^{-2} \mathbf{N}_a \otimes \mathbf{N}_a. \end{aligned} \tag{2.18}$$

REMARKS

1. Just like the nine dyadic products of the rectangular base vectors $\mathbf{e}_i \otimes \mathbf{e}_j$ form a fixed global basis for general second order tensors, the nine dyadic products of the principal vectors $\mathbf{N}_a \otimes \mathbf{N}_b$ form a variable local basis attached to each particle. Since the material tensors derived from $\mathbf{C}$ reduce to a diagonal form on this latter basis, only the three main unit dyads $\mathbf{N}_a \otimes \mathbf{N}_a$ are necessary for a complete representation of these tensors. The space spanned by the self-dyads $\mathbf{N}_a \otimes \mathbf{N}_a$ is called the commutator of $\mathbf{C}$ because it is also characterized by $\{\mathbf{X},\ \mathbf{CX} = \mathbf{XC}\}$.

2. The summation convention for repeated indices appearing in a single expression applies throughout, i.e. $\lambda_a^2 \mathbf{N}_a \otimes \mathbf{N}_a = \sum_{a=1}^{3} \lambda_a^2 \mathbf{N}_a \otimes \mathbf{N}_a$.

Accordingly, the stretch tensor is properly defined as

$$\mathbf{U} = \lambda_a \mathbf{N}_a \otimes \mathbf{N}_a \tag{2.19}$$

since

$$\mathbf{U}^2 = (\lambda_a \mathbf{N}_a \otimes \mathbf{N}_a)(\lambda_b \mathbf{N}_b \otimes \mathbf{N}_b) = \lambda_a \lambda_b \delta_{ab} \mathbf{N}_a \otimes \mathbf{N}_b = \lambda_a^2 \mathbf{N}_a \otimes \mathbf{N}_a = \mathbf{C}.$$

The square roots λ_a of the principal deformations λ_a^2 are called the *principal stretches.* They are the direct generalization of the homogeneous one-dimensional stretch ratio $\lambda = l/L = d\mathrm{x}/d\mathrm{X}$ introduced in the previous section to inhomogeneous three-dimensional deformations.

Now, using the polar decomposition theorem, the (nonsingular) deformation gradient $\mathbf{F}$ can be decomposed into the product of a pure *rotation* and a pure *stretch*

$$\mathbf{F} = \mathbf{RU}, \tag{2.20}$$

where $\mathbf{R}$ is orthogonal ($\mathbf{R}^T\mathbf{R} = \mathbf{I}$, $\mathbf{R}\mathbf{R}^T = \mathbf{i}$) and $\mathbf{U}$ is the symmetric right stretch tensor just introduced ($\mathbf{U} = \mathbf{U}^T$). Indeed it can easily be checked that $\mathbf{R}^T\mathbf{R} = \mathbf{U}^{-T}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-1} = \mathbf{I}$. It must be kept in mind that the polar decomposition is a *local* operation (defined pointwise), devoid of any global meaning in general. The image of a global stretched configuration, such as the one sketched forth in Fig. 3 does not exist in a strict continuum sense. (At most, it could be pictured as an "aggregate" of incompatible pieces). As already implied and sketched in Fig. 4, $\mathbf{U}$ stretches each unit principal fiber $\mathbf{N}_a$ by an amount λ_a along $\lambda_a \mathbf{N}_a = \mathbf{U}\mathbf{N}_a$ (no sum over a). Separately, $\mathbf{R}$ rotates the unit orthogonal triad $\mathbf{N}_a$ embedded into the undeformed configuration into the corresponding unit orthogonal triad $\mathbf{n}_a$ embedded into the deformed configuration $\mathbf{n}_a = \mathbf{R}\mathbf{N}_a$. In the

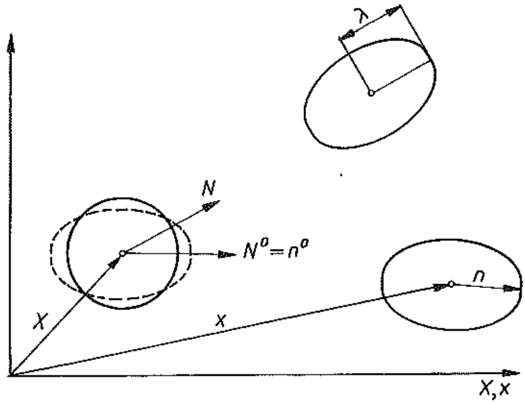

FIG. 4.

spatial description, the unit vectors $\mathbf{n}_a$ would be recognized as the principal directions of the spatial stretch and deformation tensors

$$\mathbf{u} = \sqrt{\mathbf{c}} = \sqrt{\mathbf{F}\mathbf{F}^T} = \lambda_a \mathbf{n}_a \otimes \mathbf{n}_a.$$

The composition of $\mathbf{U}$ and $\mathbf{R}$ indicates that the deformation gradient first stretches the unit cube delimited by the principal edges $\mathbf{N}_a$ into a rectangular parallelepiped with edges $\lambda_a \mathbf{N}_a$, and next rotates it in the spatial principal directions $\mathbf{n}_a$

$$\lambda_a \mathbf{n}_a = \mathbf{F}\mathbf{N}_a \qquad \text{(no sum)}. \tag{2.21}$$

Accordingly, the spectral forms of these three tensors and their inverses are

$$\begin{aligned} \mathbf{U} &= \lambda_a \mathbf{N}_a \otimes \mathbf{N}_a, & \mathbf{R} &= \mathbf{n}_a \otimes \mathbf{N}_a, & \mathbf{F} &= \lambda_a \mathbf{n}_a \otimes \mathbf{N}_a, \\ \mathbf{U}^{-1} &= \lambda_a^{-1} \mathbf{N}_a \otimes \mathbf{N}_a, & \mathbf{R}^T &= \mathbf{N}_a \otimes \mathbf{n}_a, & \mathbf{F}^{-1} &= \lambda_a^{-1} \mathbf{N}_a \otimes \mathbf{n}_a. \end{aligned} \tag{2.22}$$

Tensorial forms of these tensors are also available but, as a rule, they are even more complicated to obtain than the spectral forms (2.22). For instance, once $\mathbf{U}$ is extricated, as indicated earlier, it may be inverted to derive the rotation from $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. The rotation $\mathbf{R}$ has one real eigenvalue equal to $+1$ along the principal axis of rotation $\mathbf{j}$ defined by $\mathbf{R}\mathbf{j} = \mathbf{j}$ and two complex conjugate eigenvalues $\cos\omega \pm i \sin\omega$, involving the angle of rotation ω, along two principal radii $\mathbf{j}_1 \perp \mathbf{j}$ and $\mathbf{j}_2 = \mathbf{j} \times \mathbf{j}_1$. Consequently the rotation tensor may be equivalently represented by the rotation vector $\mathbf{r} = \omega\mathbf{j}$ whenever favorable. The rotation tensor $\mathbf{R}$ can then be recovered from

$$\mathbf{R} = \cos\omega(\mathbf{j}_1 \otimes \mathbf{j}_1 + \mathbf{j}_2 \otimes \mathbf{j}_2) - \sin\omega(\mathbf{j}_1 \otimes \mathbf{j}_2 - \mathbf{j}_2 \otimes \mathbf{j}_1) + \mathbf{j} \otimes \mathbf{j}.$$

The polar decomposition (2.20) applied to the transpose of the inverse deformation gradient, $\mathbf{F}^{-T} = \mathbf{R}\mathbf{U}^{-1}$, reveals the close (but otherwise hidden) similarity between the line and surface transformations (2.11) and (2.12) (see Fig. 5 for an illustration),

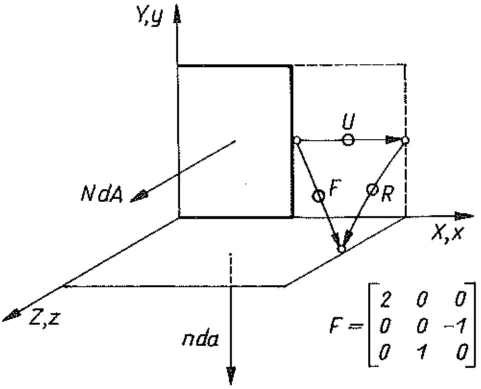

FIG. 5.

$$\begin{aligned} d\mathbf{x} &= \mathbf{R}\mathbf{U}\,d\mathbf{X}, \\ d\mathbf{s} &= \mathbf{R}(J\mathbf{U}^{-1})\,d\mathbf{S}. \end{aligned} \tag{2.23}$$

The above similarity confirms the analogy already mentioned between the length and area relations (2.15) and (2.16).

In the hypothesis of small deformations, the decomposition of the deformation gradient into the product of a symmetric stretch tensor and an orthogonal rotation tensor may be replaced by the (exact) decomposition into the sum of a symmetric, but approximate, stretch tensor $\tilde{\mathbf{U}} = \tilde{\mathbf{U}}^T \simeq \mathbf{U}\ (= \mathbf{U}^T)$ and an antisymmetric, also approximate, rotation tensor $\tilde{\mathbf{R}} = -\tilde{\mathbf{R}}^T \simeq \mathbf{R}\ (= \bar{\mathbf{R}}^T)$ defined by

$$\mathbf{F} = \tilde{\mathbf{U}} + \tilde{\mathbf{R}} = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) + \frac{1}{2}(\mathbf{F} - \mathbf{F}^T). \tag{2.24}$$

Note that the tensors $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{R}}$ are neither spatial nor material, nor even mixed like $\mathbf{F}$, which makes their very existence questionnable, not to grapple with their components (unless the actual rotations are so small that the deformed and undeformed principal directions $\mathbf{n}_a$ and $\mathbf{N}_a$ cannot be distinguished, i.e. when $\mathbf{F} = \tilde{\mathbf{U}} = \mathbf{U}$, $\mathbf{R} = \mathbf{I}$, $\tilde{\mathbf{R}} = 0$). In the event of large deformations, the additive decomposition (2.24) becomes mathematically and physically meaningless, i.e. the addition is undefined and the factors lose their interpretation of stretch or rotation.

To close this section, note that the mixed deformation gradient $\mathbf{F}$ is not objective, whereas the material stretch tensor $\mathbf{U}$ and its by-products $\mathbf{C}$ and $\mathbf{C}^{-1}$ are.

2.3. Generalized strain

The fact that neither the deformation gradient $\mathbf{F}$ nor its byproducts $\mathbf{U}$, $\mathbf{C}$ or even $\tilde{\mathbf{U}}$ vanish in the reference configuration ($\mathbf{x} = \mathbf{X}$, $\mathbf{F} = \mathbf{U} = \mathbf{C} = \tilde{\mathbf{U}} = \mathbf{I} \neq \mathbf{0}$) disqualifies them as pure deformation measures.

The most classical substitute is the small strain formula introduced by Cauchy, based on the approximate stretch $\tilde{\mathbf{U}}$:

$$\mathbf{E}^C = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \mathbf{I}, \qquad \text{(Cauchy)}. \tag{2.25}$$

Unfortunately, this small strain approximation, besides its peculiar tensorial nature already mentioned, does not remain invariant in a finite rotation of the body ($\mathbf{x} = \mathbf{R}\mathbf{X}$, $\mathbf{F} = \mathbf{R}$, $\mathbf{E}^C = \frac{1}{2}(\mathbf{R} + \mathbf{R}^T) - \mathbf{I} \neq \mathbf{0}$). This deficiency rules it out as an objective measure of large deformation.

The generalization of the classical strain measures introduced in Sect. 2.1. is not subject to this criticism because they are based on the objective byproducts $\mathbf{U}$ and $\mathbf{C}$ (rather than $\tilde{\mathbf{U}}$). They take the material forms

$$\begin{aligned}
\mathbf{E}^G &= \frac{1}{2}(\mathbf{C} - \mathbf{I}), && \text{(Green)},\\
\mathbf{E}^B &= \mathbf{U} - \mathbf{I}, && \text{(Biot)},\\
\mathbf{G} &= \operatorname{Log}\mathbf{U} = \frac{1}{2}\operatorname{Log}\mathbf{C}, && \text{(natural)},\\
\mathbf{E}^H &= \mathbf{I} - \mathbf{U}^{-1}, && \text{(Hill)},\\
\mathbf{E}^K &= \frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1}), && \text{(Karni)}.
\end{aligned} \tag{2.26}$$

REMARK. It must be emphasized that the Hill and Karni strains are simply the material forms of the spatial strains of Swainger and Almansi, respectively, defined by $\mathbf{e}^S = \mathbf{i} - \mathbf{u}^{-1}$ and $\mathbf{e}^A = \frac{1}{2}(\mathbf{i} - \mathbf{c}^{-1})$, where $\mathbf{c}^{-1} = \mathbf{u}^{-2} = \mathbf{F}^{-T}\mathbf{F}^{-1}$ and $\mathbf{i}$ is the spatial identity. In particular, the interpretation of the Karni strain may be facilitated by the series of identities

$$\begin{aligned}
\frac{1}{2}(d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X}) &= d\mathbf{X} \cdot \mathbf{E}^G d\mathbf{X} = d\mathbf{X} \cdot \mathbf{U}\mathbf{E}^K\mathbf{U} d\mathbf{X}\\
&= d\mathbf{x} \cdot \mathbf{R}\mathbf{E}^K\mathbf{R}^T d\mathbf{x} = d\mathbf{x} \cdot \mathbf{e}^A d\mathbf{x}
\end{aligned}$$

which show its connection with the Green and Almansi strains.

With a provision for the natural strain, all the classical measures (2.26) are included in the one-parameter family [19, 4]

$$\text{(2.27)} \qquad \mathbf{E}^{(m)} = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}), \qquad m = +2, +1, -1, -2, \qquad \text{(Seth)}.$$

The Biot strain ($m = 1$) is the closest objective substitute for Cauchy's small strain approximation. However, the extraction of the square root $\mathbf{U} = \sqrt{\mathbf{C}}$ makes it difficult to use in practice.

REMARK. Conversely, an approximation to the objective Hill strain (2.26) could be defined by analogy with Eq. (2.25) as $\tilde{\mathbf{E}}^H = \mathbf{I} - \frac{1}{2}(\mathbf{F}^{-1} + \mathbf{F}^{-T})$. In fact, it seems that the spatial strain advocated by Swainger was this kind of approximation rather than its objective counterpart.

The logarithm or natural strain measure ($m \to 0$) is properly defined in spectral form as [26]

$$\text{(2.28)} \qquad \mathbf{G} = \frac{1}{2}\mathrm{Log}\mathbf{C} = \mathrm{Log}\mathbf{U} = \mathrm{Log}\lambda_a \mathbf{N}_a \otimes \mathbf{N}_a.$$

It follows from this definition of the logarithm of a symmetric tensor that the property of "additivity" is preserved under the exclusive condition of "parallel" (or "irrotational" or "pure") defomations along fixed principal directions (to guarantee the product commutativity):

$$\mathrm{Log}\mathbf{AB} = \mathrm{Log}\mathbf{A} + \mathrm{Log}\mathbf{B} \qquad \text{if and only if}$$

$$\mathbf{A} = \alpha_a \mathbf{N}_a \otimes \mathbf{N}_a \qquad \text{and} \qquad \mathbf{B} = \beta_a \mathbf{N}_a \otimes \mathbf{N}_a$$

along the same $\mathbf{N}_a$ (because if that is true, not only $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B}^T = \mathbf{B}$ but also $\mathbf{AB} = \mathbf{BA}$). In other words, $\mathbf{B}$ must belong to the commutator of $\mathbf{A}$.

The natural scale is even more cumbersome to evaluate than the Biot strain, and so a good approximation is sought. Instead of proceeding by mere extrapolation from the one-dimensional forms (replacing λ by $\mathbf{U}$ in Eqs. (2.5), (2.7) and (2.8)), a constructive approach which will give additional insight is preferred.

The preceding remarks and examples together with [3, 13] suggest that a generalized material strain measure can be defined as an *isotropic* tensor function of the *objective* material stretch tensor $\mathbf{U}$, which vanishes in the reference configuration, coincides with the small strain around it and remains regular away from it. More specifically, a generalized strain is characterized by

$$\begin{aligned} \mathbf{E} &= \mathbf{E}(\mathbf{U}) && \text{(objectivity)}, \\ \mathbf{A}\mathbf{E}(\mathbf{U})\mathbf{A}^T &= \mathbf{E}(\mathbf{A}\mathbf{U}\mathbf{A}^T) && \text{(isotropy)}, \\ \mathbf{E}(\mathbf{I}) = \mathbf{0}, \quad \mathbf{E}'(\mathbf{I}) &= \mathbf{I} && \text{(consistency)}, \\ \exists \mathbf{E}^{-1} & && \text{(regularity)}, \end{aligned} \tag{2.29}$$

where $\mathbf{A}$ denotes an arbitrary rotation $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. The definition of the tensor function $\mathbf{E}(\mathbf{U})$ and, more particularly, of its derivative $\mathbf{E}'$ and inverse $\mathbf{E}^{-1}$ are specified later.

The selective dependence of the strain function on the symmetric stretch tensor $\mathbf{U}$ (rather than $\mathbf{F}$ itself) reflects the basic requirement of obtaining an objective or frame-indifferent measure. Briefly, a material strain function is objective (i.e. frame-indifferent) if $\mathbf{E}(\mathbf{F}) = \mathbf{E}(\mathbf{A}\mathbf{F})$ for any rotation $\mathbf{A}$ which implies Eq. $(2.29)_1$ by choosing $\mathbf{A} = \mathbf{R}^T$. A material strain function is rotationally symmetric (i.e. body-indifferent) if $\mathbf{A}\mathbf{E}(\mathbf{F})\mathbf{A}^T = \mathbf{E}(\mathbf{F}\mathbf{A}^T)$. Objectivity plus symmetry implies isotropy as defined by $(2.29)_2$. The isotropy requirement corresponds to the *additional* deliberate uncoupling of the kinematics of deformations from the response of materials, i.e. to construct a material-indifferent measure. Compulsory for the treatment of isotropic materials, this requirement seems merely normative for the analysis of anisotropic media [7]. Note that the computationally inaccessible stretch tensor $\mathbf{U} = \sqrt{\mathbf{C}}$ is preferred to its readily available originator $\mathbf{C}$, purely for convenience of notation (provided by integer exponents).

The consistency conditions are difficult to use at this stage of generality due to their local nature. On the contrary, the global isotropy condition provides a definite restriction on the general form of a candidate deformation measure specified by the fundamental representation theorem of isotropic tensor functions, e.g. [27, 57].

Essential to the proof of this theorem are the findings that the principal directions of strain must coincide with the principal directions of stretch $\mathbf{N}_a$, and that the principal strains ε_a must be symmetric functions of the principal stretches λ_b,

$$\mathbf{E} = \varepsilon_a(\lambda_b)\mathbf{N}_a \otimes \mathbf{N}_a. \tag{2.30}$$

More technical is the fact that three symmetric scalar functions in three scalar variables can always be reduced to a three-term expansion in the form

(2.31) $$\varepsilon_a(\lambda_b) = x(\lambda_b)\lambda_a^p + y(\lambda_b) + z(\lambda_b)\lambda_a^q,$$

where $x(\lambda_b)$, $y(\lambda_b)$, $z(\lambda_b)$ are symmetric functions of $\lambda_1, \lambda_2, \lambda_3$ and p, q are any two different integer exponents (which are better taken to be of opposite sign in the interval $-2 \leq q < 0 < p \leq +2$ for the present purposes).

Merging the two results (2.30) and (2.31) produces the desired tensorial form

(2.32) $$\mathbf{E} = x\mathbf{U}^p + y\mathbf{I} + z\mathbf{U}^q \qquad (= \mathbf{E}^T).$$

Within the range recommended for the integer exponents $-2 \leq q < 0 < p \leq 2$, this representation formula is standard for the even combinations ($p = 1, q = -1$) and ($p = 2, q = -2$), but more original for the odd combinations ($p = 1, q = -2$) and ($p = 2, q = -1$). It is emphasized that the isotropy of the strain function implies the symmetry of the resulting generalized strain.

An important class of isotropic strain functions is formed by those whose each principal strain depends (in precisely the same way as the other two) on only one corresponding principal stretch i.e. $\varepsilon_a(\lambda_b) = E(\lambda_a)$, [7], leading to the generic formula

(2.33) $$\mathbf{E} = \mathbf{E}(\lambda_a)\mathbf{N}_a \otimes \mathbf{N}_a.$$

For instance, the natural strain (2.28) is a typical example of such a function with $\varepsilon_a(\lambda_b) = E(\lambda_a) = \mathrm{Log}\lambda_a$. More generally, most tensor functions extrapolated from a scalar prototype fall into this category of *simple* isotropic tensor functions [28]. The derivative and the inverse of such functions are readily obtained from their scalar analogues as

$$\mathbf{E}'(\mathbf{U}) = E'(\lambda_a)\mathbf{N}_a \otimes \mathbf{N}_a, \qquad \mathbf{E}^{-1}(\mathbf{U}) = \frac{1}{E(\lambda_a)}\mathbf{N}_a \otimes \mathbf{N}_a.$$

Even within this simple class of strain functions, the coefficients x, y, z in the expansion (2.32) remain complicated rational functions of the principal values λ_a. Explicit expressions for these coefficients are given in Appendix B. A comparison with the classical measures displayed in Eqs. (2.26) suggests that one's attention should be restricted to *constant* coefficient or trinomial expansions. It is also expedient to replace the global regularity condition $(2.29)_4$ by a local convexity control at the origin, which by analogy with Eqs. (2.7) is expressed in the form

$$E''(1) = p + q - 1.$$

Within this drastically simplified context, the local consistency and convexity conditions become deterministic.

Indeed, x, y, z being constant

$$\begin{aligned} E(\lambda) &= x\lambda^p + y + z\lambda^q, \\ E'(\lambda) &= px\lambda^{p-1} + qz\lambda^{q-1}, \\ E''(\lambda) &= p(p-1)x\lambda^{p-2} + q(q-1)z\lambda^{q-2}. \end{aligned}$$

Enforcing the local conditions

$$\left.\begin{aligned} E(1) &= x + y + z = 0, \\ E'(1) &= px + qz = 1, \\ E''(1) &= p(p-1)x + q(q-1)z = p + q - 1, \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x &= \frac{1}{p-q}, \\ y &= 0, \\ z &= \frac{1}{q-p}. \end{aligned}\right.$$

The result is the tensorial generalization of the scalar rubber family (2.7) which is confirmed to correspond to a convex combination of any two basic members $\mathbf{E}^{(p)}$ and $\mathbf{E}^{(q)}$ of the Seth family (2.27):

$$\text{(2.34)} \quad \mathbf{E} = \frac{1}{p-q}(\mathbf{U}^p - \mathbf{U}^q) = \frac{p}{p-q}\mathbf{E}^{(p)} + \frac{q}{q-p}\mathbf{E}^{(q)}, \qquad \text{(rubber)},$$
$$-2 \le q \le 0 \le p \le +2 \qquad (p \ne 0, \quad \text{or} \quad q \ne 0).$$

Schematically, the generalized strain $\mathbf{E}$ may be regarded as decomposed on the basis formed by ($\mathbf{U}^p$, $\mathbf{I}$, $\mathbf{U}^q$) provided p and q are properly selected. Now, if the alternative forms obtained with different integer values of the exponents p and q in formula (2.32) with *variable* coefficients $x(\lambda_b)$, $y(\lambda_b)$, $z(\lambda_b)$ are strictly equivalent (by virtue of the Cayley–Hamilton theorem), the corresponding forms obtained by the same procedure applied to formula (2.34) with *constant* coefficients are definitely different. In particular p and q should preferably be chosen with opposite signs to avoid erratic asymptotic behaviour as $\mathbf{U}$ shrinks to zero or expands to infinity. The new members included in the rubber family are listed below to complete the classical strains included in the Seth family:

$$\text{(2.35)} \quad \begin{aligned} \mathbf{E}^P &= \frac{1}{2}\mathbf{E}^B + \frac{1}{2}\mathbf{E}^H = \frac{1}{2}(\mathbf{U} - \mathbf{U}^{-1}) && \text{(Pelzer,} && p = -q = 1), \\ \mathbf{E}^M &= \frac{1}{2}\mathbf{E}^G + \frac{1}{2}\mathbf{E}^K = \frac{1}{4}(\mathbf{C} - \mathbf{C}^{-1}) && \text{(Mooney,} && p = -q = 2), \\ \mathbf{E}^W &= \frac{1}{3}\mathbf{E}^B + \frac{2}{3}\mathbf{E}^K = \frac{1}{3}(\mathbf{U} - \mathbf{U}^{-2}) && \text{(Wall,} && p = -\frac{q}{2} = 1), \\ \mathbf{E}^R &= \frac{2}{3}\mathbf{E}^G + \frac{1}{3}\mathbf{E}^H = \frac{1}{3}(\mathbf{U}^2 - \mathbf{U}^{-1}) && \text{(Rivlin,} && \frac{p}{2} = -q = 1). \end{aligned}$$

The Mooney strain offers definite computational advantages over the other three, just as the Green and Karni strains are more practical than the Biot and Hill members in the classical family. Moreover, the Mooney strain represents a fairly good approximation of the natural strain and therefore may be expected to remain almost "additive" up to moderate "parallel" strains. In particular the exact characterization of large isochoric deformations by the vanishing trace of the natural strain $\text{tr}\mathbf{G} = \frac{1}{2}\log(\det \mathbf{C}) = \text{Log}J = 0$ (equivalent to the unicity of the Jacobian $J = 1$) carries over to the Mooney strain within a high order of approximation $\text{tr}\,\mathbf{E}^M = \text{tr}\,\mathbf{G} + o(\varepsilon^3) = O + o(\varepsilon^3)$.

A generalized strain of the rubber family may be interpreted by a constitutive law specialist as a stable isotropic elastic gauge material characterized by the prescribed "moduli" p and q.

REMARK. An approximation to the Pelzer strain is:

$$\tilde{\mathbf{E}}^p = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T - \mathbf{F}^{-1} - \mathbf{F}^{-T}) = \frac{\mathbf{R}}{2}(\mathbf{U} - \mathbf{U}^{-1}) + (\mathbf{U} - \mathbf{U}^{-1})\frac{\mathbf{R}^T}{2}.$$

This approximate strain offers the advantage of remaining invariant in a pure rotation (since $\mathbf{U} = \mathbf{U}^{-1} = \mathbf{I}$) which elects it as a valuable candidate for the study of small strain-large rotation problems.

The derivation of the rates of the various strains introduced above is considered in the next two sections prior to an investigation of their conjugate stresses. Incidentally, this kinematic analysis will reveal the existence of a hidden rate of deformation, called the rotated rate, which will prove to play a crucial role in the investigation of conjugate stresses.

2.4. Basic kinematics

Introducing the time parameter, the discrete placement (2.10) must be replaced by the continuous *motion*

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t). \tag{2.36}$$

The *velocity* of a particle initially located at $\mathbf{X} = \mathbf{x}(\mathbf{X}, 0)$ and now occupying the position $\mathbf{x}$ at time t is defined by

$$\mathbf{v} = \mathbf{v}(\mathbf{X}, t) = \frac{\partial \mathbf{x}}{\partial t}(\mathbf{X}, t) = \dot{\mathbf{x}}. \tag{2.37}$$

The material gradient of this velocity field is found to coincide with the rate of the deformation gradient $\mathbf{F} = \partial \mathbf{x}/\partial \mathbf{X}$

$$d\mathbf{v} = \dot{\mathbf{F}}\, d\mathbf{X}, \qquad \dot{\mathbf{F}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}} = \frac{\partial^2 \mathbf{x}}{\partial \mathbf{X} \partial t} = \frac{\partial \mathbf{F}}{\partial t}. \tag{2.38}$$

REMARK. A spatial description would begin from here by referring the velocity to the spatial variable $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ and deriving the corresponding spatial velocity gradient $\mathbf{l} = \partial\mathbf{v}/\partial\mathbf{x}$ through $d\mathbf{v} = \mathbf{l}\,d\mathbf{x} = \mathbf{l}\mathbf{F}\,d\mathbf{X}$ which shows the link between the mixed and spatial forms $\dot{\mathbf{F}} = \mathbf{l}\mathbf{F}$.

The mixed (spatial-material) *velocity gradient* $\dot{\mathbf{F}}$ is taken as the basic variable and all other deformation rates of interest will eventually (or at least tentatively) be expressed in terms of it.

The polar decomposition of the deformation gradient provides additional insight into the structure of the mixed velocity gradient $\dot{\mathbf{F}} = \mathbf{R}\dot{\mathbf{U}} + \dot{\mathbf{R}}\mathbf{U}$. The rotation being orthogonal ($\mathbf{R}^T\mathbf{R} = \mathbf{I}$), the rotation rate may be referred to the rotation itself by means of a skew-symmetric tensor $\boldsymbol{\theta} = -\boldsymbol{\theta}^T$ called the material relative spin, i.e. $\dot{\mathbf{R}} = \mathbf{R}\boldsymbol{\theta}$. (For a justification, see the construction of Eq. (2.48) below). As a result, the mixed velocity gradient appears as the sum of a stretching and a spinning rate, both rotated to a material status $\dot{\mathbf{F}} = \mathbf{R}(\dot{\mathbf{U}} + \boldsymbol{\theta}\mathbf{U})$. The lack of symmetry of this deformation gradient rate, inherent in its mixed nature, reveals its lack of objectivity in a rotation and constitutes a strong incentive to look for alternative material rates of deformation.

The *rates of the material deformation* tensor[1] $\mathbf{C}$ and its inverse $\mathbf{C}^{-1}$ (obtained by straightforward differentiation of their definitions, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{C}\mathbf{C}^{-1} = \mathbf{I}$, with respect to time) fulfill part of this need,

$$\begin{aligned} \dot{\mathbf{C}} &= \mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F}, \\ -\dot{\mathbf{C}}^{-1} &= \mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{C}^{-1} + \mathbf{C}^{-1}\dot{\mathbf{F}}^T\mathbf{F}^{-T}. \end{aligned} \tag{2.39}$$

Additional insight into the inner structure of these deformation rates may be obtained by substituting the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$ and trivial

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(1) Whereas the spatial symmetric part $\mathbf{d} = \frac{1}{2}(\mathbf{l} + \mathbf{l}^T)$ due to EULER (1770) constitutes a well-defined spatial rate of deformation tensor, the hybrid counterpart $\dot{\mathbf{U}} = \frac{1}{2}(\dot{\mathbf{F}} + \dot{\mathbf{F}}^T)$ is ill-defined, which motivates the search for a fully material velocity gradient. By analogy with (2.14), a possible procedure would be to calculate the squared magnitude of the velocity increment via the self-scalar product $d\mathbf{v} \cdot d\mathbf{v} = d\mathbf{X} \cdot \dot{\mathbf{F}}^T\dot{\mathbf{F}}d\mathbf{X}$. Unfortunately, the material tensor $\dot{\mathbf{F}}^T\dot{\mathbf{F}}$ produced by this process is not the rate of any known strain tensor. Thus the alternative product $d\mathbf{x} \cdot d\mathbf{v} = d\mathbf{X} \cdot \mathbf{F}^T\dot{\mathbf{F}}d\mathbf{X}$ generating the material counterpart of the spatial velocity gradient $\mathbf{L} = \mathbf{F}^T\dot{\mathbf{F}} = \mathbf{F}^T\mathbf{l}\mathbf{F}$. Twice its symmetric part $\dot{\mathbf{C}} = (\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F})$, due to DUHEM (1904), is one material transcription of the spatial rate of deformation $2\dot{\mathbf{C}} = \mathbf{F}^T\mathbf{d}\mathbf{F}$ and is recognized as the material deformation rate more simply defined by Eqs. (2.39).

identities such as $\mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$ in their expressions (2.39) to obtain

$$\begin{aligned} \dot{\mathbf{C}} &= \mathbf{U}\mathbf{R}^T\dot{\mathbf{F}}\mathbf{U}^{-1}\mathbf{U} + \mathbf{U}\mathbf{U}^{-1}\dot{\mathbf{F}}^T\mathbf{R}\mathbf{U} = \mathbf{U}(\mathbf{R}^T\dot{\mathbf{F}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{F}}^T\mathbf{R})\mathbf{U}, \\ -\dot{\mathbf{C}}^{-1} &= \mathbf{U}^{-1}\mathbf{R}^T\dot{\mathbf{F}}\mathbf{U}^{-1}\mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{U}^{-1}\dot{\mathbf{F}}^T\mathbf{R}\mathbf{U}^{-1} \\ &= \mathbf{U}^{-1}(\mathbf{R}^T\dot{\mathbf{F}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{F}}^T\mathbf{R})\mathbf{U}^{-1}. \end{aligned}$$

The above formulas suggested the existence of an intermediate material rate called the *rotated rate of deformation*

$$2\mathbf{D} = \mathbf{R}^T\dot{\mathbf{F}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{F}}^T\mathbf{R} \qquad \text{(rotated)}. \tag{2.40}$$

This rate is so-called because it corresponds to the classical spatial rate of deformation $\mathbf{d}$, rotated to a material status i.e. $\mathbf{D} = \mathbf{R}^T\mathbf{d}\mathbf{R}$.

The proof of this assertion is straightforward:

$$\begin{aligned} 2\mathbf{D} = \mathbf{R}^T\dot{\mathbf{F}}\mathbf{U}^{-1}\mathbf{R}^T\mathbf{R} + \mathbf{R}^T\mathbf{R}\mathbf{U}^{-1}\dot{\mathbf{F}}^T\mathbf{R} &= \mathbf{R}^T(\dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}^{-T}\dot{\mathbf{F}}^T)\mathbf{R} \\ &= \mathbf{R}^T(\mathbf{l} + \mathbf{l}^T)\mathbf{R} = 2\mathbf{R}^T\mathbf{d}\mathbf{R}. \end{aligned}$$

In terms of the rotated rate (2.40), the material rates (2.39) take the form of a convected and a "contravected" rate of deformation, respectively,

$$\begin{aligned} \dot{\mathbf{C}} &= 2\mathbf{U}\mathbf{D}\mathbf{U} && \text{"convected"}, \\ -\dot{\mathbf{C}}^{-1} &= 2\mathbf{U}^{-1}\mathbf{D}\mathbf{U}^{-1} && \text{"contravected"}. \end{aligned} \tag{2.41}$$

This terminology is justified in the formulas $\dot{\mathbf{C}} = 2\mathbf{F}^T\mathbf{d}\mathbf{F}$ and $\dot{\mathbf{C}}^{-1} = -2\mathbf{F}^{-1}\mathbf{d}\mathbf{F}^{-T}$ disclosing a convection of the spatial rate of deformation by the deformation gradient and its inverse.

Although the rotated rate of deformation $\mathbf{D}$ is not the time derivative of any known strain (except for the fictitious "rotated strain" formally defined by $\int_0^t \dot{\mathbf{D}}\, dt$ or its closest substitute, the natural strain $\mathbf{G} = \mathrm{Log}\mathbf{U}$), it will be found to play a pivotal role in comparing the various candidate strain rates under study, as already apparent in Eqs. (2.41).

The derivation of an explicit formula for the *stretch rate* $\dot{\mathbf{U}} = \sqrt{\dot{\mathbf{C}}}$ (in terms of $\dot{\mathbf{F}}$ and $\mathbf{F}$ only) is rather difficult due to presence of the square root [29, 30, 61, 62, 63]. Consequently the following implicit definitions must be accepted as the best tensorial substitutes

$$\begin{aligned} \dot{\mathbf{C}} &= \mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}, \\ 2\mathbf{D} &= \mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1}. \end{aligned} \tag{2.42}$$

REMARK. The solution $\dot{\mathbf{U}}$ of Eq. $(2.42)_1$, where $\mathbf{U}$ and $\dot{\mathbf{C}}$ are considered as given, can be shown to exist and to be unique [31, 29, 30]. Idem for Eq. $(2.42)_2$.

Just as it helped constructing the notion of stretch itself, the *spectral approach* provides valuable insight into the inner structure of the stretch rate. Starting from the spectral definition (2.19) of the stretch tensor $\mathbf{U} = \sqrt{\mathbf{C}}$, the stretch rate may be derived without difficulty, provided the principal directions of stretch are recognized as varying during the deformation

$$\dot{\mathbf{U}} = \dot{\lambda}_a \mathbf{N}_a \otimes \mathbf{N}_a + \lambda_a \dot{\mathbf{N}}_a \otimes \mathbf{N}_a + \lambda_a \mathbf{N}_a \otimes \dot{\mathbf{N}}_a. \tag{2.43}$$

In fact, since the principal stretch directions are unit vectors attached to a fixed material point $\mathbf{X}$, they can rotate only around their pivot origin according to $\mathbf{N}_a(\mathbf{X},t) = \mathbf{Q}(\mathbf{X},t)\mathbf{N}_a(\mathbf{X},0)$, where $\mathbf{Q}$ is an orthogonal rotation ($\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$) called the *material stretch rotation* (cf. Fig. 4).

More concisely and conversely,

$$\mathbf{N}_a = \mathbf{Q}\mathbf{N}_a^0, \qquad \mathbf{N}_a^0 = \mathbf{Q}^T\mathbf{N}_a. \tag{2.44}$$

The uniqueness of the initial principal stretch basis $\mathbf{N}_a^0 = \mathbf{N}_a(\mathbf{X},0)$ also called the background triad [13, 11] is questionable since the undeformed configuration is by definition exempt from any strain. It is easiest to see it as the incipient basis obtained as the limit orientation of the current basis as time approaches zero going backwards.

The "tangential" velocity of each "radial" stretch vector may be referred to itself as follows

$$\dot{\mathbf{N}}_a = \dot{\mathbf{Q}}\mathbf{N}_a^0 = \dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{N}_a = \mathbf{\Omega}\mathbf{N}_a. \tag{2.45}$$

The instantaneous rotation rate of the principal stretch triad in relation to itself is called the *material stretch spin*, e.g. [13],

$$\mathbf{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T = -\mathbf{\Omega}^T, \qquad \dot{\mathbf{Q}} = \mathbf{\Omega}\mathbf{Q}. \tag{2.46}$$

The spin tensor is skewsymmetric: this is easily checked by differentiating the identity $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$. In particular its decomposition on the principal triad may be written

$$\mathbf{\Omega} = \Omega_{bc}\mathbf{N}_b \otimes \mathbf{N}_c \qquad \text{with} \qquad \Omega_{bc} = -\Omega_{cb}.$$

It follows that

$$\dot{\mathbf{N}}_a = \mathbf{\Omega}\mathbf{N}_a = (\Omega_{bc}\mathbf{N}_b \otimes \mathbf{N}_c)\mathbf{N}_a = \Omega_{bc}\mathbf{N}_b\delta_{ac} = \Omega_{ba}\mathbf{N}_b.$$

Therefore to summarize:

$$\dot{\mathbf{N}}_a = \Omega_{ba}\mathbf{N}_b, \qquad \Omega_{ba} = -\Omega_{ab}. \tag{2.47}$$

Finally, the stretch rate (2.43) takes the spectral form

$$\dot{\mathbf{U}} = [\dot{\lambda}_a\delta_{ab} + (\lambda_b - \lambda_a)\Omega_{ab}]\mathbf{N}_a \otimes \mathbf{N}_b. \tag{2.48}$$

This formula is fundamental in showing the inner structure of the stretch tensor, namely its decomposition into the sum of a "radial" stretch rate and a "hoop" rate. However, its practical interest is limited by the fact that the material stretch spin tensor remains out of reach. Consult Appendix C for a similar spectral form of the deformation gradient rate.

Repeating the same process for the spectral forms (2.18) of the deformation tensor $\mathbf{C}$ and its inverse $\mathbf{C}^{-1}$ leads to

$$\begin{aligned} \dot{\mathbf{C}} &= [2\lambda_a\dot{\lambda}_a\delta_{ab} + (\lambda_b^2 - \lambda_a^2)\Omega_{ab}]\mathbf{N}_a \otimes \mathbf{N}_b, \\ -\dot{\mathbf{C}}^{-1} &= \left[\frac{2\dot{\lambda}_a}{\lambda_a^3}\delta_{ab} + (\lambda_a^{-2} - \lambda_b^{-2})\Omega_{ab}\right]\mathbf{N}_a \otimes \mathbf{N}_b. \end{aligned} \tag{2.49}$$

Finally, a similar procedure applied to the rotated rate expression (2.41) gives the spectral formula

$$\mathbf{D} = \left[\frac{\dot{\lambda}_a}{\lambda_a}\delta_{ab} + \frac{1}{2}\left(\frac{\lambda_b}{\lambda_a} - \frac{\lambda_a}{\lambda_b}\right)\Omega_{ab}\right]\mathbf{N}_a \otimes \mathbf{N}_b. \tag{2.50}$$

The intermediate role played by $2\mathbf{D}$ between $\dot{\mathbf{C}}$ and $-\dot{\mathbf{C}}^{-1}$ already disclosed by Eqs. (2.41) is confirmed in a different guise (no summation over repeated indices)

$$\begin{aligned} \frac{1}{\lambda_a^2}\dot{C}_{aa} &= 2D_{aa} = -\lambda_a^2\dot{C}_{aa}^{-1}, \\ \frac{1}{\lambda_a\lambda_b}\dot{C}_{ab} &= 2D_{ab} = -\lambda_a\lambda_b\dot{C}_{ab}^{-1} \qquad (a \neq b). \end{aligned}$$

2.5. Strain rates

Following [7, 33] the rate $\dot{\mathbf{E}}$ of the generalized strain (2.29) can be formally expressed in terms of the stretch rate $\dot{\mathbf{U}}$, or the deformation gradient rate $\dot{\mathbf{F}}$ by means of the chain rule, or even in terms of the rotated rate $\mathbf{D}$ simply by inspection:

$$\dot{\mathbf{E}} = \frac{d\mathbf{E}}{d\mathbf{U}}\dot{\mathbf{U}} = \frac{d\mathbf{E}}{d\mathbf{F}}\dot{\mathbf{F}} = \mathbb{E}\mathbf{D}, \tag{2.51}$$

where $\frac{d\mathbf{E}}{d\mathbf{U}}$, $\frac{d\mathbf{E}}{d\mathbf{F}}$ and $\mathbb{E}$ are fourth order linear operators. For instance, the derivative of the Green strain with respect to the stretch can be shown to be

$$\frac{dE^G_{IJ}}{dU_{KL}} = \frac{1}{4}(\delta_{IK}U_{LJ} + U_{IK}\delta_{LJ} + U_{IL}\delta_{KJ} + \delta_{IL}U_{KJ}).$$

See Appendix D for the proof of this derivative and others.

Of course these fourth order tensors inherit the minor symmetries of their respective inputs and outputs (just like an elasticity tensor)

$$\frac{dE_{IJ}}{dU_{KL}} = \frac{dE_{IJ}}{dU_{LK}} = \frac{dE_{JI}}{dU_{KL}} \qquad \text{and the like for} \qquad \mathbb{E}_{IJKL}, \qquad \frac{dE_{IJ}}{dF_{kL}} = \frac{dE_{JI}}{dF_{kL}}.$$

Moreover, as noted by [7], for the class of *simple* isotropic strain functions (2.33) with the property $\frac{\partial E}{\partial \lambda_a}(\lambda_b) = \frac{\partial E}{\partial \lambda_b}(\lambda_a)$ even for $a \neq b$, the tangent operator $\frac{d\mathbf{E}}{d\mathbf{U}}$ possesses the additional major symmetry (just like a hyperelasticity tensor)

$$\frac{dE_{IJ}}{dU_{KL}} = \frac{dE_{KL}}{dU_{IJ}}.$$

For $d\mathbf{E}/d\mathbf{F}$, the existence of a strain potential results in a more complex symmetry not shown here. This kind of symmetry is questionable for $\mathbb{E}$. Furthermore these derivatives are invertible by definition of the strain function, meaning that

$$\text{(2.52)} \qquad \dot{\mathbf{U}} = \left(\frac{d\mathbf{E}}{d\mathbf{U}}\right)^{-1}\dot{\mathbf{E}}; \qquad \dot{\mathbf{F}} = \left(\frac{d\mathbf{E}}{d\mathbf{F}}\right)^{-1}\dot{\mathbf{E}}; \qquad \mathbf{D} = \mathbb{E}^{-1}\dot{\mathbf{E}}.$$

However, the general formulas (2.51) and their formal inverses (2.52) are misleading because, except in a few particular cases, they cannot be explicitly written down.

For instance the only *strain rates of the classical family* (2.26) which can be easily related to (at least one of) the reference rates $\dot{\mathbf{U}}$, $\dot{\mathbf{F}}$ or $\mathbf{D}$ (either in direct or in inverse form) are

$$\text{(2.53)} \qquad \begin{aligned} \dot{\mathbf{E}}^G &= \frac{1}{2}(\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}) = \frac{1}{2}(\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F}) = \mathbf{UDU}, \\ \dot{\mathbf{E}}^B &= \dot{\mathbf{U}}, \qquad\qquad 2\mathbf{D} = \mathbf{U}^{-1}\dot{\mathbf{E}}^B + \dot{\mathbf{E}}^B\mathbf{U}^{-1}, \\ \dot{\mathbf{E}}^H &= \mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}^{-1}, \qquad 2\mathbf{D} = \mathbf{U}\dot{\mathbf{E}}^H + \dot{\mathbf{E}}^H\mathbf{U}, \\ \dot{\mathbf{E}}^K &= \frac{1}{2}\mathbf{U}^{-1}(\mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1})\mathbf{U}^{-1} \\ &= \frac{1}{2}\mathbf{C}^{-1}(\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F})\mathbf{C}^{-1} = \mathbf{U}^{-1}\mathbf{D}\mathbf{U}^{-1}. \end{aligned}$$

Therefore the Green and Karni strains are the only ones with fully explicit tensorial expressions for their rates. Their expressions in terms of the rotated rate $\mathbf{D}$ are even invertible. It follows that all the classical strain rates (2.53) may be incorporated into a one-parameter family, called the Seth strain rate family, characterized by the implicit formulas

(2.54)
$$\begin{aligned} \mathbf{D} &= \mathbf{U}^{\frac{-m}{2}}\dot{\mathbf{E}}^{(m)}\mathbf{U}^{\frac{-m}{2}} && \text{if} \quad m = +2, -2,\\ \mathbf{D} &= \frac{1}{2}[\mathbf{U}^{-m}\dot{\mathbf{E}}^{(m)} + \dot{\mathbf{E}}^{(m)}\mathbf{U}^{-m}] && \text{if} \quad m = +1, -1. \end{aligned}$$

Similarly, the expicit *strain rates of the rubber family* (2.35) are limited to

(2.55)

$$\begin{aligned} \dot{\mathbf{E}}^P &= \frac{1}{2}\dot{\mathbf{E}}^B + \frac{1}{2}\dot{\mathbf{E}}^H = \frac{1}{2}(\dot{\mathbf{U}} + \mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}^{-1}),\\ \dot{\mathbf{E}}^M &= \frac{1}{2}\dot{\mathbf{E}}^G + \frac{1}{2}\dot{\mathbf{E}}^K = \frac{1}{4}[\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U} + \mathbf{U}^{-1}(\mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1})\mathbf{U}^{-1}]\\ &= \frac{1}{4}[\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F} + \mathbf{C}^{-1}(\mathbf{F}^T\dot{\mathbf{F}} + \dot{\mathbf{F}}^T\mathbf{F})\mathbf{C}^{-1}] = \frac{1}{2}(\mathbf{U}\mathbf{D}\mathbf{U} + \mathbf{U}^{-1}\mathbf{D}\mathbf{U}^{-1}),\\ \dot{\mathbf{E}}^W &= \frac{1}{3}\dot{\mathbf{E}}^B + \frac{2}{3}\dot{\mathbf{E}}^K = \frac{1}{3}[\dot{\mathbf{U}} + \mathbf{U}^{-1}(\mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1})\mathbf{U}^{-1}],\\ \dot{\mathbf{E}}^R &= \frac{2}{3}\dot{\mathbf{E}}^G + \frac{1}{3}\dot{\mathbf{E}}^H = \frac{1}{3}(\mathbf{U}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U} + \mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}^{-1}). \end{aligned}$$

Here also, the Mooney strain is the only one with explicit tensorial rate expressions. For the others, spectral forms remain the most feasible.

The spectral derivation developed in the previous section to elucidate the stretch rate is also applicable to the *generalized strain expression* (2.33) resulting in

(2.56)
$$\dot{\mathbf{E}} = [E'(\lambda_a)\dot{\lambda}_a\delta_{ab} + (E(\lambda_b) - E(\lambda_a))\Omega_{ab}]\mathbf{N}_a \otimes \mathbf{N}_b.$$

For instance, a *strain rate of the Seth family* (2.27) can now be expressed in direct form as

(2.57)
$$\dot{\mathbf{E}}^{(m)} = \left[\lambda_a^{m-1}\dot{\lambda}_a\delta_{ab} + \frac{\lambda_b^m - \lambda_a^m}{m}\Omega_{ab}\right]\mathbf{N}_a \otimes \mathbf{N}_b$$

with an exception for the *natural strain* corresponding to $m = 0$:

(2.58)
$$\dot{\mathbf{G}} = \left[\frac{\dot{\lambda}_a}{\lambda_a}\delta_{ab} + (\log\lambda_b - \log\lambda_a)\Omega_{ab}\right]\mathbf{N}_a \otimes \mathbf{N}_b.$$

An approximation to the above rate may be found by applying the trapezoidal rule to the integral definition of the logarithm as before:

$$\operatorname{Log}\lambda_b - \operatorname{Log}\lambda_a = \int_{\lambda_a}^{\lambda_b} \frac{d\lambda}{\lambda} \simeq \frac{1}{2}(\lambda_b - \lambda_a)\left(\frac{1}{\lambda_b} - \frac{1}{\lambda_a}\right) = \frac{1}{2}\left(\frac{\lambda_b}{\lambda_a} - \frac{\lambda_a}{\lambda_b}\right)$$

to obtain

$$\dot{\mathbf{G}} \simeq \left[\frac{\dot{\lambda}_a}{\lambda_a}\delta_{ab} + \frac{1}{2}\left(\frac{\lambda_b}{\lambda_a} - \frac{\lambda_a}{\lambda_b}\right)\Omega_{ab}\right]\mathbf{N}_a \otimes \mathbf{N}_b \quad (= \mathbf{D}). \tag{2.59}$$

This approximation of the natural strain rate $\dot{\mathbf{G}}$ is readily recognized as the spectral form of the rotated rate (2.50) and not as the Pelzer strain rate, as a similarity in the approximation procedure utilized could mislead: in other words, the rate of the approximation differs from the approximation of the rate.

It follows that the *natural strain* may be regarded as an *approximate primitive* for the unintegrable *rotated rate of deformation*

$$\mathbf{G} \simeq \int_0^t \mathbf{D}\,dt. \tag{2.60}$$

Finally, *a strain rate of the rubber family* (2.34) takes the form

$$\dot{\mathbf{E}} = \left[\frac{p\lambda_a^{p-1} - q\lambda_a^{q-1}}{p-q}\dot{\lambda}_a\delta_{ab} + \frac{\lambda_b^p - \lambda_b^q - \lambda_a^p + \lambda_a^q}{p-q}\Omega_{ab}\right]\mathbf{N}_a \otimes \mathbf{N}_b. \tag{2.61}$$

Most important of all, fully explicit and invertible relationships between different strain rates such as Eq. (2.51) or their inverses (2.52) are readily obtained in spectral component form, e.g. (no summation over the repeated indices)

$$\begin{aligned}
\dot{E}_{aa} &= E'(\lambda_a)\dot{U}_{aa} &&= \lambda_a E'(\lambda_a) D_{aa},\\
\dot{E}_{ab} &= \frac{E(\lambda_b) - E(\lambda_a)}{\lambda_b - \lambda_a}\dot{U}_{ab} &&= \frac{2\lambda_a\lambda_b}{\lambda_b^2 - \lambda_a^2}(E(\lambda_b) - E(\lambda_a))D_{ab} \quad (a \neq b).
\end{aligned} \tag{2.62}$$

Note that the first relationship between the diagonal components $\dot{E}_{aa}$ and $\dot{U}_{aa}$ is a strict duplicate of the uniaxial elementary formula (2.8). The addition of the second relationship between the offdiagonal components $\dot{E}_{ab}$ and $\dot{U}_{ab}$ $(a \neq b)$ is the correct generalization to arbitrary deformations. The offdiagonal relationship is consistent with the diagonal one through the nice properties of symmetry ($\dot{E}_{ab} = \dot{E}_{ba}$), regularity ($\dot{E}_{ab} \to \dot{E}_{aa}$ as $\lambda_b \to \lambda_a$) and consistency ($\dot{E}(1,1) = 1$).

For instance, all the strain rates (2.61) of the rubber family can be expressed in terms of the rotated rate as follows (no summation over the repeated indices):

$$\begin{aligned} \dot{E}_{aa} &= \frac{p\lambda_a^p - q\lambda_a^q}{p-q} D_{aa}, \\ \dot{E}_{ab} &= \frac{2}{p-q}\frac{\lambda_a\lambda_b}{\lambda_a+\lambda_b}\frac{\lambda_b^p - \lambda_b^q - \lambda_a^p + \lambda_a^q}{\lambda_b - \lambda_a} D_{ab} \quad (a \neq b). \end{aligned} \tag{2.63}$$

To close this section note that a tensorial form for the stretch rate $\dot{\mathbf{U}}$ may be formally constructed in the light of its spectral decomposition (2.48) by defining

$$\dot{\lambda}_a \mathbf{N}_a \otimes \mathbf{N}_a = \dot{\lambda}_a \mathbf{Q}\mathbf{N}_a^0 \otimes \mathbf{N}_a^0 \mathbf{Q}^T = \mathbf{Q}\dot{\mathbf{U}}^D\mathbf{Q}^T = \overset{\circ}{\mathbf{U}},$$

and observing that

$$\lambda_b \Omega_{ab} \mathbf{N}_a \otimes \mathbf{N}_b = \Omega_{ab}\mathbf{N}_a \otimes \mathbf{N}_b \lambda_b \mathbf{N}_b \otimes \mathbf{N}_b = \mathbf{\Omega}\mathbf{U},$$

therefore

$$\dot{\mathbf{U}} = \overset{\circ}{\mathbf{U}} + \Omega\mathbf{U} - \mathbf{U}\Omega. \tag{2.64}$$

Accordingly, a similar procedure applied to the generalized strain (2.55) leads to

$$\dot{\mathbf{E}} = \overset{\circ}{\mathbf{E}} + \Omega\mathbf{E} - \mathbf{E}\Omega, \tag{2.65}$$

where

$$\overset{\circ}{\mathbf{E}} = \mathbf{Q}\dot{\mathbf{E}}^D\mathbf{Q}^T = E'(\lambda_a)\dot{\lambda}_a\mathbf{Q}\mathbf{N}_a^0 \otimes \mathbf{N}_a^0\mathbf{Q}^T = E'(\lambda_a)\dot{\lambda}_a\mathbf{N}_a \otimes \mathbf{N}_a.$$

The above formula (2.65) confirms the symmetry of the generalized strain rate taken for granted in Eq. (2.51). A similar relationship between the rotated rate $\mathbf{D}$ and the natural rate $\dot{\mathbf{G}}$ has been established by [34] along this line.

The most significant expressions of the eight members of the rubber family are summarized in Table 2 together with those of their rates.

Table 2. The "rubber" strain family and its rates.

p \ q	0	−1	−2
0	$\mathbf{E}^0 = 0$ $\dot{\mathbf{E}}^0 = 0$	$\mathbf{E}^H = \mathbf{I} - \mathbf{U}^{-1}$ $\dot{\mathbf{E}}^H = \mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}^{-1}$	$\mathbf{E}^K = \frac{1}{2}(\mathbf{I} - \mathbf{C}^{-1})$ $\dot{\mathbf{E}}^K = \frac{1}{2}\mathbf{C}^{-1}\dot{\mathbf{C}}\mathbf{C}^{-1}$
1	$\mathbf{E}^B = \mathbf{U} - \mathbf{I}$ $\dot{\mathbf{E}}^B = \dot{\mathbf{U}}$	$\mathbf{E}^P = \frac{1}{2}(\mathbf{U} - \mathbf{U}^{-1})$ $\dot{\mathbf{E}}^P = \frac{1}{2}(\dot{\mathbf{U}} + \mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}^{-1})$	$\mathbf{E}^W = \frac{1}{3}(\mathbf{U} - \mathbf{U}^{-2})$ $\dot{\mathbf{E}}^W = \frac{1}{3}(\dot{\mathbf{U}} + \mathbf{C}^{-1}\dot{\mathbf{C}}\mathbf{C}^{-1})$
2	$\mathbf{E}^G = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ $\dot{\mathbf{E}}^G = \frac{1}{2}\dot{\mathbf{C}}$	$\mathbf{E}^R = \frac{1}{3}(\mathbf{U}^2 - \mathbf{U}^{-1})$ $\dot{\mathbf{E}}^R = \frac{1}{3}(\dot{\mathbf{C}} + \mathbf{U}^{-1}\dot{\mathbf{U}}\mathbf{U}^{-1})$	$\mathbf{E}^M = \frac{1}{4}(\mathbf{C} - \mathbf{C}^{-1})$ $\dot{\mathbf{E}}^M = \frac{1}{4}(\dot{\mathbf{C}} + \mathbf{C}^{-1}\dot{\mathbf{C}}\mathbf{C}^{-1})$

2.6. Simple extension and shear

Changes in lengths and angles are the two basic modes of deformation which affect the shape of a body. Their study is at the origin of the classical experiments of simple extension and simple shear. The analysis of these two homogeneous deformations (characterized by a constant deformation gradient) represents a necessary test for a strain measure.

A simple extension along the X axis is characterized by (cf. Fig. 1),

$$\begin{aligned} x &= \lambda X, \\ y &= \lambda^{-\nu} Y, \\ z &= \lambda^{-\nu} Z, \end{aligned} \qquad \mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-\nu} & 0 \\ 0 & 0 & \lambda^{-\nu} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-2\nu} & 0 \\ 0 & 0 & \lambda^{-2\nu} \end{bmatrix},$$

where ν is a generalized transverse contraction factor (reminiscent of Poisson's ratio). In a pure extension particularized by $y = Y$ and $z = Z$ (cf., Sect. 2.1) this contraction factor vanishes $\nu = 0$. In an isochoric extension characterized by $J = |\mathbf{F}| = \lambda^{(1-2\nu)} = 1$, it equals one half, $\nu = 1/2$.

The principal strain directions coincide with the rectangular axes. The generalized strain (2.34) becomes

$$\mathbf{E} = \frac{1}{p-q} \begin{bmatrix} \lambda^p - \lambda^q & 0 & 0 \\ 0 & \lambda^{-\nu p} - \lambda^{-\nu q} & 0 \\ 0 & 0 & \lambda^{-\nu p} - \lambda^{-\nu q} \end{bmatrix}. \tag{2.66}$$

It is pointed out that the generalized contraction factor ν corresponds to the usual contraction ratio at infinitesimal strains only

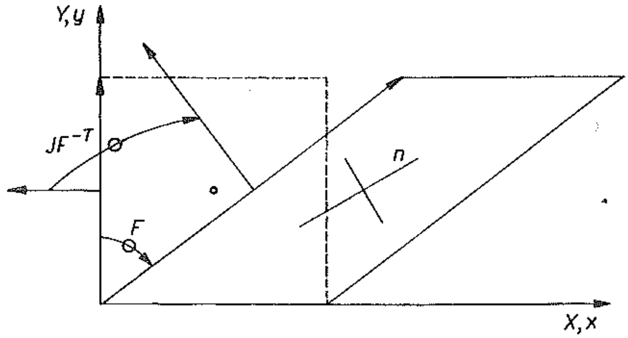
$$\frac{E_{22}}{E_{11}} = \nu - (\nu+1)(p^2 - q^2)\frac{\varepsilon}{2} + \dots .$$

For the Pelzer and the Mooney strains the convergence is quadratic.

A *simple shear* in the $X-Y$ plane is defined by (cf. Fig. 6),

$$\begin{aligned} x &= X + 2\gamma Y, \\ y &= Y, \\ z &= Z, \end{aligned} \qquad \mathbf{F} = \begin{bmatrix} 1 & 2\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 2\gamma & 0 \\ 2\gamma & 1+4\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The slope 2γ is called the amount of shear and the arc $\theta = \text{Arctan } 2\gamma$ is known as the angle of shear.



REMARK. Note that the composition of two shear deformations specified by $\mathbf{F}(\gamma)$ and $\mathbf{F}(\beta)$ is a shear deformation characterized by $\mathbf{F}(\gamma+\beta) = \mathbf{F}(\beta)\mathbf{F}(\gamma)$. In particular, the inverse of a shear deformation $\mathbf{F}(\gamma)$ is given by $\mathbf{F}(-\gamma)$. The reflexion of a shear deformation $\mathbf{F}(\gamma)$ with respect to the bisector corresponds to the transpose $\mathbf{F}(\gamma)^T$.

Simple shear is an isochoric deformation since $J = |\mathbf{F}| = \sqrt{|\mathbf{C}|} = 1$. The principal stretches and their directions are complicated functions of the amount of shear. The principal squared stretches and corresponding unit vectors are

$$\begin{aligned} \lambda_1^2 &= 1 + 2\gamma^2 + 2\gamma\sqrt{1+\gamma^2}\,, & \mathbf{N}_1 &= \frac{\mathbf{i} + \mathbf{j}(\gamma + \sqrt{1+\gamma^2})}{\sqrt{2+\gamma^2+2\gamma\sqrt{1+\gamma^2}}}\,, \\ \lambda_2^2 &= 1 + 2\gamma^2 - 2\gamma\sqrt{1+\gamma^2}\,, & \mathbf{N}_2 &= \frac{\mathbf{i} + \mathbf{j}(\gamma - \sqrt{1+\gamma^2})}{\sqrt{2+\gamma^2+2\gamma\sqrt{1+\gamma^2}}}\,, \\ \lambda_3 &= 1\,, & \mathbf{N}_3 &= \mathbf{k}\,. \end{aligned}$$

The rotation and stretch tensors are

$$\mathbf{R} = \begin{bmatrix} 1/\sqrt{1+\gamma^2} & \gamma/\sqrt{1+\gamma^2} & 0 \\ -\gamma/\sqrt{1+\gamma^2} & 1/\sqrt{1+\gamma^2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} 1/\sqrt{1+\gamma^2} & \gamma/\sqrt{1+\gamma^2} & 0 \\ \gamma/\sqrt{1+\gamma^2} & \dfrac{1+2\gamma^2}{\sqrt{1+\gamma^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As stated, the Green, Karni and Mooney strains, regrouped in $\mathbf{E} = \frac{1+k}{2}\mathbf{E}^G + \frac{1-k}{2}\mathbf{E}^K$ (with $k = 1, -1$ and 0), are the only ones which are easy to compute

$$\mathbf{E}^{(k)} = \begin{bmatrix} (k-1)\gamma^2 & \gamma & 0 \\ \gamma & (k+1)\gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.67}$$

where

$$\mathbf{E}^G = \mathbf{E}(k=1), \quad \mathbf{E}^K = \mathbf{E}(k=-1) \quad \text{and} \quad \mathbf{E}^M = \mathbf{E}(k=0).$$

Therefore the Mooney strain is distinguished by a simultaneous contraction of the $\mathbf{X}$ fibers and elongation of the $\mathbf{Y}$ fibers. Finally, note that $\mathrm{tr}\mathbf{E} = 2k\gamma^2$ vanishes for the Mooney strain, confirming its capacity to characterize an isochoric deformation by the simple constraint $\mathrm{tr}\mathbf{E}^M = 0$ for all practical purposes.

3. Generalized stress

3.1. Elementary definition

The stress enforced in a deformed bar by an external load applied at its extremity (as already shown in Fig. 1) is defined in rudimentary terms as the ratio of the load magnitude(2) q to the original area A of the cross-section of the bar

$$P = \frac{q}{A} \qquad \text{(nominal)}. \tag{3.1}$$

(2) The load q must not be mistaken for the parameter q introduced in the previous section at Eq. (2.7) or Eq. (2.31), in spite of the notation overlap.

In fact, this elementary definition presumes the existence of an internal force, equal to the external load, acting along the whole axis of the bar. By further assuming this force to be evenly distributed over each cross-section(3), the relative notion of *stress ratio* or force density per unit of (undeformed) area naturally occurs as a basic quantity for assessing the strength of the bar, for instance.

This stress is called "*nominal*" in contrast to the "true" stress defined per unit of deformed area. The nominal stress is also called the first Piola-Kirchhoff stress. A brief history of stress, providing the key to most stress names used in this paper, is given in Appendix A. The obvious true stress formula $t = q/a$ is deliberately put aside here, in spite of its classicism, because it corresponds to the spatial form of this measure: the standard Cauchy stress. The derivation of the material form of the true stress usually called the *rotated* stress is postponed until the necessary notions become available. Nevertheless the accepted true stress and the presumed rotated stress provide clear evidence that stresses are not unique in nonlinear mechanics.

As a matter of fact, alternate measures of stress may be constructed at will by scaling both the force q and the area A by means of two appropriate adimensional factors, say H and K, according to the generic formula

$$S = \frac{Hq}{KA} = \frac{H}{K}p. \tag{3.2}$$

These factors may even be taken to vary with the deformation, provided this variation starts from unity and remains monotone elsewhere. The inverse stretch ratio λ^{-1}, for instance, is a legitimate scaling factor for either the force (second Piola-Kirchhoff contravected stress) or the area (Noll rotated stress) variation, since it remains positive.

Paradoxically, the initial guess and subsequent interpretation of the factors H and K are less natural in the present simplistic one-dimensional context than in a fully three-dimensional setting (due to the neglected coupling which exists between longitudinal and transversal deformations, even in a bar). Consequently their systematic study is deferred to later sections. Meanwhile, it is taken for granted that a *generalized stress* S may be defined as a linear function of the nominal stress ratio P with a variable coefficient $V(\lambda)$, function of the stretch ratio λ, satisfying pertinent consistency conditions

$$S = V(\lambda)P, \qquad V(1) = 1, \qquad V(\lambda) > 0. \tag{3.3}$$

(3)This assumption is difficult to satisfy if the bar is of an anisotropic material [35].

The monotonicity condition guarantees the existence of an inverse coefficient $1/V(\lambda)$, and thereby the equivalence between S and P.

The main drawback of a static formulation such as Eq. (3.1) is that it fails to reveal the one-to-one correspondence presumed to exist between the stress produced (here the nominal stress ratio) and a certain strain (by anticipation, the stretch ratio) and *vice versa.* To this end, a less intuitive but more systematic way to introduce the concept of stress is to invoke the principle of virtual *power* or, equivalently, the balance of kinetic energy. In essence, these principles postulate the equality of the external power supplied by the applied load q along a (virtual or real) displacement $(l - L)$ of its point of application with the internal power developed by internal pointwise resistances S, called *stresses*, along the corresponding pointwise deformations E, called *strains*, throughout the volume of the bar, which is naturally taken as the reference volume $V = AL$ in a material description

$$S\dot{E}V = q\dot{l}. \tag{3.4}$$

In the above formula, a dot denotes either a variation or a rate.

REMARK. A spatial description would begin from here by using the stress power developed throughout the deformed volume $v = al$, namely $s\,\overset{\circ}{e}\,v = q\,\overset{\circ}{l}$ where all variables are referred to the stretched abscissa x but $q(l) = q(L)$ and $\overset{\circ}{e}$ denotes a correct objective rate.

This joint approach provides an infallible guide for the consistent definition of *conjugate* stress-strain pairs $(S - E)$ while leaving an infinite number of possibilities (in nonlinear mechanics).

For instance, dividing both sides of Eq. (3.4) by $V = AL$ gives the material *stress power* per unit of reference volume

$$S\dot{E} = \frac{q}{A}\frac{\dot{l}}{L} = P\dot{\lambda}. \tag{3.5}$$

By direct identification of the terms, based on the most natural partition, the nominal stress ratio is found again $(S = q/A = P)$ and *shown* to be conjugate to the stretch ratio $(\dot{E} = \dot{l}/L = \dot{\lambda})$.

REMARK. Note that the adjective "natural" usurps the one of "arbitrary" since nothing except tradition prevents us from reversing the order of the factors A and L and defining a generalized strain as a change of length per unit area $(l - L)/A$, which would lead us to an associated stress per unit length q/L. A dual theory of continuum mechanics based on these original ideas would perhaps shed additional light on the structure of the primal theory blindly accepted by us. Note also the use of the areal strain $(A-a)/A$ (in wire drawing) as reported by Tabor in the Hardness of Metals.

The unique association (3.5) of P with the basic stretch variable λ, added to its natural occurrence in the formulation of the material equation of equilibrium ($dP/dX = 0$) elect the nominal stress to the status of a fundamental stress measure to be used as a reference for other candidate stresses.

More generally, recalling the rate expression of the generalized strain in terms of the stretch rate (2.8), the stress power equivalence relationship (3.5) provides a reciprocal definition of the conjugate generalized stress in terms of the nominal stress

$$
(3.6) \qquad \begin{aligned} \dot{E} &= E'(\lambda)\dot{\lambda}, && \text{Eq. (2.8)} \\ P &= E'(\lambda)S, && \text{generalized.} \end{aligned}
$$

Remark. A more intrinsic definition of a generalized stress could be the postulation of a relationship of the type $S = S(P)$. Invoking the principle of virtual forces or of complementary energy $\dot{S}EV = \dot{q}(l - L)$ would then lead to a conjugate strain implied by $S'(P)E = \lambda - 1$.

In addition to its parallelism with the strain rate relation (2.8), the fact that the nominal stress P is well adapted for stating the static equilibrium equation and that it is difficult to invert the derivative $E'(\lambda)$ in three dimensions are the main reasons for keeping the definition of the generalized stress S in the implicit form (3.6). Another advantage of the stress power approach is to provide a rigorous link between material and spatial measures by equating the corresponding powers

$$
S(X)\dot{E}(X)V = q(L)\dot{l}(L) = q(l)\dot{l}(l) = s(x)\overset{\circ}{e}(x)v \quad \rightarrow \quad S\dot{E} = \frac{v}{V}s\,\overset{\circ}{e},
$$

where $\overset{\circ}{e}$ denotes an objective spatial rate of e.

In particular, since the material and spatial forms of the natural strain rate happen to coincide, $\dot{E} = \overset{\circ}{e} = \dot{l}/l$, the material and spatial forms of the true stress, namely the rotated stress T and the Cauchy stress t, are related by $T = (v/V)t$, in the absence of rotation.

A comparison of the static and dynamic stress definitions (3.3) and (3.6) reveals that, in order for the generalized stress $S = V(\lambda)P$ to be conjugate to the generalized strain $E(\lambda)$, the inverse stress coefficient $V^{-1}(\lambda)$ must be exactly equal to the strain derivative $E'(\lambda)$

$$
(3.7) \qquad \frac{1}{V(\lambda)} = E'(\lambda).
$$

Because the natural logarithm strain $G = \operatorname{Log} \lambda$ plays a pivotal role among candidate strains, it is worthwhile finding its conjugate stress T before the others. Substitution of its derivative $G' = 1/\lambda$ in Eq. (3.6) gives

$$(3.8) \qquad P = \frac{T}{\lambda} \quad \text{or} \quad T = \lambda P, \qquad \text{rotated.}$$

This so-called rotated stress is recognized as the material form of the "true" stress earlier looked for in vain, since

$$T = \lambda P = \frac{l}{L}\frac{q}{A} = \frac{al}{AL}\frac{q}{a} = \frac{v}{V}t,$$

where t is the classical spatial true stress of Cauchy. In the absence of rotation, the rotated stress T coincides with the so-called Kirchhoff stress [13, 7] defined as the Cauchy stress t multiplied by the Jacobian $J = v/V$. A more fundamental difference will appear in the presence of rotation, i.e. $\mathbf{T} = J\mathbf{R}^T\mathbf{t}\mathbf{R}$.

In terms of the rotated stress T, the generalized stress S conjugate to the generalized strain E takes the implicit form:

$$(3.9) \qquad \begin{aligned} \dot{E} &= \lambda E'(\lambda)\dot{G}, && \text{Eq. (2.9)}, \\ T &= \lambda E'(\lambda)S, && \text{generalized.} \end{aligned}$$

Note that by the consistency condition $E'(1) = 1$, all generalized stresses S will coincide with both the nominal stress P and the rotated stress T for sufficiently small deformations

$$S \sim P \sim T \qquad \text{for} \qquad \lambda \sim 1.$$

Applying the general stress definitions (3.6) and (3.9) to the classical strain family (2.2) gives conjugate stresses. See Appendix A for a justification of the following stress names.

$$(3.10) \qquad \begin{array}{lll} \underline{P = E'(\lambda)S} & \underline{T = \lambda E'(\lambda)S} & \underline{(S \text{ name} - E \text{ name})} \\ P = \lambda S^G, & T = \lambda^2 S^G, & (\text{2nd P.K.} - \text{G.Green}), \\ P = 1 S^B, & T = \lambda S^B, & (\text{Biot} - \text{Biot}), \\ P = \dfrac{1}{\lambda} T, & T = T, & (\text{rotated} - \text{natural}), \\ P = \dfrac{1}{\lambda^2} S^H, & T = \dfrac{1}{\lambda} S^H, & (\text{Hill} - \text{Hill}), \\ P = \dfrac{1}{\lambda^3} S^K, & T = \dfrac{1}{\lambda^2} S^K, & (\text{E.Green} - \text{Karni}), \end{array}$$

$$\left(P = \frac{q}{A}, \ \lambda = \frac{l}{L} \right), \ \left(T = \frac{l}{L}\frac{q}{A} = \frac{v}{V}\frac{q}{a} \right).$$

Here also, the Biot stress is distinguished from the nominal stress because their appearing to be identical is a mere coincidence in the absence of rotation in the deformation. The pivotal role played by the rotated stress is quite clear and justifies the special treatment previously given to this measure. However, its conjugacy to the natural strain will be shown to be only approximate in the presence of rotations.

Similarly the stress conjugate to the generic member of the Seth family (2.3) takes the form

$$P = \lambda^{m-1} S^{(m)}, \qquad T = \lambda^m S^{(m)} \qquad \text{(Seth)}. \tag{3.11}$$

Repeating this operation for the *rubber* family (2.5) one obtains

$$\begin{aligned}
P &= \frac{1}{2}\left(1+\frac{1}{\lambda^2}\right) S^P, & T &= \frac{1}{2}\left(\lambda+\frac{1}{\lambda}\right) S^P & &\text{(Pelzer)},\\
P &= \frac{1}{2}\left(1+\frac{1}{\lambda^3}\right) S^M, & T &= \frac{1}{2}\left(\lambda^2+\frac{1}{\lambda^2}\right) S^M & &\text{(Mooney)},\\
P &= \frac{1}{3}\left(1+\frac{2}{\lambda^3}\right) S^W, & T &= \frac{1}{3}\left(\lambda+\frac{2}{\lambda^3}\right) S^W & &\text{(Wall)},\\
P &= \frac{1}{3}\left(2\lambda+\frac{1}{\lambda^2}\right) S^R, & T &= \frac{1}{3}\left(2\lambda^2+\frac{1}{\lambda}\right) S^R & &\text{(Rivlin)}.
\end{aligned} \tag{3.12}$$

All these measures are plotted in Fig. 7 to facilitate their comparison. Parallel to the double definition of the rubber strain family (2.7) (first as an

Table 3. The "rubber" stress family (Eq. (3.10) and (3.12)).

p \ q	0	−1	−2
0	$P = \forall$ $T = \forall$	$P = \frac{1}{\lambda^2} S^H$ $T = \frac{1}{\lambda} S^H$	$P = \frac{1}{\lambda^3} S^K$ $T = \frac{1}{\lambda^2} S^K$
1	$P = 1 S^B$ $T = \lambda S^B$	$P = \frac{1}{2}(1+\frac{1}{\lambda^2}) S^P$ $T = \frac{1}{2}(\lambda+\frac{1}{\lambda}) S^P$	$P = \frac{1}{3}(1+\frac{2}{\lambda^3}) S^W$ $T = \frac{1}{3}(\lambda+\frac{2}{\lambda^2}) S^W$
2	$P = \lambda S^G$ $T = \lambda^2 S^G$	$P = \frac{1}{3}(2\lambda+\frac{1}{\lambda^2}) S^R$ $T = \frac{1}{3}(2\lambda^2+\frac{1}{\lambda}) S^R$	$P = \frac{1}{2}(\lambda+\frac{1}{\lambda^3}) S^M$ $T = \frac{1}{2}(\lambda^2+\frac{1}{\lambda^2}) S^M$

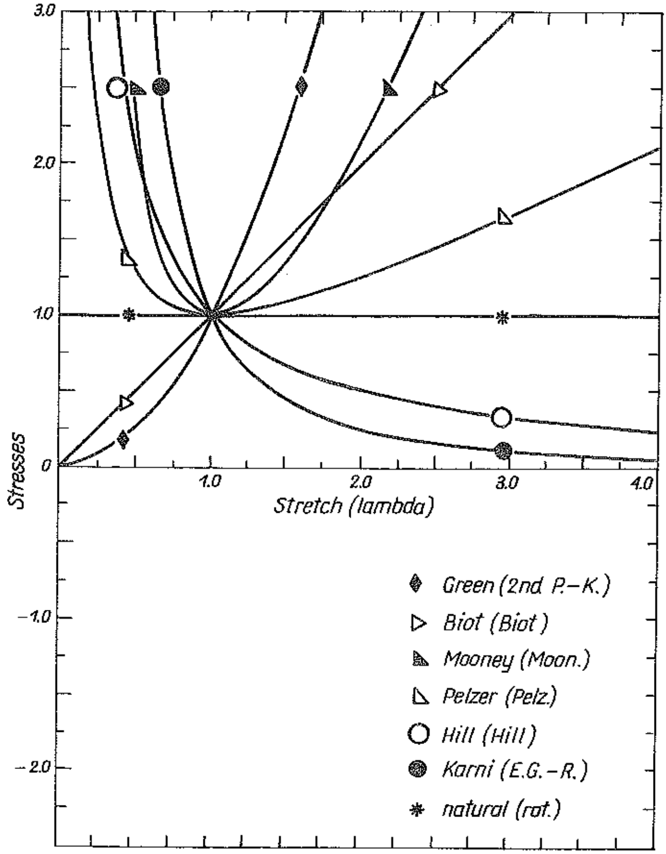


FIG. 7.

intrinsic function of the stretch ratio and then as a weighted average of two basic strains), follows a double definition of the conjugate stress family (from the definition of the derivative (2.9) and from its linearity respectively):

$$
\begin{aligned}
P &= \frac{p\lambda^{p-1} - q\lambda^{q-1}}{p-q} S = \left(\frac{p}{p-q} E^{(p)\prime} + \frac{q}{p-q} E^{(q)\prime}\right) S, \\
T &= \frac{p\lambda^{p} - q\lambda^{q}}{p-q} S = \left(\frac{p\lambda}{p-q} E^{(p)\prime} + \frac{q\lambda}{p-q} E^{(q)\prime}\right) S.
\end{aligned}
\tag{3.13}
$$

The "symmetry" of the Pelzer and Mooney strains carries over to their conjugate stresses since for $q = -p$, Eq. $(3.13)_2$ for instance reduces to $T = \frac{1}{2}(\lambda^p + \lambda^{-p})S$. The expressions of the nominal stress P and the rotated stress T in terms of the conjugate stresses of the rubber family are collected in Table 3 for completeness.

3.2. Basic statics of stress

Consider a body in static equilibrium under the single action of external distributed contact loads $d\mathbf{q}(\mathbf{x})$ applied to its deformed surface a, but translated back abstractly onto its undeformed surface A according to $d\mathbf{q}(\mathbf{X}) = d\mathbf{q}[\mathbf{x}(\mathbf{X})]$, for the sake of the material description. By the definition of the static equilibrium, both the resultant of all these elementary forces and their moment about (say) the origin must vanish

$$\begin{aligned} \int_A d\mathbf{q} &= 0, \\ \int_A \mathbf{x} \times d\mathbf{q} &= 0. \end{aligned} \tag{3.14}$$

Through the action of these external loads, internal forces develop throughout the deformed volume v of the body and they may again be transferred back into the undeformed volume V by an abstract translation and still be denoted by the same symbol $d\mathbf{q}(\mathbf{X})$. It is further postulated that the internal forces acting over any imagined surface enclosing an arbitrary part of the body satisfy the same principles of equilibrium (3.14) as the external loads.

The (external or internal) force per unit of undeformed area (of the external or any internal surface of the body) is called the *nominal stress vector* or traction and is implicitly defined by the differential relationship

$$d\mathbf{q} = \mathbf{P}_N \, dA. \tag{3.15}$$

REMARK. A spatial formulation would begin from here by posing $d\mathbf{q} = \mathbf{t}_n da$, where $\mathbf{t}_n$ is called the true stress vector.

As suggested by the subscript N, the stress vector $\mathbf{P}_N$ is traditionally postulated as depending not only on the location X, but also on the orientation N of the surface element on which it is assumed to act $\mathbf{P}_N = \mathbf{P}_N[\mathbf{X}, \mathbf{N}(\mathbf{X})]$. This selective dependence of the internal stress vector on the surface orientation (but not on its curvature for instance) constitutes the fundamental hypothesis of stress analysis. It was first advanced by Cauchy in its spatial form and may be compared with the restriction of the kinematic description to local deformations described by the single deformation gradient (2.11).

REMARK. In a remarkable attempt to axiomize continuum mechanics, Noll (1959) has succeeded in deriving this already basic postulate from even more elementary facts. Using a more standard technique, Germain

(1973) derives this basic postulate from the principle of virtual power, which amounts to postulating the existence of the stress tensor itself, so that the gain is arguable.

Application of the principle of action and reaction (which is simply a corollary of the balance of forces (3.14)) to the stress vectors acting on the opposite faces of any imagined diaphragm stretched across the reference volume, further requires this dependence to be an *odd* function $\mathbf{P}_N(-\mathbf{N}) = -\mathbf{P}_N(\mathbf{N})$. Finally Cauchy's famous theorem, based on the balance of the elementary forces acting on the facets of a shrinking tetrahedron and adapted to the material description, asserts that a *linear* dependence is sufficient to express local equilibrium. By definition, the linear operator mapping the material surface element unit normal $\mathbf{N}$ into the actual stress vector $\mathbf{P}_N$ applied to it is a (second order) tensor called the *nominal stress* and denoted by $\mathbf{P}$

(3.16) $$\mathbf{P}_N = \mathbf{P}\,\mathbf{N}, \qquad \text{(nominal)}.$$

This mixed (spatial-material) tensor is the fundamental measure of stress adopted in this study to construct alternate options. The nominal stress is also called the first Piola-Kirchhoff stress and sometimes defined as its transpose. A typical coefficient P_{iI} of this tensor decomposed on the mixed rectangular basis $\mathbf{e}_i \otimes \mathbf{E}_I$, represents the i-th component of the force $\mathbf{P}_N$ presently acting on a surface element $\mathbf{N}dA$, initially normal to the I-th coordinate axis direction.

Using the stress principle (3.16) folowed by the divergence theorem, the statements of static equilibrium (3.14) take the alternative forms

(3.17) $$\begin{aligned} \int_A \mathbf{PN}\, dA &= \int_V \operatorname{Div} \mathbf{P}\, dV = 0, \\ \int_A \mathbf{x} \times \mathbf{PN}\, dA &= \int_A \operatorname{Div}(\mathbf{x} \times \mathbf{P})\, dV = 0, \end{aligned}$$

where $\mathbf{x} \times \mathbf{P}$ is the mixed tensor given by $(\mathbf{x} \times \mathbf{P})_{iI} = \varepsilon_{ijk} x_j P_{kI}$ with ε_{ijk} denoting the usual permutation symbol [24].

Using the identity

$$[\operatorname{Div}(\mathbf{x} \times \mathbf{P})]_i = \frac{\partial}{\partial X_I}(\varepsilon_{ijk} x_j P_{kI}) = (\mathbf{x} \times \operatorname{Div} \mathbf{P})_i + \varepsilon_{ijk} F_{jI} P_{kI},$$

the local forms of these principles reduce to

(3.18) $$\begin{aligned} \operatorname{Div} \mathbf{P} &= 0, \\ \mathbf{FP}^T &= \mathbf{PF}^T. \end{aligned}$$

The balance of moments confirms that the nominal stress tensor is not symmetric, a fact inherent in its nature as a mixed tensor.

3.3. Generalized stress

The lack of symmetry of the nominal stress **P** related to its mixed nature is a sufficient incentive to look for alternative stress measures.

The most common (and perhaps the most fundamental) stress measure is due to *Cauchy* and herein denoted by **t**. It is also called the *true* stress because, by definition, it transforms a spatial element of deformed surface into the current elementary force acting on it, in accordance with $d\mathbf{q} = \mathbf{t}_n da$. It may therefore be related to the nominal stress (3.16) by means of the surface transformation (2.12) by the formula

$$\text{(3.19)} \qquad \mathbf{t} = \frac{1}{J}\mathbf{P}\mathbf{F}^T \qquad (= \mathbf{t}^T), \qquad \text{(Cauchy)}.$$

In spite of its symmetry (which results from the balance of moments (3.18), this classical stress measure is deliberately avoided in this study (except as a standard of reference) because of its spatial nature. Its best material substitute is the *rotated* stress **T** apparently due to Noll, defined by

$$\text{(3.20)} \qquad \mathbf{T} = \mathbf{R}^T J\mathbf{t}\mathbf{R} = \mathbf{R}^T\mathbf{P}\mathbf{U} \qquad (= \mathbf{T}^T), \qquad \text{(rotated)}.$$

It is thus the true Cauchy stress scaled by the Jacobian and rotated to a material status, hence its name. It transforms the contra-rotated deformed surface element into the contra-rotated actual force. It may also be interpreted as the "stretched" stress, meaning the stress naturally associated with the "stretched (but unrotated) reference configuration", implicitly disclosed by the polar decomposition of the line and surface transformations (2.23). The product of the true Cauchy stress **t** by the Jacobian J is sometimes called the Kirchhoff stress [13].

Besides the rotated stress, several other *material* stresses have been proposed in the literature, e.g. [36, 6, 34, 13, 58], among which the most classical ones are (See Appendix A for additional comments concerning the terminology adopted here):

(3.21)

$$
\begin{array}{rclrcll}
\mathbf{S}^G & = & \mathbf{F}^{-1}\mathbf{P} = \mathbf{U}^{-1}\mathbf{T}\mathbf{U}^{-1}, & (\mathbf{S}^G & = & \mathbf{S}^{GT}) & \text{(2nd P.-K.)},\\
\mathbf{S}_1^B & = & \mathbf{R}^T\mathbf{P} = \mathbf{T}\mathbf{U}^{-1}, & (\mathbf{S}_1^B\mathbf{U} & = & \mathbf{U}\mathbf{S}_1^{BT}) & \text{(Biot)},\\
\mathbf{T} & = & \mathbf{R}^T\mathbf{P}\mathbf{U}, & (\mathbf{T} & = & \mathbf{T}^T) & \text{(rotated)},\\
\mathbf{S}_1^H & = & \mathbf{F}^T\mathbf{P}\mathbf{U} = \mathbf{U}\mathbf{T}, & (\mathbf{U}^{-1}\mathbf{S}_1^H & = & \mathbf{S}_1^{HT}\mathbf{U}^{-1}) & \text{(Hill)},\\
\mathbf{S}^K & = & \mathbf{F}^T\mathbf{P}\mathbf{C} = \mathbf{U}\mathbf{T}\mathbf{U}, & (\mathbf{S}^K & = & \mathbf{S}^{KT}), & \text{(E.G.-R.)},\\
(d\mathbf{q} & = & \mathbf{P}\mathbf{N}dA), & (\mathbf{P}\mathbf{F}^T & = & \mathbf{F}\mathbf{P}^T). &
\end{array}
$$

The unnatural superscript and subscript identification will reveal its significance later. The equation of balance of moments is given in parentheses to enhance the symmetry of certain stresses and the lack of symmetry of the others. The interpretation of these stresses in terms of surface and force elements is postponed until a general treatment becomes available.

The above classical stresses may be conveniently collected into a one-parameter family analogous to the Seth strain family, called the *"classical"* family:

(3.22)

$$
\begin{array}{ll}
\mathbf{S}^{(m)} = \mathbf{U}^{-\frac{m}{2}}\mathbf{R}^T\mathbf{P}\mathbf{U}\mathbf{U}^{-\frac{m}{2}} = \mathbf{U}^{-\frac{m}{2}}\mathbf{T}\mathbf{U}^{-\frac{m}{2}} & \text{if } m = 2, 0, -2,\\
\mathbf{S}^{(m)} = \mathbf{U}^{-\frac{m-1}{2}}\mathbf{R}^T\mathbf{P}\mathbf{U}\mathbf{U}^{-\frac{m+1}{2}} = \mathbf{U}^{-\frac{m-1}{2}}\mathbf{T}\mathbf{U}^{-\frac{m+1}{2}} & \text{if } m = 1, -1,
\end{array}
$$

where the factor 2 is introduced simply for later consistency. Explicit inverses are available for all m.

The pivotal role played by the rotated stress, already acknowledged by the special symbol $\mathbf{T}$, is ratified by the central value $m = 0$. The cases $m = 2$ and $m = -2$ are recognized as the classical second Piola-Kirchhoff stress $\mathbf{S}^G$ and the *E.* Green-Rivlin stress $\mathbf{S}^K$, respectively. The Green-Rivlin stress is often called the *convected* stress upon inspection of its relationship to the spatial true stress, $\mathbf{S}^K = \mathbf{F}^T(J\mathbf{t})\mathbf{F}$, disclosing a transport operated by the deformation gradient. Conversely, the second Piola-Kirchhoff stress may be interpreted as the "contravected" stress since $\mathbf{S}^G = \mathbf{F}^{-1}(J\mathbf{t})\mathbf{F}^T$. It is thus the true stress pulled back to the reference configuration. The cases $m = 1$ and $m = -1$ are identified with the less known Biot stress $\mathbf{S}_1^B$ and Hill stress $\mathbf{S}_1^H$, respectively, (the latter appelation is coined herein). These last two stresses are not symmetric as acknowledged by the subscript 1. This lack of symmetry represents a serious impediment to their practical use. However, by analogy with the stretch formula $\mathbf{U} = \mathbf{R}^T\mathbf{F}$, the Biot stress $\mathbf{S}_1^B = \mathbf{R}^T\mathbf{P}$

is expected to play a similar role in statics to that of stretch in geometry, hence its importance. It is the closest objective substitute to the nominal stress P.

The presumption of the existence of other stresses, as a counterpart to the rubber strains, motivates the search for a more general definition of stress.

A *generalized* stress measure S may be constructed by altering both the force vector $d\mathbf{q}$ and the surface vector $\mathbf{N}dA$ (entering the basic definition of the nominal stress $d\mathbf{q} = \mathbf{P}\mathbf{N}dA$) by means of two adequate transformations, denoted(4) by H and K, respectively, according to [36, 10]:

$$\mathbf{H}d\mathbf{q} = \mathbf{S}\mathbf{K}\mathbf{N}dA\,. \tag{3.23}$$

It is understood that (at least one of) the two transformations H and K

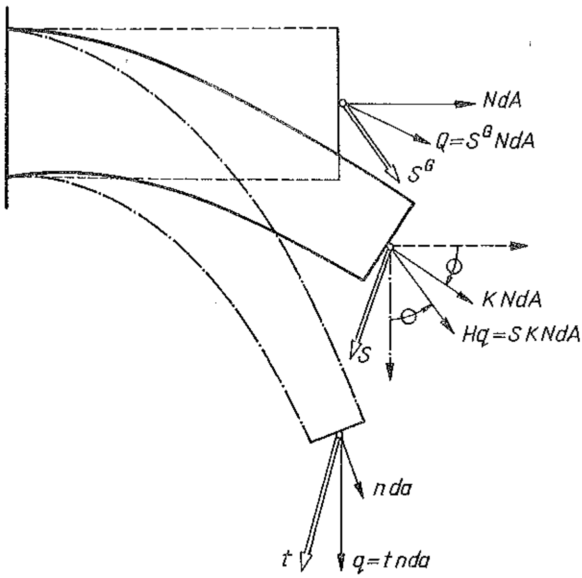


FIG. 8.

must depend on the deformation (say on the deformation gradient F) for the above definition not to be trivial (since it would otherwise reduce to a sterile fixed change of bases). The force and surface transformations H and K are illustrated in Fig. 8. In fact, the stress enforced in a body may

(4)By analogy with Eq. (2.12), the surface transformation could be written as an adjoint $|\mathbf{K}|\mathbf{K}^{-T}$ in order to prevent an awkward definition of the corresponding line transformation.

be interpreted as a linear transformation from the vector space of oriented surfaces into the vector space of forces. From this standpoint, the nominal stress **P** and the generalized stress **S** are just two different representations of that same linear stress operator corresponding to two different choices of pairs of bases for these spaces. The equivalence transformation (3.24) $\mathbf{S} = \mathbf{HPK}^{-1}$, provides the missing link between the two representations **P** and **S**, given the matrices of change of bases **H** and **K**, whether they are fixed or variable.

Relation between the resulting generalized stress **S** and the reference nominal stress **P** is easily established, by identification, to be

$$\text{(3.24)} \qquad \mathbf{P} = [\mathbf{H}(\mathbf{F})]^{-1}\mathbf{S}\mathbf{K}(\mathbf{F}) \quad \text{or} \quad \mathbf{S} = \mathbf{H}(\mathbf{F})\mathbf{P}[\mathbf{K}(\mathbf{F})]^{-1} .$$

It is clear that both the force and surface transformations **H** and **K** must be *regular* and *adimensional* in order for the above definition to make sense from a mathematical and a physical standpoint, respectively. In terms of linear algebra, a linear transformation such as Eq. (3.24), where both the factors **H** and **K** possess an inverse, is called an *equivalence* transformation. Accordingly, the generalized stress **S** is said to be equivalent to the nominal stress **P**. The term "equivalence" implies the invariance of a certain intrinsic "valence" during the transformation. Indeed it can be shown [37] that the two related stresses are equivalent if and only if they have the same *rank*, that is, if and only if they both possess three independent principal stress directions (which seems a sensible requirement). The adimensionality of the factors **H** and **K** is dictated by the imperative necessity to generate a tensor **S** with the correct dimension of stress ($ML^{-1}T^{-2}$) in view of the linearity of the transformation in terms of **P**.

Even with these stipulations, the generalized stress definition (3.24) remains too vague to be fruitful, a fact which calls for additional restrictions.

A basic aim is to arrive at a *material* measure of stress, $\mathbf{S}(\mathbf{P}, \mathbf{F})$ which is *objective*, i.e. one which will produce a (frame-indifferent) material force when acting on a (frame-indifferent) material surface, in order to keep our marked preference for the material description. A quick look at definition (3.23) or (3.24) indicates that the force transformation **H** must be a mixed material-spatial tensor, whereas the surface transformation **K** must be a pure material tensor to achieve this aim.

This conjecture may be proved as follows. Given the mixed spatial-material nature of the deformation gradient and the nominal stress, and the material nature of the generalized stress, the principle of objectivity imposes the specific restriction:

$$\mathbf{S}(\mathbf{P},\mathbf{F}) = \mathbf{S}(\mathbf{AP},\mathbf{AF}), \qquad \forall \mathbf{A}, \quad \mathbf{AA}^T = \mathbf{I}, \qquad \mathbf{A}^T\mathbf{A} = \mathbf{i}\,,$$

for any mixed material-spatial rotation $\mathbf{A}$.

In particular, the above identity must be true for the transpose rotation $\mathbf{R}^T$ occurring in the polar decomposition of the deformation gradient. Therefore, the generalized stress necessarily takes the form:

$$\text{(3.25)} \quad \mathbf{S} = \mathbf{S}(\mathbf{R}^T\mathbf{P},\mathbf{U}) = \mathbf{V}(\mathbf{U})\,\mathbf{R}^T\mathbf{P}\,[\mathbf{K}(\mathbf{U})]^{-1} \qquad \text{(objectivity)}.$$

In other words, it is shown that the force transformation is a mixed material-spatial tensor, which allows a transposed polar decomposition, whereas the surface transformation is a pure material stretch function:

$$\text{(3.26)} \qquad \begin{aligned} \mathbf{H}(\mathbf{F}) &= \mathbf{V}(\mathbf{U})\mathbf{R}^T, \\ \mathbf{K}(\mathbf{F}) &= \mathbf{K}(\mathbf{U}). \end{aligned}$$

The material stress $\mathbf{R}^T\mathbf{P}$ naturally disclosed in the process is recognized as the (asymmetric) Biot stress $\mathbf{S}_1^B$.

Another important aim is to obtain an objective measure of stress which is *isotropic* i.e. one which will use the same force and surface scaling factors in all material directions. It is emphasized that isotropy of the stress measure does not mean isotropy of the material of course! It simply guarantees that in case of an isotropic material the generalized stress will not show an anisotropic response. Added to the objective formula (3.25), this hypothesis of isotropy imposes the cumulated restriction:

$$\mathbf{AS}(\mathbf{R}^T\mathbf{P},\,\mathbf{U})\mathbf{A}^T = \mathbf{S}(\mathbf{AR}^T\mathbf{PA}^T,\,\mathbf{AUA}^T), \quad \forall \mathbf{A}, \quad \mathbf{AA}^T = \mathbf{A}^T\mathbf{A} = \mathbf{I},$$

for any material rotation $\mathbf{A}$. Material isotropy alone requires $\mathbf{AS}(\mathbf{P},\mathbf{F})\mathbf{A}^T = \mathbf{S}(\mathbf{PA}^T,\mathbf{FA}^T)$, $\forall \mathbf{A}$. When added to spatial objectivity, it produces the indicated restriction. Given the linearity of the objective stress function (3.25) in terms of the Biot stress (which guarantees isotropy with respect to that variable) the material force and surface transformations taken separately must be isotropic:

$$\text{(3.27)} \qquad \begin{aligned} \mathbf{AV}(\mathbf{U})\mathbf{A}^T &= \mathbf{V}(\mathbf{AUA}^T), \\ \mathbf{AK}(\mathbf{U})\mathbf{A}^T &= \mathbf{K}(\mathbf{AUA}^T) \qquad \text{(isotropy)}. \end{aligned}$$

The isotropy of $\mathbf{V}(\mathbf{U})$ and $\mathbf{K}(\mathbf{U})$ implies their symmetry $\mathbf{V} = \mathbf{V}^T$ and $\mathbf{K} = \mathbf{K}^T$ as well as their ability to commute $\mathbf{VK} = \mathbf{KV}$.

A final aim is to ensure that all stress measures will coincide for evanescent deformations, which means the following *consistency* conditions are required:

$$\mathbf{V}(\mathbf{I}) = \mathbf{K}(\mathbf{I}) = \mathbf{I} \qquad \text{(consistency)}. \tag{3.28}$$

To complete this series of restrictions, it is useful to state the balance of moments equation $\mathbf{S}_1^B\mathbf{U} = \mathbf{U}\mathbf{S}_1^{BT}$ in terms of the generalized stress $\mathbf{S}$:

$$V^{-1}\mathbf{SKU} = \mathbf{UKS}^T\mathbf{V}^{-1} \qquad \text{(balance)}. \tag{3.29}$$

Because the stretch tensor $\mathbf{U}$ is contiguous to the surface transformation $\mathbf{K}(\mathbf{U})$, they should be merged into a single operator $[\mathbf{W}(\mathbf{U})]^{-1}$ (best written as an inverse for convenience) in order to simplify the above equation:

$$[\mathbf{W}(\mathbf{U})]^{-1} = \mathbf{K}(\mathbf{U})\mathbf{U}. \tag{3.30}$$

Of course the modified inverse surface transformation $\mathbf{W}(\mathbf{U})$ inherits the properties of isotropy, symmetry, commutativity (with $\mathbf{V}(\mathbf{U})$), and consistency of its originator $\mathbf{K}(\mathbf{U})$.

Substitution of the above modification into the generalized stress formula (3.25) produces the alternative form:

$$\mathbf{S} = \mathbf{VUR}^T\mathbf{PUW}(\mathbf{U}) = \mathbf{V}(\mathbf{U})\mathbf{TW}(U), \tag{3.31}$$

aiming at a standard definition of the corresponding line transformation.

The pivot stress obtained for $\mathbf{V}(\mathbf{U}) = \mathbf{W}(\mathbf{U}) = \mathbf{I}$ is recognized as the rotated stress $\mathbf{T}$ defined by Eq. (3.20). In other words the Biot stress $\mathbf{S}_1^B$ is traded for the rotated stress $\mathbf{T}$ in the process. Accordingly, the *reference* surface transformation $\mathbf{K}(\mathbf{U})$ is replaced by the *stretched* surface transformation $[\mathbf{W}(\mathbf{U})]^{-1}$. As predicted, the balance of moments simplifies to

$$\mathbf{V}^{-1}\mathbf{SW}^{-1} = \mathbf{W}^{-1}\mathbf{S}^T\mathbf{V}^{-1} \qquad \text{(balance)}. \tag{3.32}$$

To recapitulate, a *generalized* stress (not necessarily symmetric) may be defined as an *equivalence* transformation of the (objective, symmetric) *rotated* stress $\mathbf{T}$, resulting from its premultiplication by a (regular) *force* transformation $\mathbf{V}(\mathbf{U})$ and its postmultiplication by a (regular) inverse *surface* transformation $\mathbf{W}(\mathbf{U})$. In addition, both transformations are assumed to be *isotropic* functions of the (objective, symmetric, positive-definite) *stretch* tensor U (which implies that V and W are also symmetric and that they commute). Finally these two transformations are taken to reduce to the identity in the reference configuration. More specifically, a generalized stress may be defined as

(3.33)

$$\begin{aligned}
&\mathbf{S} = \mathbf{V}(\mathbf{U})\mathbf{T}\mathbf{W}(\mathbf{U}) && && \text{(objectivity);}\\
&\mathbf{A}\mathbf{V}(\mathbf{U})\mathbf{A}^T = \mathbf{V}(\mathbf{A}\mathbf{U}\mathbf{A}^T), \quad (= \mathbf{V}^T), && (\mathbf{V}\mathbf{W} = \mathbf{W}\mathbf{V}) && \text{(isotropy);}\\
&\mathbf{A}\mathbf{W}(\mathbf{U})\mathbf{A}^T = \mathbf{W}(\mathbf{A}\mathbf{U}\mathbf{A}^T), \quad (= \mathbf{W}^T), && && \\
&\mathbf{V}(\mathbf{I}) = \mathbf{W}(\mathbf{I}) = \mathbf{I}, && && \text{(consistency);}\\
&\mathbf{T} = [\mathbf{V}(\mathbf{U})]^{-1}\mathbf{S}[\mathbf{W}(\mathbf{U})]^{-1}, \quad (= \mathbf{T}^T), && && \text{(regularity),}
\end{aligned}$$

where $\mathbf{A}$ is an arbitrary material rotation $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} = \mathbf{I}$ and relevant symmetries are indicated in parentheses.

The connection between the above definition (3.33) and the original definition (3.25), in terms of the nominal stress P and the deformation gradient F, is repeated here for completeness:

$$\begin{aligned}
\mathbf{S} &= \mathbf{H}(\mathbf{F})\mathbf{P}[\mathbf{K}(\mathbf{F})]^{-1},\\
\mathbf{H}(\mathbf{F}) &= \mathbf{V}(\mathbf{U})\mathbf{R}^T,\\
[\mathbf{K}(\mathbf{F})]^{-1} &= \mathbf{U}\mathbf{W}(\mathbf{U}).
\end{aligned} \tag{3.34}$$

Interesting categories of generalized stresses may be obtained by imposing additional optional constraints between the force and the surface transformations, in correlation with special classes of equivalence transformations found in linear algebra (see Fig. 9)

$$\begin{aligned}
\mathbf{V}(\mathbf{U}) &= \mathbf{W}(\mathbf{U}) && \text{(congruence),}\\
\mathbf{V}(\mathbf{U}) &= \mathbf{W}(\mathbf{U})^{-1} && \text{(similarity),}\\
\mathbf{V}(\mathbf{U}) &= \mathbf{I} \quad \text{or} \quad \mathbf{W}(\mathbf{U}) = \mathbf{I} && \text{(left or right identity).}
\end{aligned} \tag{3.35}$$

Congruences are the most important because they preserve *symmetry*, meaning that they produce symmetric stresses $\mathbf{S} = \mathbf{S}^T$. Hereby they guarantee not only the automatic satisfaction of the balance of moments (3.32) but also the reality of the principal stresses and the orthogonality of the principal directions, which greatly simplify the formulation and solution of problems.

REMARK. Note that the true stress $J\mathbf{t}$ (Kirchhoff) and the rotated stress $\mathbf{T} = \mathbf{R}^T J\mathbf{t}\mathbf{R}$ have the same principal stresses, whereas their principal directions differ by R in orientation. It follows by transitivity of equivalence

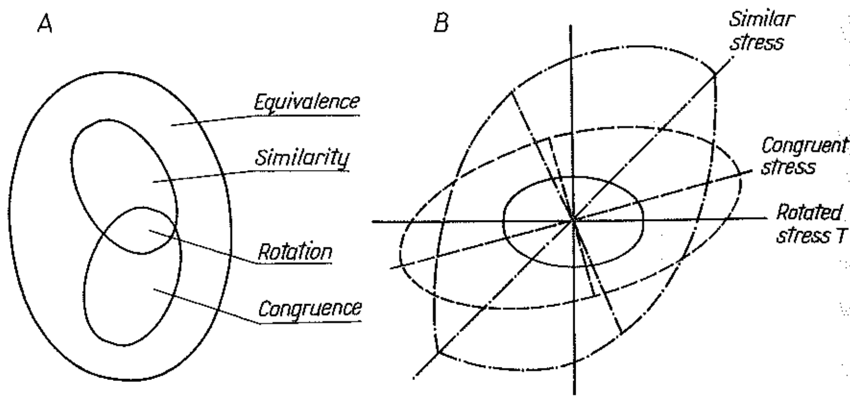


FIG. 9.

relations that stresses congruent to the rotated stress are also congruent to the Cauchy stress.

Moreover, it can be shown [37] that two symmetric stresses are congruent if and only if they have the same rank and the same *signature*, which seems a sensible requirement. The signature of a symmetric stress tensor is the number of positive principal stresses minus the number of negative ones. *Similarities* preserve the principal stresses (which adds nothing), but destroy the orthogonality of the principal directions (which is a severe loss). Therefore their interest is limited. *Half identities*, also called right and left equivalences, are awkward because they preserve nothing. In different terms, two stresses are congruent if forces are transformed as *lines* (within a Jacobian). On the contrary, two stresses are similar if forces are treated as *surfaces*. The former association is more attractive than the latter, both in principle and in practice.

Incidentally, note that the regularity of $\mathbf{V}$ and $\mathbf{W}$ may be favorably replaced by their positive-definiteness to preserve the sign and order of the principal stresses and directions.

Among the classical stresses introduced in Eq. (3.21), it is clear that the second Piola-Kirchhoff stress $\mathbf{S}^G = \mathbf{U}^{-1}\mathbf{T}\mathbf{U}^{-1}$ and the E.Green-Rivlin stress $\mathbf{S}^K = \mathbf{UTU}$ are congruent to the rotated stress $\mathbf{T}$. On the contrary, the Biot stress $\mathbf{S}_1^B = \mathbf{ITU}^{-1}$ and the Hill stress $\mathbf{S}_1^H = \mathbf{UTI}$, obtained by left and right identities, are barely equivalent to the rotated stress.

(Note however that the Hill stress is congruent to the Biot stress, $\mathbf{S}_1^H = \mathbf{U}\mathbf{S}_1^B\mathbf{U}$).

Although, in general, non-congruent stresses are not symmetric, two stresses $\mathbf{S}_1$ and $\mathbf{S}_2$ obtained by exchanging the force and the surface transformations $\mathbf{V}$ and $\mathbf{W}$ are observed to be transposed into each other (due to the symmetry of $\mathbf{V}$ and $\mathbf{W}$),

$$\mathbf{S}_1 = \mathbf{WTV} = (\mathbf{VTW})^T = \mathbf{S}_2^T.$$

REMARK. An important exception occurs in bodies made of isotropic materials. In such media, the stretch and stress tensor $\mathbf{U}$ and $\mathbf{T}$ have the same principal directions and therefore they commute. Since $\mathbf{V}$ and $\mathbf{W}$ are isotropic functions of $\mathbf{U}$ they also commute with $\mathbf{T}$. It follows that $\mathbf{S} = \mathbf{VTW} = \mathbf{VWT} = \mathbf{WVT} = \mathbf{WTV} = \mathbf{S}^T$ is symmetric even for $\mathbf{V} \neq \mathbf{W}$.

Moreover, two such stresses are found to be similar to each other (since $\mathbf{V}$ and $\mathbf{W}$ commute),

$$\mathbf{S}_1 = \mathbf{WTV} = (\mathbf{WV}^{-1})(\mathbf{VTW})(\mathbf{W}^{-1}\mathbf{V}) = (\mathbf{VW}^{-1})^{-1}\mathbf{S}_2(\mathbf{VW}^{-1}).$$

Combining these two observations shows that a generalized stress is similar to its transpose and the similarity transformation which links them is simply another expression of the balance of moments (3.32):

$$\mathbf{S}^T = (\mathbf{VW}^{-1})^{-1}\mathbf{S}(\mathbf{VW}^{-1}). \tag{3.36}$$

Of course, this similarity reduces to an identity in the case of congruent stresses.

The above similarity between one generalized stress and its transpose suggests a *symmetrization* should be performed in order to restore symmetry and unicity:

$$\bar{\mathbf{S}} = \frac{1}{2}(\mathbf{S} + \mathbf{S}^T) = \frac{1}{2}(\mathbf{VTW} + \mathbf{WTV}) \qquad \text{(symmetrized)}. \tag{3.37}$$

For instance, following [6, 38, 39, 40], the Biot stress and the Hill stress may be symmetrized into

$$\begin{aligned} \bar{\mathbf{S}}^B &= \frac{1}{2}(\mathbf{S}_1^B + \mathbf{S}_2^B) = \frac{1}{2}(\mathbf{TU}^{-1} + \mathbf{U}^{-1}\mathbf{T}) \qquad \text{(Jaumann)}, \\ \bar{\mathbf{S}}^H &= \frac{1}{2}(\mathbf{S}_1^H + \mathbf{S}_2^H) = \frac{1}{2}(\mathbf{UT} + \mathbf{TU}). \end{aligned} \tag{3.38}$$

However, this operation of adding one asymmetric stress tensor (better considered as a linear operator transforming a surface vector into a force vector for the present purpose) to its transpose (relating covectors) does not seem legitimate[5] (in the same way as the addition of the deformation gradient to its transpose in order to form the Cauchy strain Eq. (2.25) was limited to infinitesimal deformations). As a matter of fact, symmetrized stresses such as Eq. (3.37) lose the quality of their asymmetric originators of being invertible. They thus lose their interpretation in terms of a surface and a force transformation. In other words, they lose their equivalence with the rotated stress, which is the essential condition for their existence[6]. Finally, their artificial symmetry is, of course, not equivalent to the balance of moments, which must in any case be enforced separately. For all these reasons, such symmetrizations must be performed with extreme caution.

REMARK. A more acceptable procedure for arriving at symmetric stresses similar to their asymmetric originators may be $\mathbf{S} = \sqrt{\mathbf{UW}}\mathbf{T}\sqrt{\mathbf{WU}}$. For instance a stress similar to the Biot stress is $\mathbf{S}_0^B = \sqrt{\mathbf{U}^{-1}}\mathbf{T}\sqrt{\mathbf{U}^{-1}} = \sqrt{\mathbf{U}}\mathbf{F}^{-1}\mathbf{P}\sqrt{\mathbf{U}}$. These symmetric stresses would automatically satisfy the balance of moments. However, their utilization would remain complicated due to the presence of square roots. Note by the way that $\mathbf{P}^T\mathbf{P} = \mathbf{S}_2^B\mathbf{S}_1^B$. Note also that the symmetrized Hill stress remains congruent to the symmetrized Biot stress $\mathbf{S}^H = \mathbf{U}\mathbf{S}^B\mathbf{U}$.

Now, by analogy with the generalized strain isotropic function expansion (2.32), the force and the surface isotropic functions may be represented by

$$\begin{aligned} \mathbf{V}(\mathbf{U}) &= v_-\mathbf{U}^{-1} + v_0\mathbf{I} + v_+\mathbf{U}, \\ \mathbf{W}(\mathbf{U}) &= w_-\mathbf{U}^{-1} + w_0\mathbf{I} + w_+\mathbf{U}, \end{aligned} \tag{3.39}$$

where the coefficients v and w are (symmetric) functions of the principal stretches λ as before, and powers are restricted to $(p = 1, q = -1)$ by reference to the classical stress measures (3.21). Pursuing both comparisons suggests that investigations should be restricted to constant coefficient expansions, satisfying the inequalities $0 \leq v_- \leq 1$, etc. for positive-definiteness. Finally, the consistency conditions further restrict the choices to *convex* combinations characterized by

$$\begin{aligned} v_- + v_0 + v_+ &= 1, \qquad 0 \leq v_-, \quad v_0, \quad v_+ \leq 1, \\ w_- + w_0 + w_+ &= 1, \qquad 0 \leq w_-, \quad w_0, \quad w_+ \leq 1. \end{aligned} \tag{3.40}$$

[5] The problem consists in adding two tensors with different principal directions, better illustrated by the formula $\bar{\mathbf{S}} = [\mathbf{S} + (\mathbf{VW}^{-1})^{-1}\mathbf{S}(\mathbf{VW}^{-1})]/2$.

[6] Symmetrization destroys eigenvalues and rank as illustrated by $\mathbf{S} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\lambda_1 = 1$, $\lambda_2 = 1$, $rk = 2$; $\bar{\mathbf{S}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\lambda_1 = 2$, $\lambda_2 = 0$, $rk = 1$.

To summarize, a generalized stress measure admits the *representation*

$$(3.41)\qquad \begin{aligned} \mathbf{S} &= (v_-\mathbf{U}^{-1} + v_0\mathbf{I} + v_+\mathbf{U})\mathbf{T}(w_-\mathbf{U}^{-1} + w_0\mathbf{I} + w_+\mathbf{U}) \\ &= v_-w_-\mathbf{U}^{-1}\mathbf{T}\mathbf{U}^{-1} + v_-w_0\mathbf{U}^{-1}\mathbf{T} + v_0w_-\mathbf{T}\mathbf{U}^{-1} + v_0w_0\mathbf{T} \\ &\quad + v_-w_+\mathbf{U}^{-1}\mathbf{T}\mathbf{U} + v_+w_-\mathbf{U}\mathbf{T}\mathbf{U}^{-1} + v_+w_+\mathbf{U}\mathbf{T}\mathbf{U} + v_+w_0\mathbf{U}\mathbf{T} + v_0w_+\mathbf{T}\mathbf{U}, \end{aligned}$$

where the constant coefficients v and w are constrained by Eqs. (3.40). Consult Appendix E for an alternative derivation of this representation. An inverse formula exists, although in general it cannot be expanded,

$$(3.42)\qquad \mathbf{T} = (v_-\mathbf{U}^{-1} + v_0\mathbf{I} + v_+\mathbf{U})^{-1}\mathbf{S}(w_-\mathbf{U}^{-1} + w_0\mathbf{I} + w_+\mathbf{U})^{-1}.$$

The important class of congruent stresses is obtained with $w = v$,

$$(3.43)\qquad \begin{aligned} \mathbf{S} &= v_-^2\mathbf{U}^{-1}\mathbf{T}\mathbf{U}^{-1} + v_-v_0(\mathbf{U}^{-1}\mathbf{T} + \mathbf{T}\mathbf{U}^{-1}) + v_0^2\mathbf{T} \\ &\quad + v_-v_+(\mathbf{U}^{-1}\mathbf{T}\mathbf{U} + \mathbf{U}\mathbf{T}\mathbf{U}^{-1}) + v_+^2\mathbf{U}\mathbf{T}\mathbf{U} + v_+v_0(\mathbf{U}\mathbf{T} + \mathbf{T}\mathbf{U}). \end{aligned}$$

Symmetrized stresses have the similar representation

$$\begin{aligned} \bar{\mathbf{S}} &= v_-w_-\mathbf{U}^{-1}\mathbf{T}\mathbf{U}^{-1} + \frac{1}{2}(v_-w_0 + v_0w_-)(\mathbf{U}^{-1}\mathbf{T} + \mathbf{T}\mathbf{U}^{-1}) \\ &\quad + v_0w_0\mathbf{T} + \frac{1}{2}(v_-w_+ + v_+w_-)(\mathbf{U}^{-1}\mathbf{T}\mathbf{U} + \mathbf{U}\mathbf{T}\mathbf{U}^{-1}) \\ &\quad + v_+w_+\mathbf{U}\mathbf{T}\mathbf{U} + \frac{1}{2}(v_+w_0 + v_0w_+)(\mathbf{U}\mathbf{T} + \mathbf{T}\mathbf{U}). \end{aligned}$$

Therefore the fundamental difference between congruent and symmetrized stresses depends entirely on subtle couplings between the coefficients of their expansions.

The stress representation (3.41) reveals the existence of *nine basic stresses* which may be conveniently collected into the two-parameter family

$$(3.44)\qquad \mathbf{S} = \mathbf{U}^{-\frac{m}{2}}\mathbf{R}^T\mathbf{P}\mathbf{U}\mathbf{U}^{-\frac{n}{2}} = \mathbf{U}^{-\frac{m}{2}}\mathbf{T}\mathbf{U}^{-\frac{n}{2}} \qquad \text{(basic)},$$

where both $m = 2, 0, -2$ and $n = 2, 0, -2$. The nine members of this "basic" family are given in Table 4 for clarity. Five of these are recognized as the classic stresses introduced in Eq. (3.21). Two more are found to be the transpose of the Biot and Hill stresses. The last two are identified with the Atluri stresses introduced in [10].

The names indicated in parentheses have been attributed to the best of our knowledge. According to a more anonymous terminology, all these stresses may be referred to the spatial true stress as the left/right-contravected/ contrarotated/rotated/convected stresses. For instance, the first Atluri

Table 4. The family of basic stresses.

m	$\mathbf{V}$	$\mathbf{H}$	n $\mathbf{W}$ $\mathbf{K}$	2 $\mathbf{U}^{-1}$ $\mathbf{I}$	0 $\mathbf{I}$ $\mathbf{U}$	-2 $\mathbf{U}$ $\mathbf{C}$
$+2$	$\mathbf{U}^{-1}$	$\mathbf{F}^{-1}$		$\mathbf{S}^G = \mathbf{F}^{-1}\mathbf{P}$ $= \mathbf{U}^{-1}\mathbf{T}\mathbf{U}^{-1}$ (2nd Piola-Kirch.)	$\mathbf{S}_2^B = \mathbf{P}^T\mathbf{R}$ $= \mathbf{U}^{-1}\mathbf{T}$ (2nd Biot)	$\mathbf{S}_2^A = \mathbf{P}^T\mathbf{F}$ $= \mathbf{U}^{-1}\mathbf{T}\mathbf{U}$ (2nd Atluri)
0	$\mathbf{I}$	$\mathbf{R}^T$		$\mathbf{S}_1^B = \mathbf{R}^T\mathbf{P}$ $= \mathbf{T}\mathbf{U}^{-1}$ (1st Biot)	$\mathbf{T} = \mathbf{R}^T\mathbf{P}\mathbf{U}$ (rotated-Noll)	$\mathbf{S}_2^H = \mathbf{U}\mathbf{P}^T\mathbf{F}$ $= \mathbf{T}\mathbf{U}$ (2nd Hill)
-2	$\mathbf{U}$	$\mathbf{F}^T$		$\mathbf{S}_1^A = \mathbf{F}^T\mathbf{P}$ $= \mathbf{U}\mathbf{T}\mathbf{U}^{-1}$ (1st Atluri)	$\mathbf{S}_1^H = \mathbf{F}^T\mathbf{P}\mathbf{U}$ $= \mathbf{U}\mathbf{T}$ (1st Hill)	$\mathbf{S}^K = \mathbf{F}^T\mathbf{P}\mathbf{C}$ $= \mathbf{U}\mathbf{T}\mathbf{U}$ (Green-Rivlin)

stress $\mathbf{S}_1^A = \mathbf{F}^T(J\mathbf{t})\mathbf{F}^{-1}$ becomes the left convected – right contravected stress, whereas the second Hill stress $\mathbf{S}_2^H = \mathbf{R}^H(J\mathbf{t})\mathbf{F}^T$ may be called the left rotated – right convected true stress. Anticipating stress-strain conjugacy, the superscripts used to identify these stresses correspond to the ones on their conjugate strains, in order to avoid a profusion of symbols.

Of course, the diagonal members in Table 4, obtained for $m = n$, are symmetric as congruent products of the rotated stress $\mathbf{T} = \mathbf{T}^T$, whereas the off-diagonal members, obtained by permuting m and n, are checked to be transposed into each other $\mathbf{S}(m,n) = \mathbf{S}(n,m)^T$. Finally, the Atluri stresses, characterized by $m = -n$, are observed to be similar to the rotated stress, which relegates them to the status of curiosities.

The fact that, with the addition of the Atluri stresses, all the classical stresses and their transpose form a basis for the representation of the generalized stress (3.41) is reassuring. (It is sufficient to take one v and one w equal to one and the others equal to zero in sequence to scan the basis). On the contrary, the finding that, in general, the arithmetic mean of two classical stresses is not a generalized stress in the sense of Eq. (3.33) is very disappointing. (This is due to the unavoidable coupling between the v and w coefficients in Eq. (3.41)). In particular, it is confirmed that the symmetrized stresses of Biot, Hill and Atluri are not admissible stresses in the sense of Eq. (3.33). The same criticism applies to the inverse representation (3.42) (even though it cannot be expanded). For instance, the arithmetic mean of the second Piola-Kirchhoff and the E.Green - Rivlin inverse stress functions ($\mathbf{T} = \mathbf{U}\mathbf{S}^G\mathbf{U}$ and $\mathbf{T} = \mathbf{U}^{-1}\mathbf{S}^K\mathbf{U}^{-1}$) is not an acceptable generalized inverse:

$$\mathbf{T} = \frac{1}{2}\mathbf{U}\mathbf{S}^M\mathbf{U} + \frac{1}{2}\mathbf{U}^{-1}\mathbf{S}^M\mathbf{U}^{-1} .$$

This is unfortunate since the above stress $\mathbf{S}^M$ will shortly be shown to be conjugate to the Mooney strain (2.35).

These criticisms indicate one should give up the idea of a force and a surface transformation as a basis for a generalized stress definition and accept instead of Eq. (3.41) *arbitrary* convex combinations of the form

$$\mathbf{S} = s_a\mathbf{S}_a, \qquad \sum_{a=1}^{9} s_a = 1, \qquad 0 \le s_a \le 1, \tag{3.45}$$

where s_a are now nine independent (instead of coupled) coefficients, and $\mathbf{S}_a$ are the nine basic stresses (3.43), sometimes also called generators (see Appendix E for another justification).

A similar expansion may be written for the inverse relationship

$$\mathbf{T} = t_a\mathbf{T}_a, \qquad \sum_{a=1}^{9} t_a = 1, \qquad 0 \le t_a \le 1,$$

where $\mathbf{T}_a$ are the nine inverse stress functions defined by

$$\mathbf{T} = \mathbf{U}^{n/2}\mathbf{S}\mathbf{U}^{m/2} .$$

This approach has not been pursued further.

3.4. Stress power

The main weakness of static definitions of stress, such as Eq. (3.24) or Eq. (3.33), is their failure to reveal the definite correspondence between any given stress measure and a certain strain (and vice versa). Only a dynamic picture relying on the concepts of virtual power or real energy can remedy this shortcoming.

A statement equivalent to the balance laws (3.18) is the principle of virtual power [41] (or, [42], the principle of objectivity of the balance of energy [43, 44]). In essence, this principle states that the virtual (or real) power of deformation of a body must vanish in any virtual (or superposed) rigid body motion. More specifically, in the absence of body forces,

$$(3.46) \qquad \int_A \mathbf{v} \cdot \mathbf{PN}\, dA = \int_V \mathrm{Div}(\mathbf{v} \cdot \mathbf{P})\, dV = 0,$$

whenever the virtual (or superposed) velocity $\mathbf{v}$ is in the form $\mathbf{v}(\mathbf{X}) = \mathbf{c}(\mathbf{X}) + \mathbf{\Omega}(\mathbf{X})\mathbf{x}$, where $\mathbf{c}$ and $\mathbf{\Omega} = -\mathbf{\Omega}^T$ are two arbitrary velocities of translation and rotation, respectively. (The proof of equivalence relies upon the identity $\mathbf{\Omega}\mathbf{x}\cdot\mathbf{P} = \omega\times\mathbf{x}\cdot\mathbf{P} = \omega\cdot\mathbf{x}\times\mathbf{P}$, where $\omega(\mathbf{X})$ is the rotation vector associated with the skew rotation matrix $\mathbf{\Omega}(\mathbf{X})$).

REMARK. According to [45] the origins of the principle of virtual power may be traced back to Leibniz (1686), d'Alembert (1743), Lagrange (1788), Coriolis (1829) in connection with rigid body dynamics. According to [2], its first formulation in continuum mechanics is due to Piola (1833).

REMARK. The skew rotation tensor $\mathbf{\Omega}$ has nothing to do with the material stretch spin $\mathbf{\Omega}$ introduced in Eq. (2.46).

By distributivity of the divergence with respect to the scalar product and enforcement of the local statement of equilibrium, the integrand in Eq. (3.46) may be reduced as follows

$$\mathrm{Div}(\mathbf{v} \cdot \mathbf{P}) = \frac{\partial \mathbf{v}}{\partial \mathbf{X}} : \mathbf{P} + \mathbf{v} \cdot \mathrm{Div}\mathbf{P} = \mathrm{tr}(\dot{\mathbf{F}}^T\mathbf{P}),$$

where the trace operator is defined as

$$\mathrm{tr}(\dot{\mathbf{F}}^T\mathbf{P}) = \dot{\mathbf{F}} : \mathbf{P} = \dot{F}_{iI}P_{iI} \qquad (= \mathrm{tr}(\mathbf{P}^T\dot{\mathbf{F}})).$$

The familiar form ($W_{\mathrm{ext}} = W_{\mathrm{int}}$) of the principle of virtual power follows:

$$(3.47) \qquad \int_A \mathbf{v} \cdot \mathbf{P}_N\, dA = \int_V \mathrm{tr}(\dot{\mathbf{F}}^T\mathbf{P})\, dV,$$

which is true for any velocity field $\mathbf{v}$ and corresponding deformation gradient rate $\dot{\mathbf{F}}$ and vanishes if and only if $\mathbf{v} = \mathbf{c} + \mathbf{\Omega}\mathbf{x}$.

REMARK. Our notation does not distinguish between applied tractions $\mathbf{P}_N$ on the outside of the body surface and reaction stress vectors $\mathbf{P}_N = \mathbf{PN}$ on the inside of the same surface because of our *a priori* acceptance of the action-reaction and stress principles. Conversely, one can postulate the existence of an internal power in the form (3.39), and thereby of a stress tensor in the form $\mathbf{P}$ and derive the stress and the action-reaction principles [46].

The main advantage of this principle is not so much to constitute a concise statement of the equilibrium equations but rather to provide a *functional link* between the deformation gradient $\mathbf{F}$ and the nominal stress $\mathbf{P}$ in particular, and between kinematics and dynamics in general, as advocated by Lagrange (1788) and more recently by [46]. For this reason, the principle of virtual power may legitimately be regarded as an integral part of constitutive theory.

As a matter of fact, the right-hand side of statement (3.47) constitutes a definition of the internal power developed in the body during its deformation. By further postulating that *this internal power must remain invariant under a replacement of strain measure*, a rigorous (though implicit) definition of the *conjugate* stress is obtained

$$\int_V \mathrm{tr}(\dot{\mathbf{E}}^T \mathbf{S})\, dV = \int_V \mathrm{tr}(\dot{\mathbf{F}}^T \mathbf{P})\, dV. \tag{3.48}$$

It is emphasized that the above definition of conjugacy apparently due to [13] is biased and limiting. A more general (and in many respects more satisfactory) statement of conjugacy would read

$$\int_{V^0} \mathrm{tr}(\sigma^T \varepsilon^0)\, dv = \int_V \mathrm{tr}(\mathbf{P}^T \dot{\mathbf{F}})\, dV,$$

where ε^0 would denote an objective strain rate associated with the volume of integration v^0 or, more exactly, with its mode of deformation, and would be the corresponding stress. For instance, the spatial deformed volume v could be retained together with the convected rate of Rivlin. The stress power per unit of deformed spatial volume is related to Eq. (3.41) by $\mathrm{tr}(\mathbf{s}\mathbf{e}^0)\, dv^0 = \mathrm{tr}(\mathbf{t}\mathbf{d})\, dv = \mathrm{tr}(\mathbf{P}^T \dot{\mathbf{F}})\, dV$. This equality provides a rigorous definition for alternative spatial stresses provided objective strain rates are used. It is most important to realize that *different definitions of conjugacy may apparently associate different stresses to the same strains.* For instance, anticipating a little, the rotated stress (divided by the Jacobian J), instead of the Green-Rivlin stress, would be found to be conjugate to the Karni strain if the "stretched configuration", instead of the reference configuration, were used in the definition of conjugacy. Accordingly, the Cauchy stress would be shown to be conjugate to the Almansi strain [47] rather than to the logarithm strain [8] in a spatial description, if the deformed configuration together with the Rivlin-Ericksen rate were used instead of the reference configuration and the material rate.

The *material* bias of definition (3.48) being well understood, a local definition of conjugacy follows by assuming sufficient regularity

$$\mathrm{tr}(\mathbf{S}^T \dot{\mathbf{E}}) = \mathrm{tr}(\mathbf{P}^T \dot{\mathbf{F}}) = \dot{W}\,. \tag{3.49}$$

Put into words, the stress $\mathbf{S}$ conjugate to any given generalized strain $\mathbf{E}$ is implicitly defined by the requirement that the *stress power per unit of material volume* developed by this (still hypothetical) stress at the generalized strain rate $\dot{\mathbf{E}}$ must be equal to the reference power developed by the nominal stress $\mathbf{P}$ at the deformation gradient rate $\dot{\mathbf{F}}$.

It is pointed out that the symmetry of the generalized strain rate $\dot{\mathbf{E}} = \dot{\mathbf{E}}^T$ implies neither the symmetry nor the uniqueness of the conjugate stress $\mathbf{S}$, a priori, as illustrated by(7)

$$\mathrm{tr}(\mathbf{S}\dot{\mathbf{E}}) = \mathrm{tr}(\mathbf{S}^T\dot{\mathbf{E}}) = \mathrm{tr}\left[\frac{1}{2}(\mathbf{S}+\mathbf{S}^T)\dot{\mathbf{E}} + \frac{1}{2}(\mathbf{S}-\mathbf{S}^T)\dot{\mathbf{E}}\right], \quad \mathbf{S} \neq \mathbf{S}^T, \quad \dot{\mathbf{E}} = \dot{\mathbf{E}}^T.$$

In other words, statement (3.49) completed by the symmetry ($\dot{\mathbf{E}} = \dot{\mathbf{E}}^T$) defines a class of conjugate stresses (differing by a skewsymmetric part) rather than a single item. This absence of both symmetry and uniqueness is precluded in [6, 33, 13] by selecting the *symmetric* part $\bar{\mathbf{S}} = \frac{1}{2}(\mathbf{S}+\mathbf{S}^T)$ of the stress emanating from Eq. (3.49) as the canonical representative of the class. This choice is based on the realization that the skewsymmetric stress $\mathbf{S}' = \frac{1}{2}(\mathbf{S}-\mathbf{S}^T)$ will not develop any power when acting at a symmetric deformation rate, and is motivated by the wish to obtain a reflexive definition of conjugacy. In other words, a skewsymmetric stress is orthogonal to a symmetric rate in the sense of the stress power scalar product. With this convention, the conjugate stress definition becomes

$$\text{(3.50)} \quad \mathrm{tr}(\bar{\mathbf{S}}\dot{\mathbf{E}}) = tr(\mathbf{P}^T\dot{\mathbf{F}}) = \dot{W}, \qquad \bar{\mathbf{S}} = \frac{1}{2}(\mathbf{S}+\mathbf{S}^T), \qquad \dot{\mathbf{E}} = \dot{\mathbf{E}}^T.$$

However, because certain symmetrized stresses $\bar{\mathbf{S}}$ show a number of failings denounced in the previous section, it is advisable to check, in each particular case, whether the produced stress is genuinely or only artificially symmetric (by checking whether it is *congruent* to the rotated stress or not, for example). Another danger of untimely symmetrization has been recently pointed out by [48] in a different context.

Now, unlike the *static* notion of generalized stress (3.33), which is free of any strain connotation, the *dynamic* concept of conjugate stress (3.50) is

(7)The trace of a matrix product is unaltered by a cyclic permutation of the factors, e.g. $\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}(\mathbf{BCA})$, and is equal to the trace of the transpose of the product, i.e. $\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}(\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T) = \mathrm{tr}(\mathbf{CBA})$ for symmetric matrices. Moreover, the trace is a linear operator, i.e. $\mathrm{tr}(\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha\mathrm{tr}\mathbf{A} + \beta\mathrm{tr}\mathbf{B}$.

related to a definite strain. This link may be further specified by substituting the rate dependence (2.51) in the trace statement (3.50)

$$\mathrm{tr}(\mathbf{P}^T\dot{\mathbf{F}}) = \mathrm{tr}(\bar{\mathbf{S}}\dot{\mathbf{E}}) = \mathrm{tr}\left[\bar{\mathbf{S}}\left(\frac{d\mathbf{E}}{d\mathbf{F}}\dot{\mathbf{F}}\right)\right] = \mathrm{tr}\left[\left(\bar{\mathbf{S}}\frac{d\mathbf{E}}{d\mathbf{F}}\right)\dot{\mathbf{F}}\right].$$

An explicit definition of the *conjugate stress* follows by identification.

(3.51) $$\mathbf{P} = \left(\frac{d\mathbf{E}}{d\mathbf{F}}\right)^T \bar{\mathbf{S}} \qquad \text{or} \qquad \bar{\mathbf{S}} = \left(\frac{d\mathbf{E}}{d\mathbf{F}}\right)^{-T} \mathbf{P},$$

where the transposition means $P_{kL} = \frac{dE_{IJ}}{dF_{kL}} S_{IJ}$.

Hence the formulation of the equation of equilibrium (Div$\mathbf{P}$ = 0) involves the expression of the generalized strain derivative with respect to the deformation gradient. Conversely, the conjugate stress definition is tied to the inverse strain derivative. However compact and neat this definition may appear, it remains obscure and sometimes deceptive because, as already pointed out, the strain derivative $d\mathbf{E}/d\mathbf{F}$ is often difficult (if not impossible) to express and even more to invert. A more reliable procedure for obtaining a specific conjugate stress is the substitution of the relevant strain rate in the stress power equivalence (3.50) and the identification of the terms explicitly after a few trace manipulations. As a preliminary and crucial illustration, consider the rotated rate explicitly given by Eq. (2.40). The conjugate stress is defined by

$$\begin{aligned}\mathrm{tr}(\mathbf{P}^T\dot{\mathbf{F}}) &= \mathrm{tr}(\bar{\mathbf{S}}\mathbf{D}) = \mathrm{tr}\left[\bar{\mathbf{S}}\frac{1}{2}\left(\mathbf{R}^T\dot{\mathbf{F}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{F}}^T\mathbf{R}\right)\right]\\ &= \frac{1}{2}\mathrm{tr}(\mathbf{U}^{-1}\bar{\mathbf{S}}\mathbf{R}^T\dot{\mathbf{F}}) + \frac{1}{2}\mathrm{tr}(\dot{\mathbf{F}}^T\mathbf{R}\bar{\mathbf{S}}\mathbf{U}^{-1}) = \mathrm{tr}[(\mathbf{U}^{-1}\bar{\mathbf{S}}\mathbf{R}^T)\dot{\mathbf{F}}].\end{aligned}$$

It follows by identification that

$$\mathbf{P} = \mathbf{R}\bar{\mathbf{S}}\mathbf{U}^{-1} \qquad \text{or} \qquad \bar{\mathbf{S}} = \mathbf{R}^T\mathbf{P}\mathbf{U} = \mathbf{T}.$$

Therefore *the rotated stress* $\mathbf{T}$ *is conjugate to the rotated rate* $\mathbf{D}$

(3.52) $$\dot{W} = \mathrm{tr}(\mathbf{P}^T\dot{\mathbf{F}}) = \mathrm{tr}(\mathbf{T}\mathbf{D}).$$

Readers are reminded (to eliminate any ambiguity) that the symmetry of the rotated stress results from the balance of moments.

To illustrate another type of derivation, consider next the stretch rate implicitly defined by Eq. (2.42). The conjugate stress satisfies

$$\begin{aligned}\mathrm{tr}(\bar{\mathbf{S}}\dot{\mathbf{U}}) = \mathrm{tr}(\mathbf{T}\mathbf{D}) &= \mathrm{tr}\left[\mathbf{T}\frac{1}{2}(\mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1})\right]\\ &= \mathrm{tr}\left[\frac{1}{2}(\mathbf{T}\mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{T})\dot{\mathbf{U}}\right] = \mathrm{tr}(\bar{\mathbf{S}}^B\dot{\mathbf{U}}).\end{aligned}$$

It follows by identification that *Biot's symmetrized stress is conjugate to the stretch rate and therefore to the Biot strain*

$$\dot{W} = \mathrm{tr}(\mathbf{TD}) = \mathrm{tr}(\bar{\mathbf{S}}^B\dot{\mathbf{U}}) = \mathrm{tr}(\bar{\mathbf{S}}^B\dot{\mathbf{E}}^B). \tag{3.53}$$

It is again emphasized that the asymmetric Biot stress $\mathbf{S}_1^B$ (or its transpose $\mathbf{S}_2^B$) are also conjugate to the stretch rate $\dot{\mathbf{U}}$, as stated in [10]. More generally, if a stress $\bar{\mathbf{S}}$ is found to be conjugate to a symmetric rate $\dot{\mathbf{E}}$ via Eq. (3.50), and if this stress is decomposed into a sum of the form $\bar{\mathbf{S}} = \frac{1}{2}(\mathbf{S} + \mathbf{S}^T)$ with $\mathbf{S} \neq \mathbf{S}^T$ (which is always possible), then $\mathbf{S}$ and $\mathbf{S}^T$ are also conjugate to the strain $\mathbf{E}$. A definite choice must rely on an additional relationship such as $\mathrm{tr}(\mathbf{SB}) = \mathrm{tr}(\mathbf{TA}) = 0$, where $2\mathbf{A} = \dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}}$ is the rotated spin tensor and $\mathbf{B}$ the generalized spin tensor corresponding to $\dot{\mathbf{E}}$.

At this point it is instructive to look at the inverse problem of finding a strain conjugate to a given stress. Substitution of the symmetrized stress expression (3.37) in the symmetrized stress power (3.50) produces by identification the rate relationships

$$\begin{aligned} \mathbf{D} &= \frac{1}{2}(\mathbf{W}\dot{\mathbf{E}}\mathbf{V} + \mathbf{V}\dot{\mathbf{E}}\mathbf{W}), \\ \dot{\mathbf{F}}^T &= \frac{1}{2}(\mathbf{K}^{-1}\dot{\mathbf{E}}\mathbf{H} + \mathbf{H}\dot{\mathbf{E}}\mathbf{K}^{-1}). \end{aligned} \tag{3.54}$$

It is emphasized that the substitution of the unsymmetrized stress expression (3.33)-(3.34) in the unsymmetrized stress power (3.49) would produce incorrect rates even after their symmetrization (except for congruent stresses).

The difficulties presented by the inversion and the time integration of the rate formulas (3.54) make the inverse problem much harder to solve than the direct one. For congruent rates (conjugate to congruent stresses characterized by $\mathbf{V} = \mathbf{W}$) the inversion is trivial, but the integration remains exceptional

$$\dot{\mathbf{E}} = \mathbf{V}^{-1}\mathbf{D}\mathbf{V}^{-1} = \mathbf{V}^{-1}(\mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1})\mathbf{V}^{-1}\,. \tag{3.55}$$

No such simplifications emerge from similarity or half-identity transformations, which demonstrates once again the superiority of congruences.

Of course, the representations (3.39) of the force and the surface transformations $\mathbf{V}$ and $\mathbf{W}$ may be used to derive corresponding representations of the generalized strain rates (3.54) or (3.55). Such developments are omitted here.

3.5. Conjugate stresses

In terms of the reference pairs of stress-strain rate measures $(\bar{\mathbf{S}}^B - \mathring{\mathbf{U}})$, $(\mathbf{P}^T - \dot{\mathbf{F}})$ or $(\mathbf{T} - \mathbf{D})$, the definition of the generalized stress $\mathbf{S}$ conjugate to a generalized strain $\mathbf{E}$ takes the alternative forms

$$
\begin{aligned}
\dot{W} &= \mathrm{tr}(\bar{\mathbf{S}}\dot{\mathbf{E}}) && = \mathrm{tr}(\bar{\mathbf{S}}^B\mathring{\mathbf{U}}) && = \mathrm{tr}(\mathbf{P}^T\dot{\mathbf{F}}) && = \mathrm{tr}(\mathbf{T}\mathbf{D}), \\
\dot{\mathbf{E}} &= \frac{d\mathbf{E}}{d\mathbf{U}}\mathring{\mathbf{U}} && = \frac{d\mathbf{E}}{d\mathbf{F}}\dot{\mathbf{F}} && = \mathsf{E}\mathbf{D} && (= \dot{\mathbf{E}}^T), \\
\bar{\mathbf{S}} &= \left(\frac{d\mathbf{E}}{d\mathbf{U}}\right)^{-1}\bar{\mathbf{S}}^B && = \left(\frac{d\mathbf{E}}{d\mathbf{F}}\right)^{-1}\mathbf{P}^T && = \mathsf{E}^{-1}\mathbf{T} && (= \bar{\mathbf{S}}^T).
\end{aligned}
\tag{3.56}
$$

Using the above definition and the usual properties of the trace operator, it is simple to show that the classical stresses (3.21), properly symmetrized according to Eq. (3.38), are conjugate to the classical strains (2.26) via the strain rates (2.53) (with one exception for the rotated stress (3.20) which is strictly conjugate to the rotated rate (2.40), but only approximately conjugate to the natural strain (2.28)). More concisely,

$$
\begin{aligned}
\dot{W} &= \mathrm{tr}(\mathbf{S}^G\dot{\mathbf{E}}^G) = \mathrm{tr}(\bar{\mathbf{S}}^B\dot{\mathbf{E}}^B) = \mathrm{tr}(\mathbf{T}\mathbf{D}) = \mathrm{tr}(\bar{\mathbf{S}}^H\dot{\mathbf{E}}^H) \\
&= \mathrm{tr}(\mathbf{S}^K\dot{\mathbf{E}}^K) \cong \mathrm{tr}(\mathbf{T}\mathring{\mathbf{G}}).
\end{aligned}
\tag{3.57}
$$

Relationships between the classical stresses such as Eq. (3.21) follow at once from Eq. (3.57) and are not repeated here.

Accordingly the symmetrized stress of the Seth family (3.22) is conjugate to the corresponding strain (2.27) via its rate (2.54) (with the same provision as above for the case $m = 0$)

$$
\dot{W} = \mathrm{tr}(\bar{\mathbf{S}}^{(m)}\dot{\mathbf{E}}^{(m)}), \qquad m = 2, 1, -1, -2. \tag{3.58}
$$

Now, it follows from the linearity of both the time and the directional derivative on the one hand and the linearity of the trace operator on the other hand that

$$
\begin{aligned}
&\text{if} && \mathbf{E} = \frac{p}{p-q}\mathbf{E}^{(p)} + \frac{q}{q-p}\mathbf{E}^{(q)}, \\
&\text{and} && \dot{\mathbf{E}}^{(p)} = \mathsf{E}^{(p)}\mathbf{D}; \qquad \dot{\mathbf{E}}^{(q)} = \mathsf{E}^{(q)}\mathbf{D}, \\
&\text{then} && \dot{\mathbf{E}} = \left[\frac{p}{p-q}\mathsf{E}^{(p)} + \frac{q}{q-p}\mathsf{E}^{(q)}\right]\mathbf{D}, \\
&\text{and} && \mathbf{T} = \left[\frac{p}{p-q}\mathsf{E}^{(p)} + \frac{q}{q-p}\mathsf{E}^{(q)}\right]\bar{\mathbf{S}}.
\end{aligned}
\tag{3.59}
$$

The proof of this proposition is straightforward simply by substitution, permutation and identification in Eq. (3.56). Several corollaries can be derived by taking different rate-stress pairs for reference. For instance, using $(\bar{\mathbf{S}}^B - \dot{\mathbf{U}})$, produces the final result

$$\bar{\mathbf{S}}^B = \left[\frac{p}{p-q}\,\frac{d\mathbf{E}^{(p)}}{d\mathbf{U}} + \frac{q}{q-p}\,\frac{d\mathbf{E}^{(q)}}{d\mathbf{U}}\right]\mathbf{S}.$$

Alternately, using the pair $(\mathbf{P} - \dot{\mathbf{F}})$, yields

$$\mathbf{P} = \left[\frac{p}{p-q}\,\frac{d\mathbf{E}^{(p)}}{d\mathbf{F}} + \frac{q}{q-p}\,\frac{d\mathbf{E}^{(q)}}{d\mathbf{F}}\right]\mathbf{S}.$$

These important results (too long to be put into words) provide the key to the determination of the stresses conjugate to the strains of the rubber family, whenever the relevant strain derivatives are available.

For instance, since all the rubber strain rates have explicit expressions in terms of the stretch rate given in Eq. (2.55), the conjugate rubber stresses are implicitly defined in terms of the Biot stress by

$$\text{(3.60)}\quad \begin{aligned}
\bar{\mathbf{S}}^B &= \frac{1}{2}(\mathbf{S}^P + \mathbf{U}^{-1}\mathbf{S}^P\mathbf{U}^{-1}) && \text{(Pelzer)},\\
\bar{\mathbf{S}}^B &= \frac{1}{4}\left[\mathbf{U}\mathbf{S}^M + \mathbf{S}^M\mathbf{U} + \mathbf{U}^{-1}(\mathbf{U}^{-1}\mathbf{S}^M + \mathbf{S}^M\mathbf{U}^{-1})\mathbf{U}^{-1}\right] && \text{(Mooney)},\\
\bar{\mathbf{S}}^B &= \frac{1}{3}\left[\mathbf{S}^W + \mathbf{U}^{-1}(\mathbf{U}^{-1}\mathbf{S}^W + \mathbf{S}^W\mathbf{U}^{-1})\mathbf{U}^{-1}\right] && \text{(Wall)},\\
\bar{\mathbf{S}}^B &= \frac{1}{3}(\mathbf{U}\mathbf{S}^R + \mathbf{S}^R\mathbf{U} + \mathbf{U}^{-1}\mathbf{S}^R\mathbf{U}^{-1}) && \text{(Rivlin)}.
\end{aligned}$$

Unfortunately, the above expressions are of little practical value since the equtions of equilibrium are different to state in terms of Biot's symmetrized stress. The situation is better for the Mooney strain since its conjugate stress is also given by

$$\text{(3.61)}\quad \begin{aligned}
\mathbf{T} &= \frac{1}{2}(\mathbf{U}\mathbf{S}^M\mathbf{U} + \mathbf{U}^{-1}\mathbf{S}^M\mathbf{U}^{-1}),\\
\mathbf{P} &= \frac{1}{2}(\mathbf{F}\mathbf{S}^M + \mathbf{F}^{-T}\mathbf{S}^M\mathbf{C}^{-1}).
\end{aligned}$$

Therefore the Mooney strain is the only one with a workable conjugate stress among the rubber family members. However, the fact that the Mooney stress is not congruent to the rotated stress (in spite of its genuine symmetry which implies the balance of moments) still condemns it to oblivion.

Finally, to complete this presentation of conjugacy, the spectral form of conjugate stresses must be derived. To this end the definitions in terms of the rotated rate $\mathbf{D}$ and the stretch rate $\dot{\mathbf{U}}$ in Eq. (3.57) are of course the most adapted. In spectral form these statements become

$$(3.62) \qquad \dot{W} = \mathrm{tr}(\bar{\mathbf{S}}\dot{\mathbf{E}}) = \bar{S}_{ab}\dot{E}_{ab} = \bar{S}^B_{ab}\dot{U}_{ab} = T_{ab}D_{ab}, \quad a,b = 1,3.$$

Substitution of the spectral relationships (2.62) between the generalized strain rate components $\dot{\mathbf{E}}_{ab}$ and the references rate components $\dot{\mathbf{U}}_{ab}$ and $\mathbf{D}_{ab}$ produces the desired formulas simply by identification of the components (no summation over the repeated indices),

$$(3.63) \qquad \begin{aligned} \bar{S}_{aa} &= \frac{1}{E'(\lambda_a)}\bar{S}^B_{aa} = \frac{1}{\lambda_a E'(\lambda_a)}T_{aa}\,, \\ \bar{S}_{ab} &= \frac{\lambda_b - \lambda_a}{E'(\lambda_b) - E(\lambda_a)}\bar{S}^B_{ab} = \frac{(\lambda_a + \lambda_b)(\lambda_b - \lambda_a)}{2\lambda_a\lambda_b[E(\lambda_b) - E(\lambda_a)]}T_{ab}\,. \end{aligned}$$

Here again the first relationship reproduces the uniaxial formula (3.9) whereas the second relationship generalizes it to three-dimensional deformations. The spectral form (3.63) offers the valuable advantage of being explicit and invertible for any generalized strain function $E(\lambda)$. For instance, the stress exactly conjugate to the natural strain (2.28) is easily found via the natural rate (2.58) to be (no summation over the repeated indices)

$$(3.64) \qquad \begin{aligned} \bar{S}_{aa} &= \lambda_a\bar{S}^B_{aa} = T_{aa}\,, \\ \bar{S}_{ab} &= \frac{\lambda_b - \lambda_a}{\mathrm{Log}\lambda_b - \mathrm{Log}\lambda_a}\bar{S}^B_{ab} = \frac{(\lambda_a + \lambda_b)(\lambda_b - \lambda_a)}{2\lambda_a\lambda_b(\mathrm{Log}\lambda_b - \mathrm{Log}\lambda_a)}T_{ab}\,. \end{aligned}$$

The offdiagonal relationship shows the difference which separates the natural stress $\bar{\mathbf{S}}$ from the rotated stress $\mathbf{T}$. Similarly, the stresses conjugate to the rubber strains (2.34) are readily derived by inversion of Eq. (2.63) to be (no summation over the repeated indices)

$$(3.65) \qquad \begin{aligned} \bar{S}_{aa} &= \frac{p-q}{p\lambda_a^p - q\lambda_a^q}\,T_{aa}\,, \\ \bar{S}_{ab} &= \frac{p-q}{2}\,\frac{\lambda_a + \lambda_b}{\lambda_a\lambda_b}\,\frac{\lambda_b - \lambda_a}{\lambda_b^p - \lambda_b^q - \lambda_a^p + \lambda_a^q}T_{ab}\,. \end{aligned}$$

Using the formal notation (2.64) the stress power may be expanded into

$$\mathrm{tr}(\bar{\mathbf{S}}\dot{\mathbf{E}}) = \mathrm{tr}[\bar{\mathbf{S}}(\overset{\circ}{\mathbf{E}} + \mathbf{\Omega}\mathbf{E} - \mathbf{E}\mathbf{\Omega})] = \mathrm{tr}(\bar{\mathbf{S}}\mathbf{E}) + \mathrm{tr}[(\mathbf{E}\bar{\mathbf{S}} - \bar{\mathbf{S}}\mathbf{E})\mathbf{\Omega}].$$

The above decompositon shows that the antisymmetric tensor $\mathbf{E}\bar{\mathbf{S}} - \bar{\mathbf{S}}\mathbf{E}$ is measure invariant as pointed out by [13]. In isotropic media, this stress "couple" vanishes since $\bar{\mathbf{S}}$ and $\mathbf{E}$ commute and the stress power reduces to the diagonal contribution $\mathrm{tr}(\bar{\mathbf{S}}\,\overset{\circ}{\mathbf{E}})$.

3.6. Stress-strain pair-law interaction

It is worthwhile at this point to give an idea of the incident of a change of stress-strain pair on the eventual form of a stress-strain law for a fixed force-deformation response of a body. Academically speaking, it is easier to address the converse question of estimating the change in the force-deformation response of a body produced by a change in the stress-strain pair definition for a given stress-strain law. For this purpose, the simplest case is to consider a linear elastic law and to look at the pure traction-elongation response of a bar. In this instance, the elastic stress-strain law reduces to a scalar relationship characterized by the single elasticity modulus

$$S = \mathsf{E}E. \tag{3.66}$$

In (3.66) the modulus of elasticity E must not be mistaken for the linear operator E involved in Eq. (2.51) in spite of the notation overlap. It is simple to show (by means of Eqs. (3.1), (3.6), (3.66), and (2.1) in that order[8] that the corresponding global force-displacement response of the bar is the nonlinear function

$$q = \mathsf{E}AE'\left(1+\frac{u}{L}\right)E\left(1+\frac{u}{L}\right), \tag{3.67}$$

where $u = l - L$ denotes the end displacement of the bar produced by the applied load q, and A is the original cross-section. Therefore, the force-displacement response of the bar is in general *nonlinear* in spite of the hypothesis of a *linear* stress strain law. It is checked however (thanks to the consistency conditions (2.4)) that all stress-strain pairs give rise to the same usual stiffness of the bar for sufficiently small deformations:

$$q = \frac{\mathsf{E}A}{L}\left[1 + E''(1)\frac{u}{L} + \ldots\right]u. \tag{3.68}$$

Instead of the force-displacement response (3.67), it is preferable to express the nominal stress $\mathbf{P}$ in terms of either the small strain ε or the stretch ratio

[8] $q = AP = AE'(\lambda)\mathrm{S} = \mathsf{E}AE'(\lambda)\mathrm{E}(\lambda) = \mathsf{E}AE'(1+u/L)\mathrm{E}(1+u/L)$.)

λ for comparison with standard experimental tests or analytical models, respectively,

$$P = \mathsf{E}E'(1+\varepsilon)E(1+\varepsilon) = \mathsf{E}E'(\lambda)E(\lambda). \tag{3.69}$$

For the "rubber" stress-strain family, the latter response function takes the form[9]

$$P = \mathsf{E}\frac{p\lambda^{p-1} - q\lambda^{q-1}}{p-q}\,\frac{\lambda^p - \lambda^q}{p-q}, \qquad -2 \le q \le 0 \le p \le 2. \tag{3.70}$$

The eight response functions implied in Eq. (3.70) are shown in Table 5 and plotted in Fig. 10 with $\mathsf{E} = 1$ for simplicity. (The corresponding stress-strain laws are added as a reminder). Similar expressions are given by [48] for the classical stress-strain pairs.

Table 5. Stress-strain ratio response functions.

p \ q	0	−1	−2
0		$S^H = E^H$ $P = \frac{1}{\lambda^2} - \frac{1}{\lambda^3}$	$S^K = E^K$ $P = \frac{1}{2}\left(\frac{1}{\lambda^3} - \frac{1}{\lambda^5}\right)$
1	$S^B = E^B$ $P = \lambda - 1$	$S^P = E^P$ $P = \frac{1}{4}\left(\lambda - \frac{1}{\lambda^3}\right)$	$S^W = E^W$ $P = \frac{1}{9}\left(\lambda + \frac{1}{\lambda^2} - \frac{2}{\lambda^3}\right)$
2	$S^G = E^G$ $P = \frac{1}{2}\left(\lambda^3 - \lambda\right)$	$S^R = E^R$ $P = \frac{1}{9}\left(2\lambda^3 - 1 - \frac{1}{\lambda^3}\right)$	$S^M = E^M$ $P = \frac{1}{8}\left(\lambda^3 - \frac{1}{\lambda^5}\right)$

Therefore the Biot stress-strain pair is the only one to produce a force-displacement response which coincides with the assumed stress-strain law at finite strains in a simple tension experiment. The Green strain – conjugate stress pair "buckles" under compression whereas the Hill and Karni pairs "fail" in tension. The rubber pairs are better behaved in that respect.

[9] The exponent q must not be mistaken for the applied load q in spite of the notation overlap.

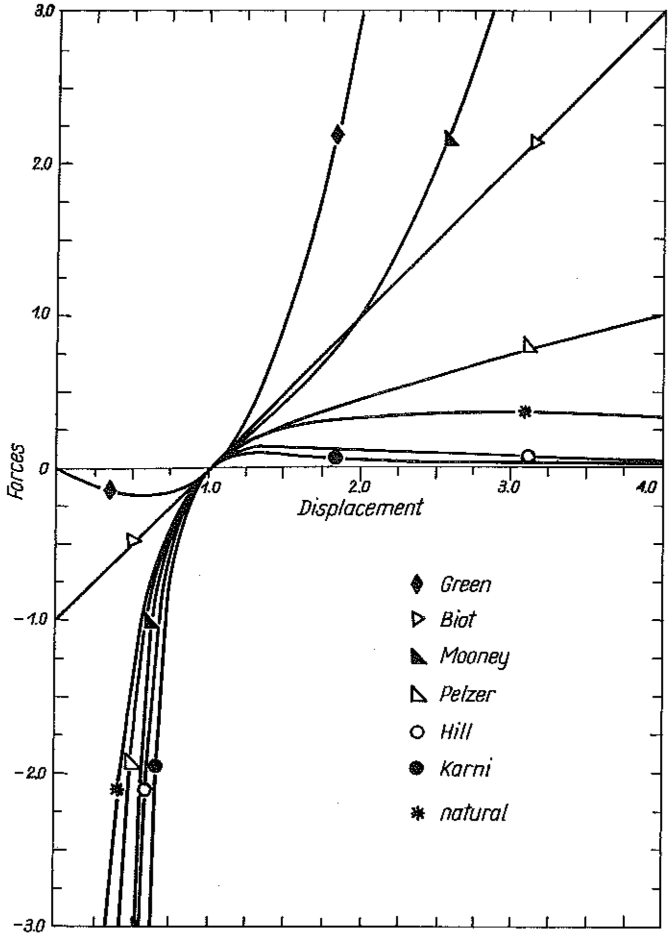


FIG. 10.

For completeness, note that a linear law between the rotated stress and the natural strain results in the almost acceptable response [49]:

$$T = G, \qquad P = \frac{\operatorname{Log} \lambda}{\lambda} . \tag{3.71}$$

This elementary analysis is sufficient to demonstrate that the simplest constitutive theoris such as linear elasticity or classical plasticity cannot be applied naively to finite strain problems unless the stress-strain pair used in the formulation is judiciously selected. A similar warning applies a fortiori to the spatial description where, as was shown by [50], a linear isotropic law between the Cauchy stress and the Almansi strain does not even define an elastic material.

APPENDIX A. A BRIEF HISTORY OF STRAIN AND STRESS

According to [2], the strain introduced by G.GREEN (1841) was independently formulated by ST. VENANT (1844) and later generalized by BRILLOUIN (1925). The herein KARNI strain, which is the material form of the well-known spatial strain introduced by ALMANSI (1911) and HAMEL (1912) and generalized by MURNAGHAN (1937), was jointly published by KARNI and REINER (1968), who also gave the spatial form of the GREEN strain (which it would be fair to call the REINER strain). An allusion to it may be found in DOYLE and ERICKSEN (1956).

The strain proposed by BIOT (1939) constitutes the proper objective generalization of the small strain formulated by CAUCHY (1822). The strain herein referred to by the name of HILL (1968) is the material form of the more familiar strain vaguely described by SWAINGER (1947) and rigorously defined by HERSHEY (1952) in spatial form. Finally the natural logarithm strain attributed to HENCKY (1928) was previously introduced by LUDWICK (1909) and even by IMBERT (1880).

All these classical strains were integrated into a one-parameter family by DOYLE and ERICKSEN (1956) and later studied by SETH (1964).

Hints leading to the PELZER (1938) and WALL (1942) strains may be found in the traction-elongation relationships established by these authors in their attempts to explain the behaviour of rubber through its microstructure. However, in none of these studies are they presented as candidate deformation measures. On the contrary, the similarity of the PELZER strain to a measure of strain magnitude suggested by TRUESDELL (1960) [2.Eq.30.9] is striking.

Hints leading to the MOONEY (1940) and the RIVLIN (1947) strains are available in their stress-stretch relationships derived from strain energy functions constructed by invoking macroscopic arguments to model the response of rubber also.

According to [2], the notion of stress *vector* can be traced to GALILEO (1638) and was gradually extended by, among others, PASCAL (1648), the BERNOULLIS (1691 and 1743), COULOMB (1776), EULER (1749) and CAUCHY (1823) who cleverly extricated the concept of stress *tensor*. The nominal stress associated with the deformation gradient and the stress conjugate to the GREEN strain are both due to PIOLA (1833) and also to KIRCHHOFF (1852), thus their alternate designation of first and second Piola-Kirchhoff stresses, respectively. Other sources attribute the nominal stress to BOUSSI-

NESQ (1872) and TREFFTZ (1928).

According to [12], the convected stress was introduced by E. GREEN and RIVLIN (1956) (within a factor equal to the Jacobian J) and it was later found by HILL (1968) to be conjugate to the KARNI (-REINER) strain (which is the material form of the ALMANSI strain). The material true stress was introduced by NOLL (1957) as the rotated CAUCHY'S spatial true stress (1823), later modified by HILL (1968) as the rotated KIRCHHOFF'S spatial stress (1852), equal to CAUCHY'S stress scaled by the Jacobian, and also found by him to be nearly conjugate to the natural logarithm strain.

The stress introduced by BIOT (1965) was clarified by ZIEGLER (1967), LURE (1968) and HILL (1968). Sometimes its symmetric part is attributed to JAUMANN (1918). The stress conjugate to the HILL strain (which is the material form of the SWAINGER strain) is also attributed to HILL (1968) although he did not give any special attention to it in his generalized definition. TRUESDELL and WANG (1973) mention it explicitly. Finally, the stresses introduced by ATLURI (1984) are alluded to by ASTARITA and MARUCCI as upper/lower – left/right convected stresses (1975).

APPENDIX B. COEFFICIENTS OF THE REPRESENTATION OF AN ISOTROPIC STRAIN FUNCTION

More specifically, assuming for the moment that the three principal stretches λ_a are distinct and equal to α, β, γ, respectively, the solution of the system (3.31) for any admissible p and q, takes the form

$$x = \frac{(\alpha\beta\gamma)^q}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\Delta^{(pq)}}\left[\frac{\beta^q-\gamma^q}{(\beta\gamma)^q}E(\alpha)+\frac{\gamma^q-\alpha^q}{(\gamma\alpha)^q}E(\beta) + \frac{\alpha^q-\beta^q}{(\alpha\beta)^q}E(\gamma)\right],$$

$$y = \frac{(\alpha\beta\gamma)^q}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\Delta^{(pq)}}\left[\frac{\gamma^{p+q}-\beta^{p+q}}{(\gamma\beta)^q}E(\alpha)+\frac{\alpha^{p+q}-\gamma^{p+q}}{(\alpha\gamma)^q}E(\beta) + \frac{\beta^{p+q}-\alpha^{p+q}}{(\beta\alpha)^q}E(\gamma)\right],$$

$$z = \frac{(\alpha\beta\gamma)^q}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\Delta^{(pq)}}\left[(\beta^p-\gamma^p)E(\alpha)+(\gamma^p-\alpha^p)E(\beta) +(\alpha^p-\beta^p)E(\gamma)\right],$$

where

$$\Delta^{(11)} = 1, \qquad \Delta^{(12)} = \alpha\beta + \beta\gamma + \gamma\alpha,$$
$$\Delta^{(21)} = \alpha + \beta + \gamma, \qquad \Delta^{(22)} = (\alpha + \beta)(\beta + \gamma)(\gamma + \alpha).$$

Of course, these functions are symmetric in α, β, γ and $x(p,q) = z(q,p)$. If two principal stretches happen to coincide, the indeterminacy is eliminated by reducing the expansion to $\mathbf{E} = x\mathbf{U}^p + y\mathbf{I}$ and the system accordingly. Finally, if all three values are equal, the solution is obtained from the more degenerated case $\mathbf{E} = y\mathbf{I}$. The case $\mathbf{E}(\lambda) = \mathrm{Log}\lambda$ provides an exact tensorial representation of the natural strain measure once the principal stretches and directions are extracted, as remarked by [8] for the spatial variant $\mathrm{Log}\,\mathbf{u} = x\mathbf{i} + y\mathbf{u} + z\mathbf{u}^2$.

Appendix C. Spectral form of the deformation gradient rate

A similar procedure applied to the spectral forms (2.22) of the mixed tensors $\mathbf{R}$ and $\mathbf{F}$ would briefly be carried out as follows, [51].

For the *rotation:*

$$\mathbf{R} = \mathbf{n}_a \otimes \mathbf{N}_a, \qquad \dot{\mathbf{R}} = \dot{\mathbf{n}}_a \otimes \mathbf{N}_a + \mathbf{n}_a \otimes \dot{\mathbf{N}}_a,$$
$$\dot{\mathbf{n}}_a = \omega_{ba}\mathbf{n}_b, \qquad \dot{\mathbf{R}} = (\omega_{ab} - \Omega_{ab})\mathbf{n}_a \otimes \mathbf{N}_b$$

where ω is the *spatial spin* defined by $\boldsymbol{\omega} = \dot{\mathbf{q}}\mathbf{q}^T = -\boldsymbol{\omega}^T$ and $\mathbf{q}$ is defined below. The spatial triad $\mathbf{n}_a$ rotates around its moving origin $\mathbf{x}$ according to

$$\mathbf{n}_a = \mathbf{q}\mathbf{n}_a^\circ \qquad \text{or} \qquad \mathbf{n}_a^\circ = \mathbf{N}_a^\circ = \mathbf{q}^T\mathbf{n}_a \qquad \text{thus} \qquad \dot{\mathbf{n}}_a = \dot{\mathbf{q}}\mathbf{q}^T\mathbf{n}_a = \boldsymbol{\omega}\mathbf{n}_a.$$

For the *deformation gradient:*

$$\mathbf{F} = \lambda_a n_a \otimes \mathbf{N}_a, \qquad \dot{\mathbf{F}} = \dot{\lambda}_a \mathbf{n}_a \otimes \mathbf{N}_a + \lambda_a \dot{\mathbf{n}}_a \otimes \mathbf{N}_a + \lambda_a \mathbf{n}_a \otimes \dot{\mathbf{N}}_a,$$
$$\dot{\mathbf{F}} = (\dot{\lambda}_a \delta_{ab} + \lambda_b \omega_{ab} + \lambda_a(\omega_{ba} - \theta_{ba})\mathbf{n}_a \otimes \mathbf{N}_b$$
$$= [\dot{\lambda}_a \delta_{ab} + (\lambda_b - \lambda_a)\omega_{ab} + \lambda_a \theta_{ab}]\mathbf{n}_a \otimes \mathbf{N}_b,$$

where $\boldsymbol{\theta}$ is the (spatial) *relative spin* of the spatial triad with respect to the material triad defined by $\boldsymbol{\theta} = \dot{\mathbf{R}}\mathbf{R}^T = -\boldsymbol{\theta}^T$. The spatial rotation $\mathbf{q}$ is recognized as the product of the material rotation $\mathbf{Q}$ by the relative rotation $\mathbf{R}$ while the rotated spatial spin $\mathbf{R}^T\boldsymbol{\omega}\mathbf{R}$ equals the sum of the material spin and the rotated relative spin

$$\mathbf{q} = \mathbf{R}\mathbf{Q}, \qquad \mathbf{R}^T(\boldsymbol{\omega} - \boldsymbol{\theta})\mathbf{R} = \boldsymbol{\Omega}.$$

Incidentally, note that at incipient deformation ($\mathbf{R} = \mathbf{I}$) the spins are simply related by $\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\theta}$.

APPENDIX D. DERIVATION OF STRAIN-STRETCH RATE OPERATORS

In this Appendix, the fourth order linear operators $d\mathbf{E}/d\mathbf{U}$ producing the strain rates $\dot{\mathbf{E}}$ when applied to the stretch rate $\dot{\mathbf{U}}$ are derived for the classical strains and the rubber strains. The main trick is to expand $\mathbf{U}$ into $\dfrac{\mathbf{U}+\mathbf{U}^T}{2}$ (and of course $\mathbf{U}^{-1}$ into $\dfrac{\mathbf{U}^{-1}+\mathbf{U}^{-T}}{2}$) in order to exhibit its symmetry which would remain hidden otherwise [48].

Green strain:

$$E^G_{IJ} = \frac{1}{2}\left(\frac{(U_{IM}+U_{MI})}{2}\,\frac{(U_{MJ}+U_{JM})}{2} - \delta_{IJ}\right)$$
$$= \frac{1}{8}(U_{IM}U_{MJ} + U_{IM}U_{JM} + U_{MI}U_{MJ} + U_{MI}U_{JM}) - \frac{1}{2}\delta_{IJ},$$

$$\dot{E}^G_{IJ} = \frac{1}{8}(\dot{U}_{IM}U_{MJ} + U_{IM}\dot{U}_{MJ} + \dot{U}_{IM}U_{JM} + U_{IM}\dot{U}_{JM}$$
$$+\dot{U}_{MI}U_{MJ} + U_{MI}\dot{U}_{MJ} + \dot{U}_{MI}U_{JM} + U_{MI}\dot{U}_{JM})$$
$$= \frac{1}{8}(\delta_{IK}\delta_{ML}U_{MJ} + U_{IM}\delta_{MK}\delta_{JL} + \delta_{IK}\delta_{ML}U_{JM} + U_{IM}\delta_{JK}\delta_{ML}$$
$$+\delta_{MK}\delta_{IL}U_{MJ} + U_{MI}\delta_{MK}\delta_{JL} + \delta_{MK}\delta_{IL}U_{JM} + U_{MI}\delta_{JK}\delta_{ML})\dot{U}_{KL},$$

$$\frac{dE^G_{IJ}}{dU_{KL}} = \frac{1}{4}(\delta_{IK}U_{LJ} + U_{IK}\delta_{LJ} + U_{IL}\delta_{KJ} + \delta_{IL}U_{KJ}).$$

Biot strain:

$$E^B_{IJ} = \frac{1}{2}(U_{IJ} + U_{JI}) - \delta_{IJ},$$
$$\dot{E}^B_{IJ} = \frac{1}{2}(\dot{U}_{IJ} + \dot{U}_{JI}) = \frac{1}{2}(\delta_{IK}\delta_{JL} + \delta_{JK}\delta_{IL})\dot{U}_{KL},$$
$$\frac{dE^B_{IJ}}{dU_{KL}} = \frac{1}{2}(\delta_{IK}\delta_{JL} + \delta_{JK}\delta_{IL}).$$

Hill strain:

$$E^H_{IJ} = \delta_{IJ} - \frac{1}{2}(U^{-1}_{IJ} + U^{-1}_{JI}),$$
$$\dot{E}^H_{IJ} = \frac{1}{2}(U^{-1}_{IK}\dot{U}_{KL}U^{-1}_{LJ} + U^{-1}_{JK}\dot{U}_{KL}U^{-1}_{LI}),$$
$$\frac{dE^H_{IJ}}{dU_{KL}} = \frac{1}{2}(U^{-1}_{IK}U^{-1}_{LJ} + U^{-1}_{JK}U^{-1}_{LI}).$$

Karni strain:

$$E^K_{IJ} = \frac{1}{2}\left[\delta_{IJ} - \left(\frac{U^{-1}_{IM} + U^{-1}_{MI})}{2}\right)\left(\frac{U^{-1}_{MJ} + U^{-1}_{JM})}{2}\right)\right],$$

$$\dot{E}^K_{IJ} = -\frac{1}{8}(\dot{U}^{-1}_{IM}U^{-1}_{MJ} + U^{-1}_{IM}\dot{U}^{-1}_{MJ} + \dot{U}^{-1}_{IM}U^{-1}_{JM} + U^{-1}_{IM}\dot{U}^{-1}_{JM}$$
$$+\dot{U}^{-1}_{MI}U^{-1}_{MJ} + U^{-1}_{MI}\dot{U}^{-1}_{MJ} + \dot{U}^{-1}_{MI}U^{-1}_{JM} + U^{-1}_{MI}\dot{U}^{-1}_{JM})$$
$$= \frac{1}{8}(U^{-1}_{IK}U^{-1}_{LM}U^{-1}_{MJ} + U^{-1}_{IM}U^{-1}_{MK}U^{-1}_{LJ} + U^{-1}_{IK}U^{-1}_{LM}U^{-1}_{JM} + U^{-1}_{IM}U^{-1}_{JK}U^{-1}_{LM}$$
$$+U^{-1}_{MK}U^{-1}_{LI}U^{-1}_{MJ} + U^{-1}_{MI}U^{-1}_{MK}U^{-1}_{LJ} + U^{-1}_{MK}U^{-1}_{LI}U^{-1}_{JM} + U^{-1}_{MI}U^{-1}_{JK}U^{-1}_{LM})\dot{U}_{KL},$$

$$\frac{dE^K_{IJ}}{dU_{KL}} = \frac{1}{4}(U^{-1}_{IK}U^{-1}_{LM}U^{-1}_{MJ}+U^{-1}_{IM}U^{-1}_{MK}U^{-1}_{LJ}+U^{-1}_{MK}U^{-1}_{LI}U^{-1}_{MJ}+U^{-1}_{IM}U^{-1}_{JK}U^{-1}_{LM}).$$

"Rubber" strains

$$\frac{dE^P_{IJ}}{dU_{KL}} = \frac{1}{2}\left(\frac{dE^B_{IJ}}{dU_{KL}} + \frac{dE^H_{IJ}}{dU_{KL}}\right),$$

$$\frac{dE^M_{IJ}}{dU_{KL}} = \frac{1}{2}\left(\frac{dE^G_{IJ}}{dU_{KL}} + \frac{dE^K_{IJ}}{dU_{KL}}\right),$$

$$\frac{dE^W_{IJ}}{dU_{KL}} = \frac{1}{3}\left(\frac{dE^B_{IJ}}{dU_{KL}} + 2\frac{dE^K_{IJ}}{dU_{KL}}\right),$$

$$\frac{dE^R_{IJ}}{dU_{KL}} = \frac{1}{3}\left(2\frac{dE^G_{IJ}}{dU_{KL}} + \frac{dE^H_{IJ}}{dU_{KL}}\right).$$

Appendix E. Generalized stress representation: an alternate approach

The analysis of Sect. 2.3 (with its limitations) leads to the definition of a *generalized* material stress measure as an *isotropic* symmetric tensor function **S** of three *objective* symmetric arguments arranged in order: a stretch tensor on the left to perform the force transformation, generically denoted by **U**, the rotated stress **T** in the center, as preferred stress representative; another stretch tensor to the right to carry out the surface transformation, distinguished by **V** to avoid any confusion with the force transformation, although finally **V** = **U**. In addition, the stress function must be *linear* in $\mathbf{T}$, coincide with $\mathbf{T}$ in the reference configuration where $\mathbf{U} = \mathbf{I}$ and remain in one to one correspondence with $\mathbf{T}$ away from it, i.e for all $\mathbf{U}$. More specifically, a generalized stress may be characterized by

$$
\text{(A.1)} \quad
\begin{aligned}
&\mathbf{S} = \mathbf{S}(\mathbf{U}, \mathbf{V}, \mathbf{T}) && \text{(objecitvity)},\\
&\mathbf{S} = \mathbf{S}^T && \text{(symmetry)},\\
&\mathbf{R}\mathbf{S}(\mathbf{U}, \mathbf{T}, \mathbf{V})\mathbf{R}^T = \mathbf{S}(\mathbf{R}\mathbf{U}\mathbf{R}^T, \mathbf{R}\mathbf{T}\mathbf{R}^T, \mathbf{R}\mathbf{V}\mathbf{R}^T) && \text{(isotropy)},\\
&\mathbf{S}(\mathbf{U}, a\mathbf{T} + \mathbf{T}_0, \mathbf{V}) = a\mathbf{S}(\mathbf{U}, \mathbf{T}, \mathbf{V}) + \mathbf{S}(\mathbf{U}, \mathbf{T}_0, \mathbf{V}) && \text{(}\mathbf{T}\text{-linearity)},\\
&\mathbf{S}(\mathbf{U}, \mathbf{T}, \mathbf{V} = \mathbf{U}; \text{ordered}) && \text{(}\mathbf{U}\text{-order)},\\
&\mathbf{S}(\mathbf{I}, \mathbf{T}, \mathbf{I}) = \mathbf{T} && \text{(consistency)},\\
&\exists \mathbf{T} = \mathbf{S}^{-1}(\mathbf{U}, \mathbf{S}, \mathbf{V}) = \mathbf{T}(\mathbf{U}, \mathbf{S}, \mathbf{V}) && \text{(regularity)},
\end{aligned}
$$

where $\mathbf{R}$ denotes an arbitrary rotation $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and a an arbitrary scalar.

The selective dependence on objective variables $\mathbf{U}$, $\mathbf{T}$ guarantees the construction of a frame-indifferent measure. Symmetry is a desirable but not indispensable feature. The isotropy requirement corresponds to the need to arrive at a body-indifferent measure. The choice of $\mathbf{T}$ (rather than $\mathbf{S}^B$) is suggested by its pivotal role. Both the force and the surface transformations $\mathbf{U}$ and $\mathbf{V} = \mathbf{U}$ are retained in the argument list of $\mathbf{S}$ because, in spite of their materialization into the same stretch tensor $\mathbf{U}$ when applied to the objective rotated stress $\mathbf{T}$, distinguishing them through their order of application remains essential; symbolically, $\mathbf{S}(\mathbf{U},\mathbf{T},\mathbf{U};$ in order$) \neq \mathbf{S}(\mathbf{U},\mathbf{T};$ in disorder$)$. The linearity in $\mathbf{T}$ is dictated by the imperative need to generate a tensor with the correct dimension of a stress (keeping in mind that the stretch tensor is adimensional). The last two conditions aim at insuring a consistent definition of stress for infinitesimal deformations and its unicity during large deformations.

Theorems of representation of symmetric isotropic tensor functions in several symmetric arguments [53, 54, 55, 35] are valuable guides for establishing general forms of such stress functions which automatically satisfy the above isotropy requirement (the most stringent of all).

In essence, they provide complete and irreducible lists of elementary combinations of the arguments (usually basic products such as $\mathbf{T}$, $\mathbf{UT}$, $\mathbf{UTV}$ etc.), called *generators* and denoted $\mathbf{S}^a$, which form a *basis* for the construction of the stress function

$$
\text{(A.2)} \qquad \mathbf{S} = s_a \mathbf{S}_a, \qquad a = 1, A \text{ finite},
$$

where the *coefficients* s_α are scalar functions of specific invariants of the arguments (usually product traces such as $\mathrm{tr}\mathbf{U}$, $\mathrm{tr}\mathbf{UV}$, etc.).

Of course, such representations are not unique. They depend on the class of functions (polynomials, rational, general functions) admitted for the coeficients as well as on the type of tensor combinations (products, "quotients", i.e. products including inverses) retained for the generators.

For polynomial coefficients and product generators, the results are well established [53] and widely used. By resorting to general coefficients (instead of polynomials) the number of product generators needed to form a basis can be significantly reduced [56, 53]. However, no explicit result could be found for "quotient" generators (such as $\mathbf{U}^{-1}\mathbf{T}, \mathbf{UTV}^{-1}$, etc.) with, say, rational coefficients. This is unfortunate because such generators seem particularly adapted to the representation of the generalized stress function with regard to the forms assumed by the classical stresses already obtained by ordinary means. A rigorous establishment of a proper rational representation for the stress function is beyond the scope of this paper, especially if we consider the length and difficulty which characterize the proofs of such theorems, even in the polynomial case.

Consequently, only the outline of a possible approach is sketched below, because it will prove instructive for comparison purposes.

A promising method for obtaining the desired representation of the stress function $\mathbf{S}(\mathbf{U}, \mathbf{T}, \mathbf{V})$ begins with the construction of a fictitious potential of the form $\tilde{\mathbf{W}} = \mathrm{tr}[\mathbf{S}(\mathbf{U},\mathbf{T},\mathbf{V})\mathbf{D}] = \tilde{\mathbf{W}}(\mathbf{U},\mathbf{T},\mathbf{V},\mathbf{D})$, where $\mathbf{D}$ is an arbitrary symmetric tensor and tr denotes the trace operator[10] defined by $\mathrm{tr}(\mathbf{SD}) = \mathbf{S}{:}\mathbf{D} = S_{IJ}D_{IJ}$. At this point, the arbitrary tensor $\mathbf{D}$ bears no relation to the rotated rate of deformation (2.40). The symmetric stress tensor can obviously be derived from this potential, linear in $\mathbf{D}$, according to

$$\text{(A.3)} \qquad \mathbf{S} = \frac{\partial \tilde{\mathbf{W}}}{\partial \mathbf{D}} = \frac{\partial \tilde{\mathbf{W}}}{\partial \mathbf{D}^T}, \qquad \tilde{\mathbf{W}} = \mathrm{tr}(\mathbf{SD}).$$

On the hypothesis that the stress function $\mathbf{S}$ is isotropic, the scalar potential $\tilde{\mathbf{W}}$ is easily shown to remain invariant under an arbitrary rotation $\mathbf{R}$, meaning that

$$\tilde{\mathbf{W}}(\mathbf{U},\mathbf{T},\mathbf{V},\mathbf{D}) = \tilde{\mathbf{W}}(\mathbf{RUR}^T,\ \mathbf{RTR}^T,\ \mathbf{RVR}^T,\ \mathbf{RDR}^T),$$

---

[10] It is recalled that the trace of a matrix product is unaltered by a cyclic permutation of the factors of the product, e.g. $\mathrm{tr}(\mathbf{UTV}) = \mathrm{tr}(\mathbf{TVU})$, and is equal to the trace of the transpose of the product, e.g. $\mathrm{tr}(\mathbf{UTV}) = \mathrm{tr}(\mathbf{V}^T\mathbf{T}^T\mathbf{U}^T)$ ($= \mathrm{tr}(\mathbf{VTU})$ for symmetric matrices).

since

$$\bar{\mathbf{W}} = \mathrm{tr}(\mathbf{RSR}^T\mathbf{RDR}^T) = \mathrm{tr}(\mathbf{SD}) = \tilde{\mathbf{W}}.$$

Therefore, the original problem is replaced by the search for a representation of an invariant *scalar* function in four arguments $\mathbf{W}(\tilde{\mathbf{U}},\mathbf{T},\mathbf{V},\mathbf{D})$, linear in $\mathbf{T}$ and $\mathbf{D}$).

At this point it is *inferred* (without proof) that a complete and irreducible (?) list of basic "quotients" which are bilinear in $\mathbf{TD}$, respect the order $\mathbf{UTVD}$, involve the three consecutive powers $\mathbf{U}$, $\mathbf{I}$, $\mathbf{U}^{-1}$ (instead of $\mathbf{I}$, $\mathbf{U}$, $\mathbf{U}^2$) for $\mathbf{U}$ and $\mathbf{V}$, and the traces of which form a rational basis valid for a rational representation of $\tilde{\mathbf{W}}$, is

$$\mathbf{TD},\ \mathbf{UTD},\ \mathbf{TUD},\ \mathbf{U}^{-1}\mathbf{TD},\ \mathbf{TU}^{-1}\mathbf{D}$$

$$\mathbf{UTUD},\ \mathbf{U}^{-1}\mathbf{TUD},\ \mathbf{UTU}^{-1}\mathbf{D},\ \mathbf{U}^{-1}\mathbf{TU}^{-1}\mathbf{D}.$$

A proof of this conjecture could follow the procedure used by [53] for polynomials by rewriting the Cayley–Hamilton theorem and its generalization due to Rivlin at one degree lower, and tracing the consequences this modification has on all the reducibility relationships of product traces which occur in the process.

A list of asymmetric "quotient" generators for $\mathbf{S}$ is readily derived from the above merely by deleting the last factor $\mathbf{D}$ in each product, as the visual result of differentiation. It is emphasized that (unlike the strain) even if the stress function (of several symmetric arguments) is isotropic, the resulting generalized stress is not necessarily symmetric, this must be brought about separately. Symmetrization of the resulting elements leads to the final *stress representation*:

$$\text{(A.4)}\qquad \begin{aligned}\mathbf{S} = {} & s_1\mathbf{U}^{-1}\mathbf{TU}^{-1} + s_2(\mathbf{U}^{-1}\mathbf{T} + \mathbf{TU}^{-1}) \\ & + s_3\mathbf{T} + s_4(\mathbf{UTU}^{-1} + \mathbf{U}^{-1}\mathbf{TU}) + s_5(\mathbf{UT} + \mathbf{TU}) + s_6\mathbf{UTU},\end{aligned}$$

where the coefficients s_a are expected to be *rational* functions of the basic invariants $\mathrm{tr}\mathbf{U}$ and $\mathrm{tr}\mathbf{U}^{-1}$. A definition of stress based on two arguments only $\mathbf{S}(\mathbf{U}, \mathbf{T})$ would lead to the incomplete representation: $\mathbf{S} = s_2(\mathbf{UT} + \mathbf{TU}) + s_3\mathbf{T} + s_5(\mathbf{U}^{-1}\mathbf{T} + \mathbf{TU}^{-1})$.

The representation (A.4) has the reassuring characteristic of being based on the symmetrized classical stresses (3.21) established separately on a firm basis:

$$\mathbf{S} = s_1\mathbf{S}^G + 2s_2\mathbf{S}^B + s_3\mathbf{T} + 2s_4\mathbf{S}^A + 2s_5\mathbf{S}^H + s_6\mathbf{S}^K.$$

The hypothesis of regularity $(A.1)_7$ assumes the existence of an inverse representation in the same form

$$\text{(A.5)} \qquad \mathbf{T} = t_1\mathbf{USU} + t_2(\mathbf{US} + \mathbf{SU}) + t_3\mathbf{S} + t_4(\mathbf{USU}^{-1} + \mathbf{U}^{-1}\mathbf{SU}) + t_5(\mathbf{U}^{-1}\mathbf{S} + \mathbf{SU}^{-1}) + t_6\mathbf{U}^{-1}\mathbf{SU}^{-1}.$$

This inverse representation is expected to be better suited to the eventual formulation of the equation of equilibrium $\mathrm{Div}(\mathbf{RTU}^{-1}) = 0$.

The generalized stress coefficients (A.4) and its inverse (A.5) may be assumed constant for simplicity. An immediate and regrettable consequence of this shortcut is that $(A.5)_5$ ceases to be the exact inverse of (A.4) except in few particular cases already encountered. By way of compensation, the consistency condition $(A.1)_6$, and its obvious corollary for the inverse become easier to use,

$$\text{(A.6)} \qquad \begin{aligned} s_1 + 2s_2 + s_3 + 2s_4 + 2s_5 + s_6 &= 1, \\ t_1 + 2t_2 + t_3 + 2t_4 + 2t_5 + t_6 &= 1. \end{aligned}$$

These relations convey that the linear combinations (A.4) and (A.5) are in fact *convex* combinations (provided $0 \leq s_a,\ t_a \leq 1$). However, the resulting representations remain indeterminate.

The resemblance between the fictitious potential $\tilde{\mathbf{W}} = \mathrm{tr}(\mathbf{SD})$ introduced above and the stress power $\dot{\mathbf{W}} = \mathrm{tr}(\bar{\mathbf{S}}\dot{\mathbf{E}})$ postulated in Section 2.4 is striking, both being defined as the trace of an invariant bilinear product. An obvious difference, however, is the replacement of the arbitrary (symmetric) *anonymous* tensor $\mathbf{D}$ by the specific (symmetric) strain *rate* $\dot{\mathbf{E}}$. Consequently, the conjugate stress is expected to have a much shorter representation than (A.4) and, in fact, be directly connected to the generalized strain rate expression by "trace reflexion".

4. Conclusion

In this paper, the concepts of generalized strain and generalized stress have been examined in detail. General representations of the corresponding strain and stress functions have been established. An interesting pair of conjugate measures, referred to by the name of Mooney, has been put forward by this approach. But more important, the analysis has demonstrated the superiority of congruent strains and stresses and revealed the coexistence of different definitions of conjugacy. However, this study has failed to disclose a satisfactory substitute for the natural strain-rotated stress misfit. The classical pair formed by the Green strain and second Piola – Kirchhoff stress remains the best choice.

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Streszczenie

UOGÓLNIONE MIARY ODKSZTAŁCENIA I NAPRĘŻENIA KRYTYCZNY PRZEGLĄD I NOWE REZULTATY

Zaproponowano cztery podstawowe zasady: obiektywności, izotropii, zgodności i regularności na ograniczenie pojęć uogólnionych odkształceń i – bardziej pierwotnie – uogólnionych naprężeń. Zasady te zastosowano do wyprowadzenia dwóch ogólnych reprezentacji odpowiednich funkcji odkształcenia i naprężenia. Opierając się na materialnej definicji skoniugowania, każde odkształcenie związano w sposób wzajemnie jednoznaczny z pewnym skoniugowanym naprężeniem i vice versa. Oprócz klasycznych, znanych z literatury, par "odkształcenie-naprężenie", otrzymano interesującą rodzinę nowych odkształceń i skoniugowanych naprężeń. Jednakże dwa główne rezultaty pracy dotyczą wykazania wyższości pewnej szczególnej klasy miar odkształcenia i naprężenia, zwanej "kongruentną" oraz unaocznienie współistnienia różnych definicji skoniugowania, co jest przyczyną nieporozumień.

Резюме

ОБОБЩЕННЫЕ МЕРЫ ДЕФОРМАЦИИ И НАПРЯЖЕНИЯ КРИТИЧЕСКОЕ ОБОЗРЕНИЕ И НОВЫЕ РЕЗУЛЬТАТЫ

Предложены четыре основные принципы: объективности, изотропии, согласованности и регулярности на ограничения понятий обобщенных деформаций и, более первично, обобщенных напряжений. Эти принципы применены для вывода дбух общих представлений соответствующих функций деформации и напряжения. Базируя на материальном определении сопряжения, каждая деформация связана взаимно однозначным образом с некоторым сопряженным напряжением и наоборот. Кроме классических, известных из литературы, пар деформация - напряжение, получено интересующее семейство новых деформаций и сопряженных напряжений. Однако два главных результата работы касаются доказательства превосходства некоторого особого класса мер деформации и напряжения, называемого "конгруэтным" и наглядного показания существования разных определений сопряжения, что дает повод для недоразумений.

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