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VIBRATION OF MULTI-CHAMBER SHELL STRUCTURES WITH DISCONTINUOUSLY VARIABLE CROSS-SECTIONS

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The paper presents the general integral of a set of differential equations describing vibrations of a multichamber prismatic shell made of a linear-elastic orthotropic material. The solution is used to construct the equations of forced vibrations of a structure consisting of a series of interconnected shells of various cross-sections. Displacement amplitudes of such shell structures subject to stationary vibrations have been found, together with the resonance frequencies and the principal vibration modes.

1. Introduction

The problem of creating a simple mathematical model allowing to describe the motion of complex engineering structure, is being extensively explored. Due to simplicity of calculations and clear description, the bar models are in common use. The possibility of obtaining the exact solution of differential equations of motion (or equilibrium) of a given structure is one the merits of these models. However, the results obtained differ from the real ones because the model cross-sections are assumed to be rigid, while in practice they are usually deformed.

Following the development of computer technology, discrete models with many degrees of freedom have been developed leading to application of the finite element method. In spite of the fact that this method made it possible to solve many practical, problems, it has many disadvantages such as, for example, long time of calculations and the necessity of using powerful computers. Application of the finite element method usually leads to approximate solutions. In the case of vibrations, description of the phenomena is not so clear as in the case of the bar models.

The disadvantages of the bar models and the discrete models of many degrees of freedom resulted in developing other calculation models, more accurate than the bar models, less time-consuming and more clear than the discrete models of many degrees of freedom. One of these models is a frame-shell model proposed by V.Z. Vlasov [1] for the static calculations of thin-walled structures of closed cross-sections. Practical advantages of the Vlasov theory decided on its application to the static calculations of shell structures of complicated multicircuit cross-sections [2, 3], and to the description of vibrations [4].

Construction of a possibly accurate and effective method of integration of the systems of differential equations which describe the motion or equilibrium of a structure is one of the basic problems connected with the applications of Vlasov's theory. The step methods [2, 3, 4] which have been used thus far proved to be misleading when applied to the structures of more complex forms and, moreover, the methods are approximate like the finite element methods.

This paper presents an analytical solution of the system of differential equations written in the matrix form and describing steady harmonic vibrations of a multi-chamber prismatic shell. Determination of this solution made it possible to extend the practical applications of the frame-shell theory to the complex structures of discontinuously (jump-like) variable cross-sections.

2. Integration of equations of motion

The basic feature of the Vlasov theory consists in projecting the displacement vector of a point of the middle surface of a shell onto three directions: binormal, tangent and normal to the cross-section contour, and in writing them in the form of polynomials of two variables:

$$u = \sum_{i=1}^{N} \nu_i \varphi_i \,, \qquad v = \sum_{k=1}^{R} \vartheta_k \psi_k \,, \qquad w = \sum_{k=1}^{R} \vartheta_k \chi_k \,, \tag{2.1}$$

where $\varphi_i, \psi_k, \chi_k$, called the shape functions, are assumed to be given, while the functions ν_i, ϑ_k of position z and time t, called the generalized Vlasov coordinates, are to be determined. Such a description, with appropriately selected shape functions, takes into account the effects of deformation of the cross-sections of a structure [4]. N is the number of the coordinates describing the longitudinal displacements, R is the number of the coordinates defining the displacements in the transverse directions.

The equation of amplitudes of the Vlasov shell structure, made of linear-elastic material and performing steady forced harmonic vibrations, is as follows [4, p.473]:

$$\mathbf{B}\mathbf{y}'' + \mathbf{C}\mathbf{y}' + \mathbf{G}\mathbf{y} = \mathbf{g} \,, \tag{2.2}$$

where $\mathbf{y}$ is a column matrix of the amplitudes of the generalized displacements $\nu_i, \vartheta_k, \mathbf{g}$ is a column matrix of the generalized excitation forces,

$$\mathbf{B} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{R} \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & \mathbf{H} \\ \mathbf{K} & 0 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & 0 \\ 0 & \mathbf{G}_2 \end{bmatrix}$$

are square matrices of real constants elements depending on the geometry of the cross-sections, the distribution of masses and the material characteristics of the structure,

$$\mathbf{G}_1 = \omega^2\mathbf{U} - \mathbf{S}\,, \qquad \mathbf{G}_2 = \omega^2\mathbf{Z} - \boldsymbol{\Xi}\,.$$

Symbol ω denotes the angular frequency of vibrations; the primes denote derivatives with respect to the variable z. The matrices $\mathbf{B}$ and $\mathbf{G}$ are symmetrical, while the matrix $\mathbf{C}$ is anti-symmetric. It is easily seen that for $\omega = 0$ Equation (2.2) describes the state of equilibrium.

By substituting $\mathbf{y}' = \mathbf{z}$, this equation can be presented in a form of the system of the first order differential equations,

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{d}\,, \tag{2.3}$$

where

$$\mathbf{x} = \{\mathbf{y}, \mathbf{z}\}^T\,, \qquad \mathbf{d} = \{\mathbf{0}, \mathbf{f}\}^T\,, \qquad \mathbf{f} = \mathbf{B}^{-1}\mathbf{g}$$

are the column matrices, and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{P}_1 & \mathbf{0} & \mathbf{0} & \mathbf{Q}_2 \\ \mathbf{0} & \mathbf{R}_1 & \mathbf{Q}_1 & \mathbf{0} \end{bmatrix}$$

is a square matrix of dimension $n = 2(N + R)$, (twice the number of the generalized Vlasov coordinates), while $\mathbf{I}$ is a unit matrix, $\mathbf{P}_1 = -\mathbf{P}^{-1}\mathbf{G}_1$, $\mathbf{R}_1 = -\mathbf{R}^{-1}\mathbf{G}_2$, $\mathbf{Q}_1 = -\mathbf{R}^{-1}\mathbf{K}$, $\mathbf{Q}_2 = \mathbf{P}^{-1}\mathbf{H}$. The elements of matrix $\mathbf{A}$ are continuous and, in the case of a prismatic shell, they are constant. A homogeneous system of the differential equations corresponding to Eq. (2.3)

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{2.4}$$

is equivalent to the system [6, p. 130]:

$$\frac{d\mathbf{X}}{dz} = \mathbf{A}\mathbf{X}\,, \tag{2.5}$$

where

$$\mathbf{X} = e^{\mathbf{A}z}\,, \tag{2.6}$$

is called the integral matrix of differential equation (2.4). The matrix exponential function on the right-hand side of Eq. (2.6) is defined as follows [5, p.105]

$$\exp\,\mathbf{A} = e^{\mathbf{A}} = \sum_{k=1}^{\infty} \frac{\mathbf{A}^k}{k!}\,. \tag{2.7}$$

The determinant of the integral matrix (2.6), according to [6, p.149], equals

$$\det \mathbf{X} = \det e^{\mathbf{A}z} = e^{\mathrm{Tr}\mathbf{A}z} = 1$$

because in our case, the trace of the matrix **A** is

$$\mathrm{Tr}\mathbf{A} = \sum_{i=1}^{n} A_{ii} = 0 \,. \tag{2.8}$$

From Eq.(2.8) it is evident that the sum of the roots of the characteristic equation (eigenvalues) of matrix **A** is equal to zero,

$$\sum_{i=1}^{n} \lambda_i = 0 \,. \tag{2.9}$$

Among many methods of calculation of the eigenvalues and the eigenvectors of matrices, the iterative method of Francis, called the QR method [7, p. 369] proves to be the most accurate and the fastest one. Application of this method is connected with the necessity of reducing the matrix **A** to the Hessenberg form [7, p. 367]. Relation (2.9) enables us to control of the accuracy of solution. It has been ascertained that real and complex numbers, and also multiple roots, can appear among the eigenvalues of the matrix **A**. The investigated system has always the eigenvalues $|\lambda| < 1$. The root $\lambda = 0$ appears only when the matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ are singular, what happens when $\omega = 0$.

The integral matrix of the Equation (2.5) may then be presented in the form [6, p. 151]

$$\mathbf{X} = \mathbf{Q}\mathbf{D} \,, \tag{2.10}$$

where **Q** is a non-singular square matrix of constant elements, selected in such a way, that

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = q \,\mathrm{diag}\, [\mathbf{J}_{\rho 1}(\lambda_1), ..., \mathbf{J}_{\rho m}(\lambda_m)] \,, \tag{2.11}$$

and where

$$\mathbf{J}_{\rho i}(\lambda_i), \qquad i = 1, 2, ..., m$$

are the elementary Jordan matrices [9, p. 357]; ρi – the dimension of a matrix, m – the number of the elementary divisors of the matrix **A**.

Substituting (2.10) into (2.5) and taking into account the relation (2.11) we obtain the equation

$$\frac{d\mathbf{D}}{dz} = q \,\mathrm{diag}\, [\mathbf{J}_{\rho 1}(\lambda_1), ..., \mathbf{J}_{\rho m}(\lambda_m)]\, \mathbf{D} \,. \tag{2.12}$$

In this way, the system of n differential equations (2.5) has been separated into m groups,

$$\frac{d\mathbf{D}_{\rho i}}{dz} = \mathbf{J}_{\rho i}(\lambda_i)\mathbf{D}_{\rho i}\,, \qquad i = 1, 2, ..., m\,. \tag{2.13}$$

In each group of the equations the diagonal elements of the matrices $\mathbf{J}_{\rho i}(\lambda_i)$ are equal to the corresponding characteristic values λ_i, the near-diagonal elements over the main diagonal are equal to 1, and all the remaining elements are equal to 0. The integral matrix of the Equation (2.12) has then the form

$$\mathbf{D} = q \operatorname{diag}\left[e^{\mathbf{J}_{\rho 1}(\lambda_1)z}, ..., e^{\mathbf{J}_{\rho m}(\lambda_m)z}\right], \tag{2.14}$$

where [6, p. 151]

$$\mathbf{D}_{\rho i} = e^{\mathbf{J}_{\rho i}(\lambda_i)z} = e^{\lambda_i z}\begin{bmatrix} 1 & z & \cdots & \dfrac{z^{(\rho i-1)}}{(\rho i-1)!} \\ & 1 & & \dfrac{z^{(\rho i-2)}}{(\rho i-2)!} \\ & & \ddots & \vdots \\ & 0 & & 1 \end{bmatrix}. \tag{2.15}$$

In order to construct the matrix $\mathbf{Q}$ defined by Eq.(2.11) it is necessary to solve the systems of equations

$$(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{q}_1^{(i)} = \mathbf{0}\,, \tag{2.16}$$

$$\begin{gathered}(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{q}_j^{(i)} = \mathbf{q}_{j-1}^{(i)}\,, \\ j = 2, ..., \rho i\,, \qquad i = 1, ..., m\,,\end{gathered} \tag{2.17}$$

in which $\mathbf{q}_1^{(i)}$ is the eigenvector corresponding to the eigenvalue λ_i, $\mathbf{q}_j^{(i)}$ are the main vectors corresponding to this eigenvalue. Vectors $\mathbf{q}_1^{(i)}, ..., \mathbf{q}_{\rho i}^{(i)}$, being the columns of the matrix $\mathbf{Q}$, constitute the basis of the space C^n for the matrix $\mathbf{A}$.

The problem of solving Eq.(2.4) consists then in calculation of the eigenvalues, the eigenvectors, the elementary divisors [8, p. 528] and the principal vectors of the matrix $\mathbf{A}$. The cage-like structure of this matrix makes the task considerably easier. Once the eigenvalues, eigenvectors and the principal vectors of the matrix $\mathbf{A}$ are known, one can construct the matrices $\mathbf{D}$ and $\mathbf{Q}$ according to the formulas (2.14) – (2.17). The following function is the solution of the differential equation (2.4):

$$\mathbf{x} = \mathbf{QDc}\,, \tag{2.18}$$

where $\mathbf{c}$ denotes a vector of arbitrary constants. In our case it is the integration constants vector following from the boundary conditions. For a prismatic shell structure, the boundary conditions are separated and written in the form of a system of first-order differential equations [4, p. 470]

$$\mathbf{M}_L\mathbf{x}(z_L) = \mathbf{p}_L\,, \qquad \mathbf{M}_P\mathbf{x}(z_P) = \mathbf{p}_P\,, \tag{2.19}$$

where $\mathbf{M}_L$, $\mathbf{M}_P$ are the rectangular matrices of constant elements, and

$$\mathbf{p} = \{p_1, ..., p_N, q_1, ..., q_R\}^T \tag{2.20}$$

is the vector of amplitudes of the generalized forces [4, p.469] acting on the end cross-section of the shell. Indices L and P correspond to the left ($z = L$) or right-hand ($z = P$) boundary of the structure. The following system of algebraic equations is obtained from the relations (2.19), (2.18)

$$\begin{bmatrix}\mathbf{M}_L\mathbf{Q}\mathbf{D}(z_L)\\ \mathbf{M}_P\mathbf{Q}\mathbf{D}(z_P)\end{bmatrix}\{\mathbf{c}\} = \begin{Bmatrix}\mathbf{p}_L\\ \mathbf{p}_P\end{Bmatrix} \tag{2.21}$$

enabling the calculation of the integration constants $\mathbf{c}$.

If the integration constants are known, one can calculate the amplitudes of generalized displacements (2.18) as well as their derivatives at a cross-section of an arbitrary coordinate z, and then the real displacements (2.1) and the stresses [4, p. 363] at arbitrarily chosen points of the structure.

3. Description of vibrations of a multi-shell structure

Once the solution of a single shell is known, it becomes possible to describe the motion of a system of shells of different cross-sections joined together. The motion of every shell representing a segment of the structure, is described by an individual set of the Vlasov generalized coordinates and, consequently, by a separate system of differential equations of different orders. The problem then consists in formulation of the boundary conditions, expressing relations among the generalized coordinates on the contact surfaces between the shells.

The natural boundary conditions (2.19) for the terminal cross-section of a given shell can be written in the following form:

$$\mathbf{M}\mathbf{x} = \mathbf{p} + \mathbf{p}^*\,, \tag{3.1}$$

where $\mathbf{p}$ is the vector of amplitudes of the applied generalized forces acting on this cross-section, while $\mathbf{p}^*$ is the vector of the generalized elastic forces exerted by the neighbouring shell. Presenting the elastic forces in the form of a distributed load acting on the cross-section contour [4, p. 463]

$$q_b = \check{E}_1^*\left(\frac{\partial u^*}{\partial z} + \nu_{21}^*\frac{\partial v^*}{\partial s}\right), \qquad q_s = \check{G}^*\left(\frac{\partial u^*}{\partial s} + \frac{\partial v^*}{\partial z}\right), \tag{3.2}$$

where

$$u^* = \sum_{j=1}^{N^*} \nu_j^* \varphi_j^* , \qquad v^* = \sum_{l=1}^{R^*} \vartheta_l^* \psi_l^* \tag{3.3}$$

denote, according to Eq.(2.1), the binormal and tangent displacements of the points of the contour of the neighbouring shell, we can calculate the generalized elastic forces [4, p. 469]

$$p_i^* = \oint_S q_b \varphi_i ds , \quad q_k^* = \oint_S q_s \psi_k ds , \quad i = 1,2,...,N , \quad k = 1,2,...,R . \tag{3.4}$$

On the basis of Eqs.(3.1) – (3.4) we obtain the relation between the generalized coordinates of the contact surface of two shells,

$$\mathbf{M}\mathbf{x} = \mathbf{N}\mathbf{x}^* + \mathbf{p} \tag{3.5}$$

in which $\mathbf{x}^*$ denote the vector of the generalized Vlasov coordinates describing the motion of the neighbouring shell; here

$$\mathrm{N} = \begin{bmatrix} \mathbf{P}^* & 0 & 0 & \Theta^* \\ 0 & \mathbf{R}^* & \mathbf{Q}^* & 0 \end{bmatrix}$$

is the rectangular matrix of dimension $(N + R) \times 2(N + R)$ and of constant elements

$$P_{ij}^* = \int_S \check{E}_1^* \varphi_i \varphi_j^* \, ds , \qquad R_{kl}^* = \int_S \check{G}^* \psi_k \psi_l^* \, ds ,$$

$$Q_{kj}^* = \int_S \check{G}^* \psi_k \frac{\partial \varphi_j^*}{\partial s} \, ds , \qquad \Theta_{il}^* = \int_S \check{E}^* \varphi_i \frac{\partial \psi_l^*}{\partial s} \, ds .$$

The asterisks indicate the material characteristics, the shape functions and the generalized coordinates of the neighbouring shell. Integration is extended over the part of the contour common for both the shells.

The rule of passage across the interface between two shells (segments of the structure) with the numbers j, $j + 1$, may be written in the following form:

$$\mathbf{M}_{Pj}\mathbf{x}_{Pj} = \mathbf{N}_{Lj+1}\mathbf{x}_{Lj+1} + \mathbf{p}_{Pj} \tag{3.6}$$

and

$$\mathbf{M}_{Lj+1}\mathbf{x}_{Lj+1} = \mathbf{N}_{Pj}\mathbf{x}_{Pj} + \mathbf{p}_{Lj+1} , \tag{3.7}$$

where symbols P and L indicate the right- or the left-hand boundary of the segment.

According to Eq.(2.18), the integral of the amplitude equation of the j-th segment of the structure can be presented in the following form:

$$\mathbf{x}_j(z) = \mathbf{X}_j(z)\mathbf{c}_j \,, \tag{3.8}$$

where $\mathbf{x}_j$ (z) is the column matrix of generalized coordinates and their derivatives at the cross-section of coordinate z; $\mathbf{X}_j = \mathbf{Q}_j\mathbf{D}_j(z)$ – the integral matrix of the differential equation describing the motion of the j-th segment; $\mathbf{c}_j$ – the column matrix of the integration constants in the j-th segment ($j = 1, 2, ..., M$; M is the number of the segments of the structure).

Substitution of the solution (3.8) into the boundary conditions (3.6), (3.7) makes it possible to construct the system of linear equations determining the integration constants $\mathbf{c}_j$

$$\tilde{\mathbf{c}} = \tilde{\mathbf{A}}^{-1}\tilde{\mathbf{p}} \,, \tag{3.9}$$

where

$$\begin{aligned} \tilde{\mathbf{c}} &= \{\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_M\}^T \,, \\ \tilde{\mathbf{p}} &= \{\mathbf{p}_{1L}, \mathbf{p}_{1P}, ..., \mathbf{p}_{MP}\}^T \,, \end{aligned}$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{M}_{L1}(0) & & & \\ \mathbf{M}_{P1}(1) & -\mathbf{N}_{L2}(1) & & \\ -\mathbf{N}_{P1}(1) & \mathbf{M}_{L2}(1) & & \\ & \mathbf{M}_{P2}(2) & -\mathbf{N}_{L3}(2) & \\ & -\mathbf{N}_{P2}(2) & \mathbf{M}_{L3}(2) & \\ & & \mathbf{M}_{P3}(3) & \\ & & & \mathbf{M}_{LN}(N-1) \\ & & & \mathbf{M}_{PN}(N) \end{bmatrix} ,$$

$$\begin{aligned} \mathbf{M}_{Lj}(k) &= \mathbf{M}_{Lj}\mathbf{X}_j(z_k) \,, & \mathbf{M}_{Pj}(k) &= \mathbf{M}_{Pj}\mathbf{X}_j(z_k) \,, \\ \mathbf{N}_{Lj}(k) &= \mathbf{N}_{Lj}\mathbf{X}_j(z_k) \,, & \mathbf{N}_{Pj}(k) &= \mathbf{N}_{Pj}\mathbf{X}_j(z_k) \,, \\ j &= 1, 2, ..., M \,, & k &= 0, 1, 2, .., M \,, \end{aligned}$$

z_k denotes the coordinate z at the right-hand boundary of the k-th segment ($k = 0$ indicates the left-hand boundary of the first segment).

When the integration constants are known, it is possible to calculate the amplitudes (3.8) of the generalized coordinates $\mathbf{x}_j(z)$ at a cross-section of an arbitrarily chosen coordinate z, and then the real displacement (2.1), the derivatives of displacements with respect to the variable z, as well as the normal and the tangential stresses at arbitrary cross-sections.

If the frequency of the excitation forces is equal to one of the natural frequencies of the structure, then the system of differential equations (2.2) has no solution; the amplitudes of the generalized displacements become infinitely large. At the frequencies of the applied forces close to the resonance frequencies, the amplitude of the node lines of the structure assume the form of the principal modes of vibrations [4]. Approximation of the results of calculations with various frequencies ω enables the determination of the resonance frequencies and the corresponding principal modes of vibrations.

4. EXAMPLES

The algorithm presented above was the basis for a computer program enabling the automatic generation of the matrix equations of motion of the particular segments, of the boundary conditions at the interfaces between the segments and at the outer boundaries of the structure, and then enabling integration of the system of equations derived. The first order Vlasov functions [4, p. 473] have been used to express the displacements. The calculations were carried out by a IBM/AT/XT microcomputer.

The constraints imposed on the motion of the cross-sections of the individual segments of the structure may be described by means of the boundary conditions (2.19), (3.5), constituting the system of differential equations in the matrix form. According to the support conditions and the types of joint between the segments, these conditions may be formulated in many ways by using the methods of the calculus of variations [4].

Let us consider the motion of a thin-walled structure with free ends composed of three shell segments, to illustrate the influence of deformability of the cross-sections on the behaviour of the system during vibrations. Dimensions of the particular segments are presented in the Fig.1. The structure is made of isotropic material with the modulus of elasticity $E = 2.1 \times 10^5 [MN/m^2]$, shear modulus $G = 8.08 \times 10^4 \, [MN/m^2]$ and density $\rho = 0.0078 [MNs^2/m^4]$.

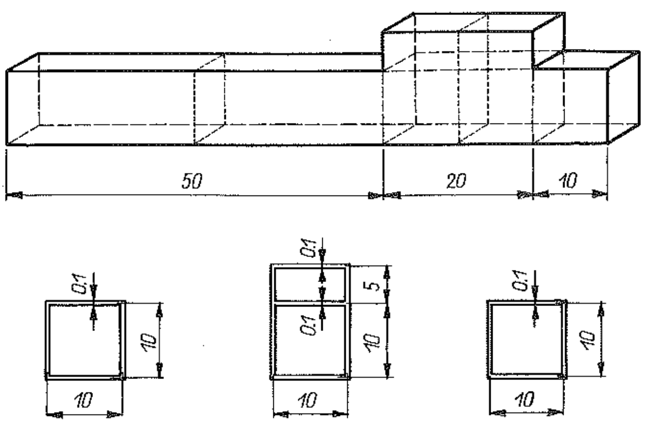

FIG. 1.

Linear distribution of displacements between the nodes of the contour is assumed, shape functions of the first kind being used. The system of differential equations (2.2) describing the vibrations of the first (and also the third) segment is a system of the 16-th order ($N = 4, R = 4$), whereas the order of the system of equations describing the vibrations of the middle segment ($N = 6, R = 5$) is equal to 22.

By imposing appropriate constraints on the node displacements and their derivatives at the end cross-sections of the segments, several variants of the boundary conditions (2.19), (3.5), corresponding to the following models of structure were formulated:

I. The structure with non-stiffened boundaries with direct (non-stiffened) joints of the segments.

II. The structure, in which the end cross-sections are undeformable (they move like rigid bodies), while the remaining cross-sections and the joints between the segments are deformable.

III. The structure, in which the end cross-sections and the cross-sections joining the segments are underformable in the transverse directions, but they are subject to warping.

IV. The structure, in which the end cross-sections and the joint cross-sections are undeformable (rigid partitions at the boundaries and at the joints of the segments).

V. The structure with the undeformable partitions at the ends and at the joints of the segments, in which the middle cross-sections of the first two segments (dotted lines in the Fig.1) are additionally stiffened by rigid partitions.

In order to demonstrate the differences between the presented frame-shell theory and the commonly known bar theories, the natural frequencies and the natural modes of the system have been determined, assuming the following calculation models of the described structure.

VI. The system of thin-walled beams taking into account shear according to the Timoshenko hypothesis;

VII. The system of thin-walled Vlasov beams (with underformable projection of the cross-section on the transversal plane, neglecting the effect of shear due to bending).

The results of calculations are shown in the Tables 1 and 2. In all the cases the first natural frequency is equal to zero – therefore it is omitted in the tables.

Table 1. Transversal vibrations in the vertical plane.

Models	Natural frequencies $[s^{-1}]$					
	2	3	4	5	6	7
I,II,III	52.48	129.03	222.20	294.57	380.85	469.20
IV,V	54.51	133.63	226.15	310.06	395.51	487.80
VI	55.16	134.44	221.40	308.35	395.31	482.26
VII	78.59	236.21	471.86	732.69	1086.58	1514.60

According to the frame-shell theory, the cross-sections of the middle segment are deformed during transversal vibrations in the vertical plane. Adoption of the calculation models (I, II, III) allowing for such deformations leads to the determination of the sequence of the lowest natural frequencies (Table 1) . Introduction of the partitions in the joints of the segments (model IV and V) increases the natural frequencies of the system by 1.2% – 5.3%.

The frame-shell model with partitions at the end cross-sections and in the joints of the segments (IV, V) leads, in the case of transversal vibrations, to results close to those based on the Timoshenko model (VI); in case of the first seven natural frequencies, the differences are not greater than 3%. On the other hand, elimination of shear distortion in the motion equation (model VII) leads to results considerably different from those based on other models, because at the small slenderness ratio of the beam structure (in this case 1 : 5.3) the effect of shear distortion is not negligible.

Table 2. Transversal vibrations in the horizontal plane coupled with torsional vibrations and with deformation of the cross-sections.

Natural freq. [1/s]	Model						Dominant form
	I	II	III	IV	V	VII	
2	35.11	—	—	—	—	—	deformations
3	52.52	52.49	53.73	53.60	53.70	75.74	bending
4	96.11	89.88	—	—	—	—	deformations
5	100.93	100.90	98.97	99.01	99.11	100.27	torsion
6	125.70	125.37	125.46	128.79	128.87	220.31	bending
7	181.22	156.63	—	—	—	—	deformations
8	200.89	196.94	224.55	225.40	226.30	234.41	torsion
9	214.09	213.22	211.92	215.28	215.58	410.07	bending
10	237.62	217.80	—	—	—	—	deformations
11	291.32	289.19	—	—	—	—	deformations
12	300.79	299.98	299.80	301.95	302.35	666.00	bending
13	314.24	311.01	310.00	308.02	309.99	321.60	torsion
14	369.87	361.84	356.76	349.50	—	—	deformations
15	385.85	385.51	382.16	388.60	387.16	991.60	bending
16	407.72	407.70	416.20	416.40	414.92	418.52	torsion

In the equations based of the frame-shell theory, the effect of bending of the examined structure in the horizontal plane is coupled with the effects of the torsion and the deformation of cross-sections. Nevertheless, looking at the principal modes of vibrations corresponding to the determined resonance frequencies, main effects may be distinguished.

The results of calculations carried out with the aid of various models are presented in the Table 2. The frame-shell models (I – V) and the Vlasov model lead to similar results concerning the natural frequency series (5, 8, 13, 16) connected with torsion. The natural frequencies (3, 6, 9, 12, 15) corresponding to bending, calculated by means of the frame-shell models, differ considerably from the frequencies determined by means of the bar model, according to which the effect of shear distortion in the equations of motion of thin-walled bars is disregarded.

In addition to the natural frequency series connected with classical bending and torsion of the structure, known from the bar theories, the frame-shell theory disclosed an additional series (2, 4, 7, 10, 11, 14) corresponding to various deformations of the cross-sections. Numerical values of these series are small in the case of non-stiffened structure (model I), and they increase when additional stiffenings are added to the cross-sections (the models II – V).

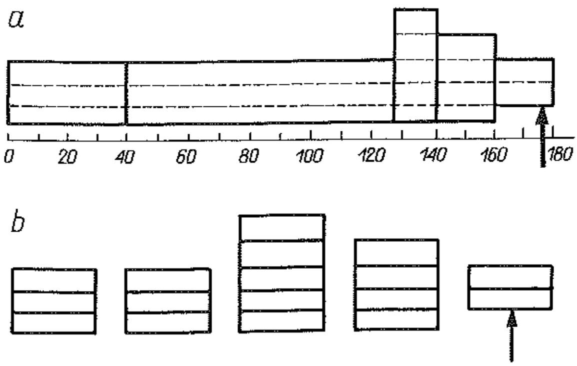

FIG. 2. Example of a multichamber thin-walled structure built from five segments: a) the view, b) the cross-sections.

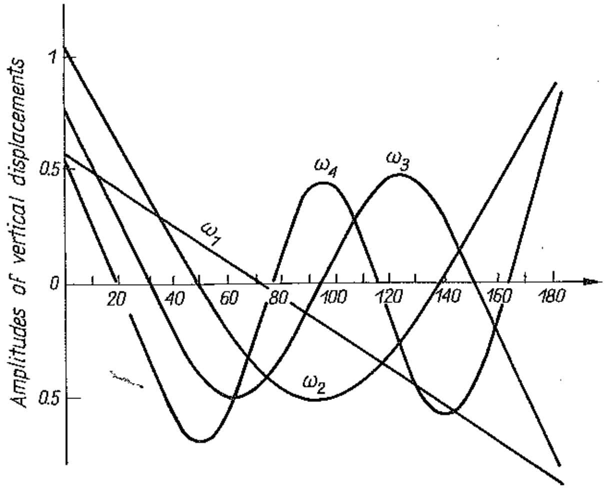

FIG. 3. Principal modes of transversal vibrations of the structure, determined on the basis of the frame-shell theory.

Let us discuss the results concerning forced vibrations of the structure with free ends representing a simplified model of a hull of a ship (Fig.2), as an example showing the practical applications of the frame-shell theory. The structure consists of five prismatic shell segments. Each segments is made of orthotropic plates of various thicknesses and material characteristics. Fig.2b shows the contours of cross-sections of these five segments.

A vertical harmonic force applied to the end part of the last segment, acting in the plane of symmetry of the structure, produces transversal vibrations of the system, coupled with symmetrical deformations of the cross-sections. The structure gets into resonance at the following frequencies of the applied forces: $\omega_1 = 0$, $\omega_2 = 14.01$, $\omega_3 = 33.00$, $\omega_4 = 55.52$, $\omega_5 = 79.42$, $\omega_6 = 98.71, \ldots, [s^{-1}]$, equal to the natural frequencies of the system. The principal modes of vibration (Fig.3) known also, as far as the vertical displacements are concerned, from the beam theory, correspond to the determined resonance frequencies. The differences between the beam models and the Vlasov frame-shell model consist in exhibiting, according to the latter theory, the longitudinal deformations of the cross-sections. Deformations of the end cross-sections of the consecutive segments of the structure, corresponding to the third resonance frequency, are shown in Fig.4.

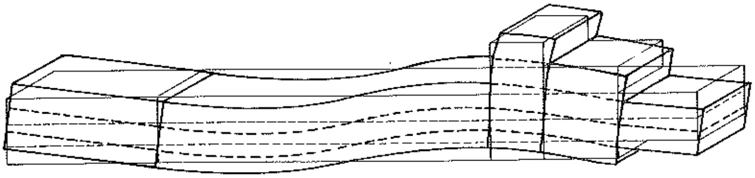

FIG. 4. Third mode of the transversal vibrations in the vertical plane.

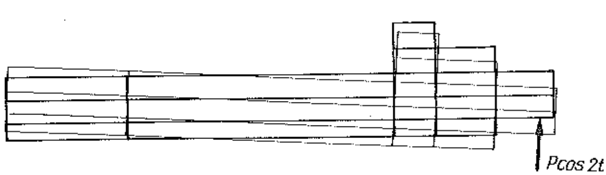

FIG. 5. Structure response to the vertical harmonic excitation of frequency $\omega = 2\ [s^{-1}]$.

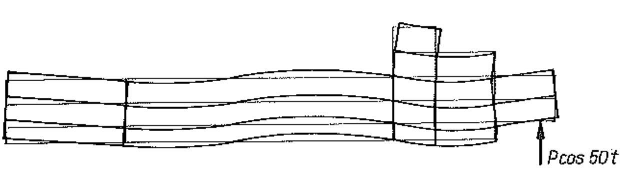

FIG. 6. Structure response to the vertical harmonic excitation of frequency $\omega = 50\ [s^{-1}]$.

Figures 5 and 6 show the responses of the structure to the excitation by a vertical harmonic force of random frequencies. At low frequencies of

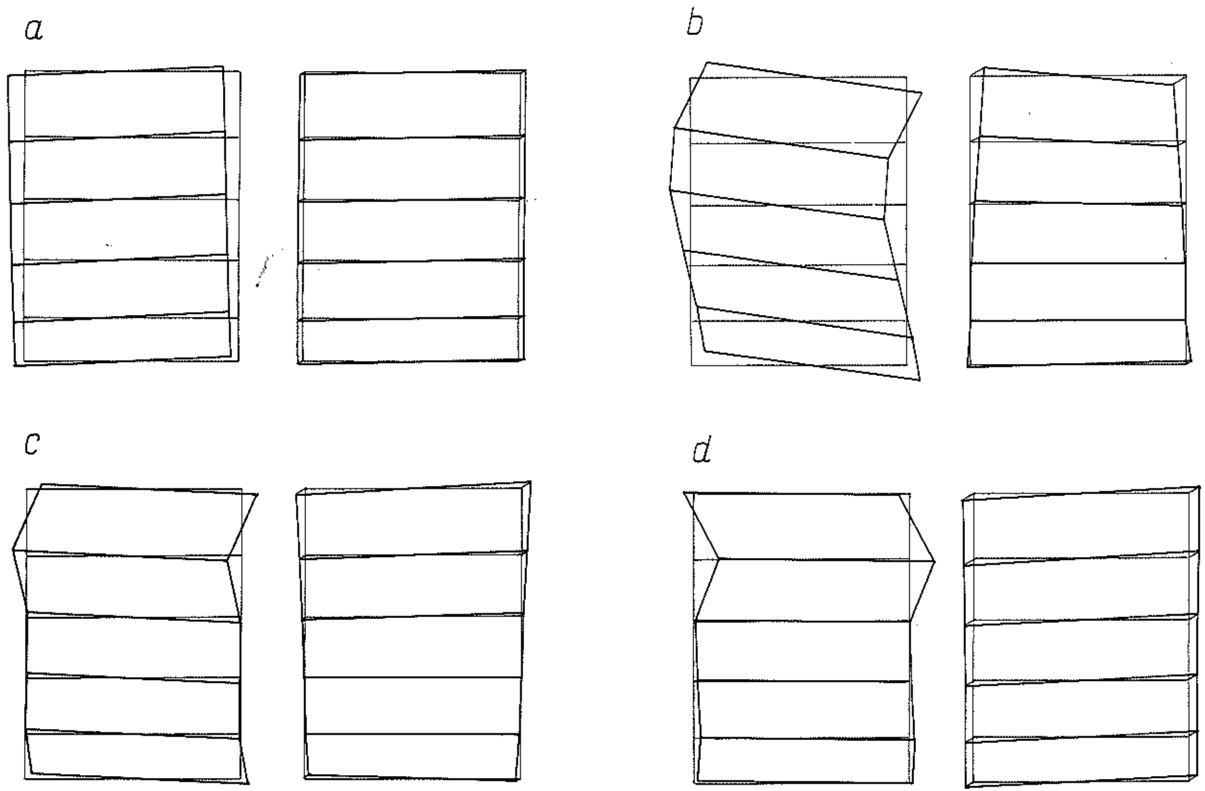

FIG. 7. Transversal and longitudinal deformations of the end cross-section ($z = 128$ m) of the third segment, corresponding to various frequencies of transeversaltorsional vibrations a/ $\omega^* = 1$, b/ $\omega^* = 10.2$, c/ $\omega^* = 16.75$, d/ $\omega^* = 21.5$ $[s^{-1}]$.

vibrations (Fig.5) one can observe swinging of the entire structure which moves as a rigid body, at the higher ones-the structure performs flexural vibrations connected with the deformations of the cross-sections (Fig.6).

The horizontal harmonic force, acting at the lower part of the fourth segment, produces flexural-torsional vibrations of the system, coupled with longitudinal and transverse deformations (warping) of the cross-sections. In this case, a new, infinite series of resonance frequencies has been obtained: $\omega_1^* = 0$, $\omega_2^* = 10.25$, $\omega_3^* = 16.75$, $\omega_4^* = 21.50$, $\omega_5^* = 36.11$, $\omega_6^* = 39.69$, $\omega_7^* = 45.22$, $\omega_8^* = 62.79$, $\omega_9^* = 65.15$, $\omega_{10}^* = 68.89$, $\omega_{11}^* = 79.62$, $\omega_{12}^* = 101.1$, $\omega_{13}^* = 103.33$, $\omega_{14}^* = 132.7$, $\omega_{15}^* = 140.95, ..., s^{-1}$.

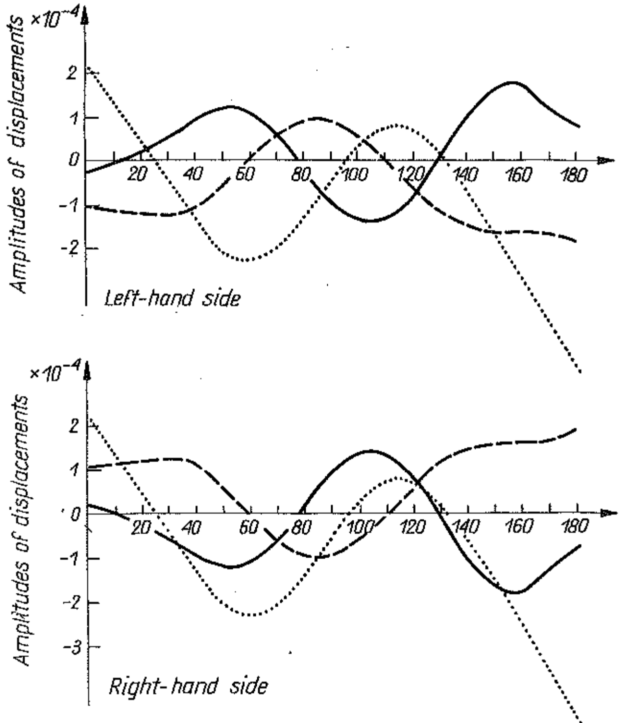

FIG. 8. Amplitudes of displacements of the upper nodal lines, caused by the horizontal harmonic force, applied at the lower part of the fourth segment ($z = 160$ m), penpendicularly to the axis of the structure. The displacements — vertical, ··· horizontal, - - - longitudinal.

Various principal modes of vibration correspond to each of the determined frequencies. Vibrations with the frequencies $\omega_2^* - \omega_4^*$ are the two-node vibrations (two points of the structure are immobile during vibrations), the modes corresonding to frequencies $\omega_5^* - \omega_8^*$ have 3 nodes, the modes corresponding to the frequencies $\omega_9^* - \omega_{12}^*$ - 4 nodes, while the modes with frequencies $\omega_{13}^* - \omega_{15}^*$ have 5 or 6 nodes. Figure 7 shows the transverse and longitudinal deformations of the end cross-section of the third segment, corresponding to the first four modes of vibration.

The response of the structure to the excitation by a horizontal harmonic force with the frequency $\omega = 50\ [s^{-1}]$, different from the resonance frequencies, is a combination of the effects of bending in the horizontal plane, torsion and deformation of the cross-sections. The amplitudes of deformations of two upper nodal lines of the structure, caused by such an excitation, are shown in Fig.8. Employing the computer program, the values of displacements of all the elements of the structure along arbitrarily chosen nodal lines or cross-sections appear on the monitor screen. The time needed for solving the presented problem, including the automatic generation of the motion equation matrices of the individual segments of the structure and integration of these equations on the IBM/AT microcomputer, amounts to 9 minutes. However, the attempts made to integrate the equations by means of the step method did not succeed due to low accuracy of these methods. The tests carried out on simpler examples (single segment structures with cross-sections containing 4-12 nodes) showed that, in addition to a high accuracy of the solution, the presented method of analytical integration of differential equations considerably reduces the time of calculations, as compared with that needed by the step method [2] consisting in expanding the solutions into the series of Tchebycheff's polynomials.

5. Conclusions

The problem of numerical integration of the ordinary differential equations describing the motion or equilibrium of the system, is frequently encountered in engineering problems. In practice, in order to solve the equations system with a large number of unknowns, the multi-step methods [2, 3] are usually applied; analytical methods are usually applied to the simplest problems only. Determination of the general integral of the equation system describing the harmonic vibrations and equilibrium of the Vlasov shell structure, was possible owing to specific properties of this set of equations. The method of analytical integration of systems of ordinary differential equations is known [6]. The novelty of the first part of the presented paper consists in adopting the theory to the computer calculations.

The method of analytical integration proved to be several times faster and more accurate in practical applications than the approximate methods. One of the basic advantages of this method is the possibility of permanent control of accuracy of the obtained solution. Contrary to the approximate methods, the analytical method does not require repeated integration of the same equations system in the case of change of the boundary conditions. The equation of motion of the Vlasov shell structure is an equation of a continuous system with the non-symmetrical stiffness matrix. The problem of integration of this equation is more general than that concerning the discrete systems with symmetrical matrices.

Once the general integral of the motion (or equilibrium) equations of a single shell is known, the equations describing the motion of the system containing several Vlasov shell structures may be coupled with each other through the boundary conditions; consequently, the analysis of dynamics or statics of a complicated engineering structure may be carried out on a personal computer. The calculations performed on a similar computer by means of the finite element method, necessitates the solution of a system of equations with several thousand degrees of freedom what, at the present state of the technology, is practically impossible. On the other hand, application of more powerful computers may prove to be expensive and highly time-consuming.

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Streszczenie

DRGANIA WIELOKOMOROWYCH KONSTRUKCJI POWLOKOWYCH O PRZEKROJACH ZMIENNYCH SKOKOWO

Znaleziono całkę ogólną układu równań różniczkowych opisujących drgania wielokomorowej, pryzmatycznej powłoki zbudowanej z liniowo-sprężystego, ortotropowego materiału. Opierając się na tym rozwiązaniu, sformułowano równania drgań wymuszonych konstrukcji złożonej z szeregu połączonych ze sobą powłok o różnych przekrojach. Wyznaczono amplitudy przemieszczeń konstrukcji podczas drgań ustalonych, częstości rezonansowe oraz główne postacie drgań.

Резюме

КОЛЕБАНИЯ МНОГОКАМЕРНЫХ ОБОЛОЧНЫХ СИСТЕМ СО СКАЧКООБРАЗНО ПЕРЕМЕННЫМ СЕЧЕНИЕМ

Получен общий интеграл матричного урабнения амплитуд гармонических колебаний многокамерной оболочки Власова. На основе этого решения были сформулированны уравнения вынужденных колебаний линейно-упругой системы состоящей из большого количества многокамерных стержней – оболочек различных сечений. Определен отклик конструкции на вынуждение гармоническими силами, собственные частоты и основные формы колебаний.

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