ENGINEERING TRANSACTIONS • Engng. Trans. • 39, 1, 97–110, 1991
Polish Academy of Sciences • Institute of Fundamental Technological Research

NONLINEAR INTERACTION BETWEEN TWO SHEAR WAVES IN ELASTIC MATERIAL

Z. WESOŁOWSKI (WARSZAWA)

The investigations are based on the successive approximation method for the Murnaghan's material. Two one-dimensional shear waves of arbitrary profiles propagate in the opposite directions. The correction terms are calculated analytically. One numerical integration is needed for the calculation of the interaction terms. The correction and the interaction terms represent the longitudinal and the transverse waves. For the wave profile of the form $\exp(-x^2)$ the correction terms and interaction terms are calculated.

1. Successive approximations

We base here on Chapter V of the monograph by Green and Adkins [1]. The displacement u_i is a function of coordinates x^k and time t, $u_i = u_i(x^k, t)$. The stored energy W is a function of the deformation tensor e_{ij} (isentropic process), and we have the following relations

$$2e_{ij} = u_{i,j} + u_{j,i} + u_{r,i}u_{r,j}, \tag{1.1}$$

$$t^{ij} = \frac{1}{2}(\delta_{rj} + u_{j,r})H^{ir}, \qquad H^{ij} = \frac{\partial W}{\partial e_{ij}} + \frac{\partial W}{\partial e_{ji}}, \tag{1.2}$$

where t^{ij} is the stress tensor. The equations of motion have the form

$$t^{ij}{}_{,i} = \rho_o \frac{\partial^2 u_j}{\partial t^2}, \tag{1.3}$$

where ρ_o is the initial density of the body. Tensor t^{ij} is, in fact, the first Piola–Kirchhoff stress tensor.

We expect the following form of the displacement $u_i(x^k, t)$:

$$(1.4)\qquad \begin{aligned} u_1 &= 2\varepsilon^2 K(x,t) + 2\varepsilon^3 P(x,t), \qquad u_2 = 0, \\ u_3 &= \varepsilon\Phi(x,t) + \varepsilon^2 M(x,t) + \varepsilon^3 R(x,t), \qquad x = x^1, \end{aligned}$$

where ε is a small parameter. The functions $K, ..., R$ will be calculated in the following chapters. Note that u_3 is of the order ε, in contrast to u_1 which is of the order ε^2. The function Φ represents the fundamental motion, and the remaining functions are the interaction and correction terms. We pass to the derivation of the equations of motion. In accord with Eqs.(1.4) and (1.2), we have

$$(1.5)\qquad \begin{aligned} u_{1,1} &= 2\varepsilon^2 K_x + 2\varepsilon^3 P_x, \qquad u_{1,t} = 2\varepsilon^2 K_t + 2\varepsilon^3 P_t, \\ u_{3,1} &= \varepsilon\Phi_x + \varepsilon^2 M_x + \varepsilon^3 R_x, \qquad u_{3,t} = \varepsilon\Phi_t + \varepsilon^2 M_t + \varepsilon^3 R_t. \end{aligned}$$

The remaining derivatives of u_i are equal to zero. The strains are

$$(1.6)\qquad \begin{aligned} 2e_{11} &= \varepsilon^2(4K_x + \Phi_x^2) + \varepsilon^3(4P_x + 2\Phi_x M_x), \\ 2e_{13} &= 2e_{31} = \varepsilon\Phi_x + \varepsilon^2 M_x + \varepsilon^3 R_x, \quad \text{remaining} \quad e_{ij} = 0. \end{aligned}$$

The strain invariants are defined to be

$$(1.7)\qquad J_1 = 2e_{rr}, \qquad J_2 = 2(e_{rr}e_{ss} - e_{rs}e_{rs}), \qquad J_3 = 8 \det e_{rs}.$$

Further calculations will be perfomed for the stored energy function proposed by Murnaghan

$$(1.8)\qquad 2W = \lambda e_{rr}e_{ss} + 2\mu e_{rs}e_{rs} + 2\alpha J_1 J_2 + 2\beta J_1^3 + 2\gamma J_3,$$

which is the simplest self-consistent generalization of the function W of linear elasticity. The material constants λ, μ, α, β, γ are assumed to be given.

Performing the calculations of the relations (1.2), we obtain

$$\begin{aligned} H^{11} &= \varepsilon^2\left\{4(\lambda+2\mu)K_x + (\lambda+2\mu-4\alpha)\Phi_x^2\right\} \\ &\quad + \varepsilon^3\left\{4(\lambda+2\mu)P_x + 2(\lambda+2\mu-4\alpha)\Phi_x M_x\right\}, \end{aligned}$$

$$(1.9)\qquad \begin{aligned} H^{22} &= \varepsilon^2\left\{4\lambda K_x + (\lambda-4\alpha-4\gamma)\Phi_x^2\right\} \\ &\quad + \varepsilon^3\left\{4\lambda P_x + 2(\lambda-4\alpha-4\gamma)\Phi_x M_x\right\}, \end{aligned}$$

(1.9) [cont.]
$$H^{33} = \varepsilon^2 \left\{4\lambda K_x + (\lambda - 4\alpha)\Phi_x^2\right\} + \varepsilon^3 \left\{4\lambda P_x + 2(\lambda - 4\alpha)\Phi_x M_x\right\},$$

$$H^{12} = \varepsilon^2 \cdot 4\mu L_1 + \varepsilon^3 4\mu Q_1, \qquad H^{23} = 0,$$

$$H^{31} = \varepsilon \cdot 2\mu\Phi_x + \varepsilon^2 2\mu M_x + \varepsilon^3 \left\{2\mu R_x - 16\alpha\Phi_x K_x - 4\alpha\Phi_x^3\right\};$$

$$2t^{11} = \varepsilon^2 \left\{4(\lambda + 2\mu)K_x + (\lambda + 2\mu - 4\alpha)\Phi_x^2\right\} + \varepsilon^3 \left\{4(\lambda + 2\mu)P_x + 2(\lambda + 2\mu - 4\alpha)\Phi_x M_x\right\},$$

$$2t^{22} = \varepsilon^2 \left\{4\lambda K_x + (\lambda - 4\alpha - 4\gamma)\Phi_x^2\right\} + \varepsilon^3 \left\{4\lambda P_x + 2(\lambda - 4\alpha - 4\gamma)\Phi_x M_x\right\},$$

$$2t^{33} = \varepsilon^2 \left\{4\lambda K_x + (\lambda + 2\mu - 4\alpha)\Phi_x^2\right\} + \varepsilon^3 \cdot 4\lambda P_x,$$

(1.10)
$$2t^{12} = 2t^{21} = \varepsilon^2 \cdot 4\mu L_x + \varepsilon^3 \cdot 4\mu Q_x,$$

$$2t^{31} = \varepsilon \cdot 2\mu\Phi_x + \varepsilon^2 \cdot 2\mu M_x + \varepsilon^3 \left\{2\mu R_x + 4(\mu - 4\alpha)\Phi_x K_x - 4\alpha\Phi_x^3\right\},$$

$$2t^{13} = \varepsilon \cdot 2\mu\Phi_x + \varepsilon^2 \cdot 2\mu M_x + \varepsilon^3 \left\{2\mu R_x + 4(\lambda + 2\mu - 4\alpha)\Phi_x K_x + (\lambda + 2\mu - 8\alpha)\Phi_x^3\right\},$$

$$t^{23} = t^{32} = 0.$$

Note that t^{ij} is not symmetric: $t^{ij} \neq t^{ji}$.

Substitute now Eqs.(1.10) and (1.4) to obtain two equations of motion (equation for $j = 2$ is satisfied identically). Due to the small parameter ε, they can be decomposed into a hierarchy of equations. As the coefficient of ε we obtain

(1.11)
$$\mu\Phi_{xx} = \rho_o\Phi_{tt}.$$

As the coefficient of ε^2 we obtain two second order partial differential equations:

(1.12)
$$\mu M_{xx} = \rho_o M_{tt},$$
$$2(\lambda + 2\mu)K_{xx} + (\lambda + 2\mu - 4\alpha)\Phi_x\Phi_{xx} = 2\rho_o K_{tt},$$

and as the coefficient of ε^3 - two equations

$$\begin{aligned} &2(\lambda+2\mu)P_{xx}+(\lambda+2\mu-4\alpha)(\Phi_{xx}M_x+\Phi_x M_{xx})=2\rho_o P_{tt}, \\ (1.13)\quad &2\mu R_{xx}+4(\lambda+2\mu-4\alpha)(\Phi_{xx}K_x+\Phi_x K_{xx}) \\ &\qquad +3(\lambda+2\mu-8\alpha)\Phi_x^2\Phi_{xx}=2\rho_o R_{tt}. \end{aligned}$$

In a purely static context Eqs.(1.1) and (1.12) were used in [3].

Introduce the notations

$$(1.14)\quad \begin{aligned} c^2&=\frac{\mu}{\rho_o}, & a^2&=\frac{\lambda+2\mu}{\rho_o}, \\ b_1&=\frac{\lambda+2\mu-4\alpha}{2\rho_o}, & b_2&=\frac{3(\lambda+2\mu-8\alpha)}{2\rho_o}. \end{aligned}$$

The above equations assume now the simple form

$$(1.15)\quad \Phi_{tt}-c^2\Phi_{xx}=0;$$

$$(1.16)\quad \begin{aligned} M_{tt}-c^2M_{xx} &= 0, \\ K_{tt}-a^2K_{xx} &= b_1\Phi_x\Phi_{xx}; \end{aligned}$$

$$(1.17)\quad \begin{aligned} R_{tt}-c^2R_{xx} &= 4b_1(\Phi_{xx}K_x+\Phi_x K_{xx})+b_2\Phi_x^2\Phi_{xx}, \\ P_{tt}-a^2P_{xx} &= b_1(\Phi_{xx}M_x+\Phi_x M_{xx}). \end{aligned}$$

Note that there exist two wave speeds: c in the equations for Φ, M, R and a in the equations for K, P. Obviously,

$$(1.18)\quad a>c.$$

2. One pulse

Let us pass to the solution of Eqs.(1.14)–(1.16) and look for the solution representing *i*) one wave + correction terms, and *ii*) two waves + correction terms.

The solution of the notation (1.14) representing a pulse has the form

$$(2.1)\quad \Phi(x,t)=f(x-ct),$$

where f is an arbitrary function with continuous first and second derivatives. This solution represents a constant profile wave running to the right ($+x$ direction) with the constant speed c. In order to find the correction terms, we must solve Eqs.(1.15). If we assume that the entire initial disturbance is described by Φ, then we must take the function

$$M = 0 \tag{2.2}$$

which satisfies Eq.$(1.15)_1$. Equation $(1.15)_2$ takes the form

$$K_{tt} - a^2 K_{xx} = b_1 f'(x - ct) f''(x - ct). \tag{2.3}$$

It is well known (cf. e.g. [2]) that the solution of Eq.(2.3) consists of two parts

$$K(x,t) = \bar{K}(x,t) + \bar{\bar{K}}(x,t). \tag{2.4}$$

The function $\bar{K}(x,t)$ takes into account the right-hand side of Eq.(2.3) and the function $\bar{\bar{K}}(x,t)$ is a solution of the corresponding homogenous equation taking into account the initial data given in advance. We have

$$\bar{\bar{K}}(x,t) = \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right] + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi(\alpha)d\alpha. \tag{2.5}$$

Here

$$\varphi(z) = \bar{\bar{K}}(z,0), \qquad \psi(z) = \bar{\bar{K}}_t(z,0). \tag{2.6}$$

Since we are looking for the correction terms only, confine the calculations for K,M,P,R to the algebraically simplest case. This case, however, does not correspond to $\varphi = \psi = 0$, as will be seen later.

Pass to the function $\bar{K}(x,t)$. We have (cf. [2])

$$\bar{K}(x,t) = \frac{b_1}{2a}\int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f'(\xi - c\tau) f''(\xi - c\tau)d\xi. \tag{2.7}$$

Integration with respect to ξ leads to the expressions

$$\begin{aligned} \bar{K}(x,t) &= \frac{b_1}{4a}\int\limits_0^t d\tau \left[f'(\xi - ct)\right]^2 \Big|_{x-a(t-\tau)}^{x+a(t-\tau)} \\ &= \frac{b_1}{4a}\int\limits_0^d d\tau \left\{\left[f'(x + a(t-\tau) - c\tau)\right]^2 - \left[f'(x - a(t-\tau) - c\tau)\right]^2\right\}. \end{aligned} \tag{2.8}$$

Define the function

$$F(z) = \int_0^t [f'(\xi)]^2 d\xi \tag{2.9}$$

and perform in Eq.(2.8) integration with respect to τ. We have

$$\begin{aligned}(2.10)\quad \bar{K}(x,t) &= \frac{b_1}{4a}\left\{-\frac{1}{a+c}F(x+a(t-\tau)-c\tau) - \frac{1}{a-c}F(x-a(t-\tau)-c\tau)\right\}\Bigg|_0^t \\ &= -\frac{b_1}{2(a^2-c^2)}F(x-ct) + \frac{b_1}{4a}\left\{\frac{1}{a+c}F(x+at) + \frac{1}{a-c}F(x-at)\right\}.\end{aligned}$$

The correction term for the pulse $f(x-ct)$ is therefore not a function of $x-ct$. The profile of the correction term changes in the course of time. Diagrams of function $K(x,t)$ for

$$f(x-ct) = \frac{1}{6}\exp[-3(x-ct)^2], \tag{2.11}$$

and for four fixed times $t = 1, 2, 5, 10$ are shown in Fig.1. For x smaller

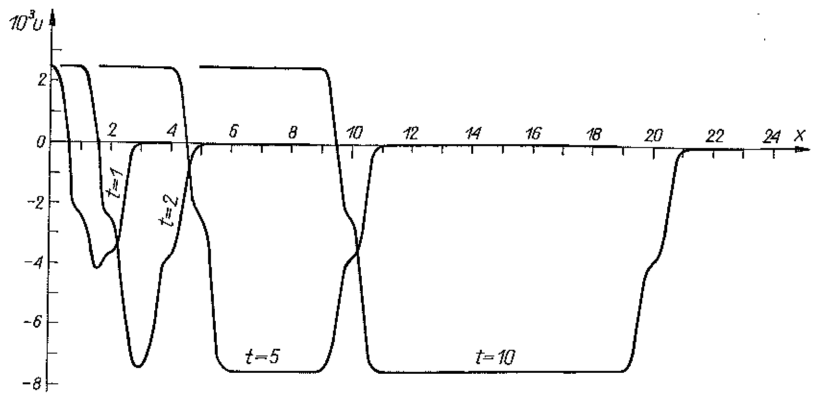

Fig. 1.

than $c(t-1)$ we have $10^3 u = 2.5$, and for x larger than $c(2t+1)$, $u = 0$. The calculations were performed for $a = 2$, $c = 1$, and the curves sketched for $b_1 = 1$. For the same values of the parameters, function $K(x,t)$ at fixed points $x = 2, 5, 10$ is given in Fig.2. For $t < x/2 - 5$ we have $u = 0$, and for $t > x+1$ we have $10^3 u = 2.5$.

Note that we may assume the functions φ and ψ in (2.5) to obtain

$$\bar{\bar{K}}(x,t) = -\frac{b_1}{4a}\left[\frac{1}{a+c}F(x+at) + \frac{1}{a-c}F(x-at)\right],$$

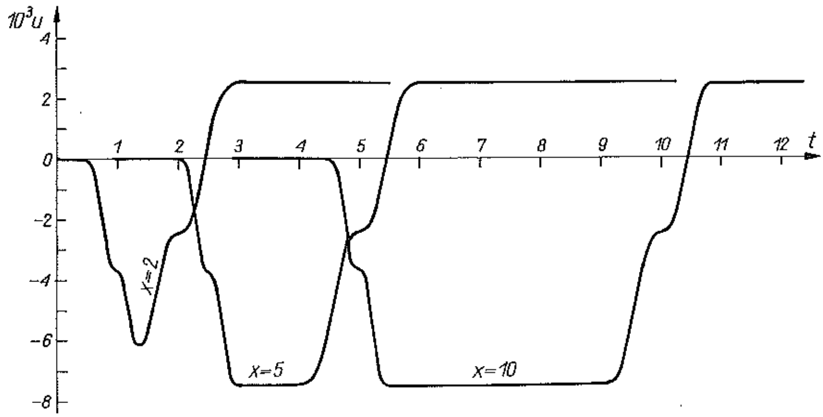

Fig. 2.

and, finally, the formula

$$K = \bar{K} + \bar{\bar{K}} = -\frac{b_1}{2(a^2 - c^2)} F(x - ct). \tag{2.12}$$

The initial values of K and K_t corresponding to the above solution are not equal to zero. Note that basing on the numerical data given in Fig.1 and Fig.2 it would make it rather difficult to arrive at the simple expression (2.12).

Let us now pass to the equation (1.17). For M given by the function (2.2), the right-hand side of Eq.$(1.17)_2$ vanishes and we assume

$$P(x, t) = 0.$$

Here again we take into account that the entire initial motion is described by Eq.(2.1).

Equation $(1.17)_1$ for K given by Eq.(2.12) assumes the form

$$\begin{aligned} R_{tt} - c^2 R_{xx} &= b_3 \left[f'(x - ct)\right]^2 f''(x - ct), \\ b_3 &= b_2 - \frac{6b_1}{a^2 - c^2}. \end{aligned} \tag{2.13}$$

As before, we represent R as $\bar{R} + \bar{\bar{R}}$ where $\bar{R}$ satisfies the homogenous equation, and

$$\bar{\bar{R}} = \frac{b_3}{2c} \int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} d\xi \left[f'(\xi - c\tau)\right]^2 f''(\xi - c\tau). \tag{2.14}$$

Integration with respect to ξ leads to

$$(2.15)\qquad \overline{\overline{R}} = \frac{b_3}{6c}\int_0^t d\tau\,[f'(x-ct)]^3\Big|_{x-c(t-\tau)}^{x+c(t-\tau)} = \frac{b_3}{6c}\int_0^t d\tau\left\{[f'(x+ct-2c\tau)]^3-[f(x-ct)]^3\right\}.$$

Denoting

$$(2.16)\qquad G(z)=\int_0^z [f'(z)]^3\,dz,$$

we have

$$(2.17)\qquad \overline{\overline{R}} = \frac{b_3}{12c^2}[-G(x-ct)+G(x+ct)]-\frac{b_3}{6c}tG'(x-ct).$$

We find now $R(x,t)$ annihilating the terms in the square brackets. Finally, taking into account Eq.(2.16), we obtain the second-order correction displacement

$$(2.18)\qquad R=-\frac{b_3}{6c}t\,[f'(x-ct)]^3.$$

The presence of the multiplier t forces us to confine the calculations to sufficiently small time. Application of similar calculations makes it possible to consider in the analysis of Eq.(2.1) the function $g(x+ct)$ instead of the function $f(x-ct)$.

3. Two pulses

Consider now two pulses running in the opposite directions

$$(3.1)\qquad \Phi(x,t)=f(x-ct)+g(x+ct),$$

where f and g are sufficiently smooth functions. It is tacitly assumed that both functions are localized near $x-ct=0$ and $x+ct=0$. The collision of both pulses takes place at $t=0$ near the point $x=0$.

For the reasons discussed above we take

$$(3.2)\qquad M=0.$$

Equation $(1.16)_2$ assumed the form

$$(3.3)\quad K_{tt} - a^2 K_{xx} = b_1[f'(x-ct) + g'(x+ct)][f''(x-ct) + g''(x+ct)].$$

The special case of that equation corresponding to $g = 0$ was considered earlier, cf. Eq.(2.3); however, the functions used here are different from those used in Chapter 2.

Decompose K into $\bar{K}$ and $\bar{\bar{K}}$ as in Eq.(2.4). We have (cf.[2])

$$(3.4)\qquad \bar{K}(x,t) = \frac{b_1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(\tau-\tau)} d\xi\, [f'(\xi - c\tau) + g'(\xi + c\tau)]\,[f''(\xi - c\tau) + g''(\xi + c\tau)]$$

Integration with respect to ξ leads to the formula

$$(3.5)\qquad \bar{K}(x,t) = \frac{b_1}{4a}\int_0^t d\tau\, [f'(\xi - c\tau) + g'(\xi + c\tau)]^2 \Big|_{x-a(t-\tau)}^{x+a(\tau-\tau)}$$

$$= \frac{b_1}{4a}\int_0^t d\tau\, \{[f'(x + a(t-\tau) - c\tau) + g'(x + a(t-\tau) + c\tau)] - [f'(x - a(t-\tau) - c\tau) + g'(x - a(t-\tau) + c\tau)]\}\,.$$

The general considerations performed in the previous chapter can not be repeated for Eq.(3.5) because the arguments of f and g are different. Simple results may be obtained for the exponential functions which, however, are not interesting. The sine function requires numerical integration for a/c not equal to an integer.

For

$$(3.6)\qquad \begin{aligned} f(z) = g(z) &= \exp(-3z^2),\\ a = 2, &\qquad c = 1, \end{aligned}$$

the numerical integration of Eq.(3.5) was perfomed. For $t = 1, 0.2, 0.5, 1$ the function K/b_1 is shown in Fig.3. At $X = 0$ we have $K = 0$. For large t the profiles are entirely different, see Fig.4. Only near the points $x = ct$ and $x = at$ the value of $\bar{K}$ is not constant. For $x < ct - m$ and $x > at + m$, m =const we obtain $K = 0$. This fact is evident from Fig.5, where the heavy lines mark the values of x, τ for which (for given t) the arguments in (3.5) are zero. Since f and g decrease rapidly with z, the functions in Eq.(3.5) are different from zero in narrow strips shown in

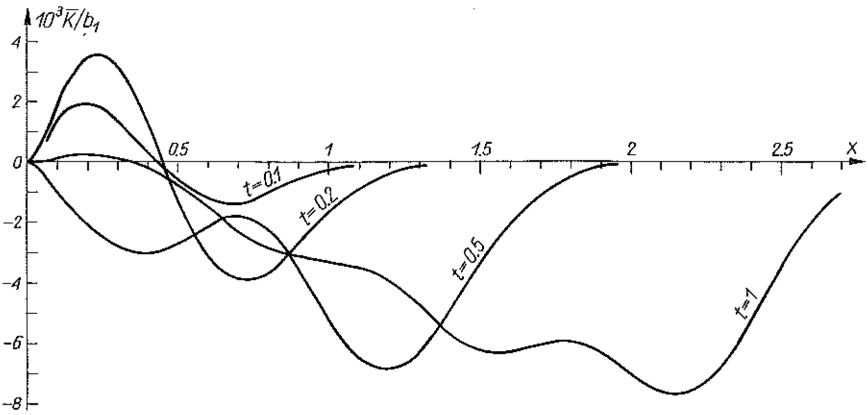

Fig. 3.

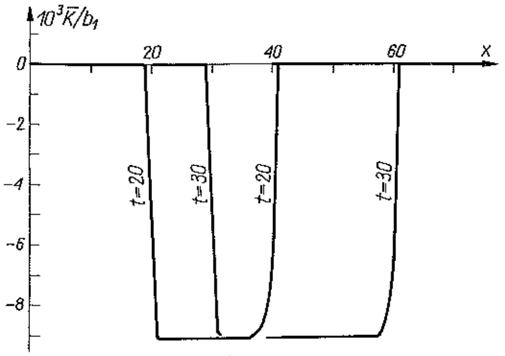

Fig. 4.

Fig.5. For x slightly larger than at, the integration path does not touch the strips and $\bar{K} = 0$. For x slightly less than ct the integration path crosses two strips but the integrals are of different signs and $\bar{K} = 0$.

Figure 4 corresponds to large values of t. For small values of t the proportions change, and already the point $\tau = t$ may be situated within one or more strips. In view of this, the curves for small t are entirely different from those for large t, see Fig.6 and Fig.7.

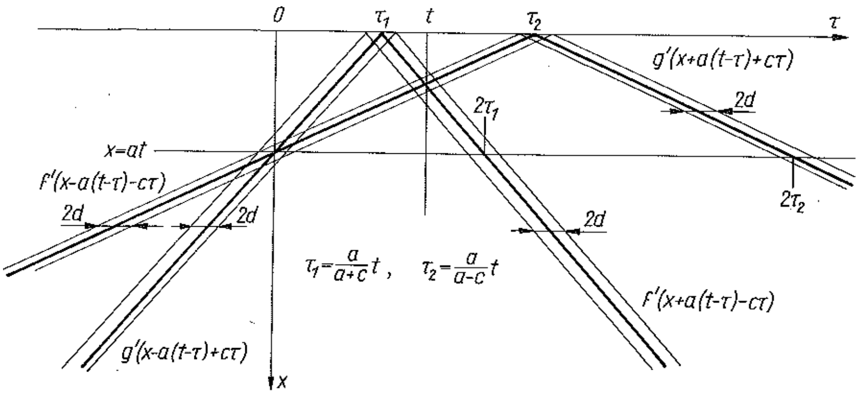

Fig. 5.

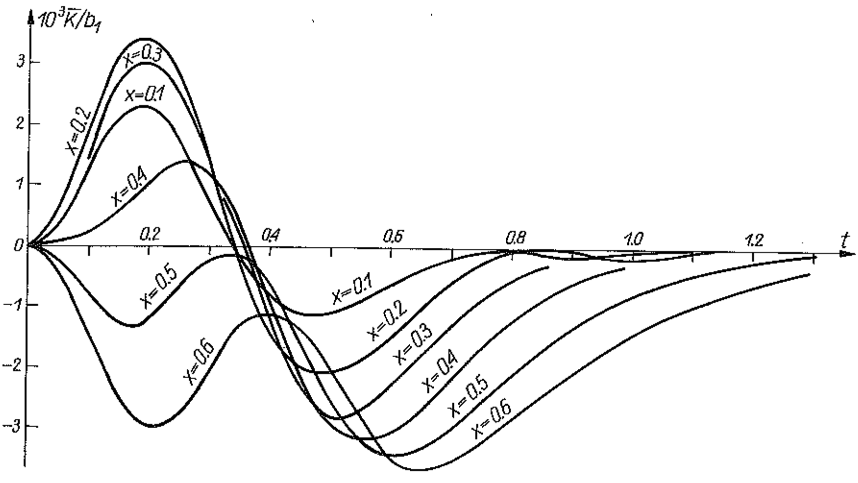

Fig. 6.

In order to eliminate the redundant displacement (cf. the discussion preceding Eq.(2.16)), consider now the cross terms of Eq.(3.5) only, denoting

$$(3.7)\qquad K_{fg} = \frac{b_1}{2a}\int_0^t d\tau\, \{f'(x+a(t-\tau)-c\tau)g'(x+a(t-\tau)+c\tau) + f'(x-a(t-\tau)-c\tau)g'(x-a(t-\tau)+c\tau)\},$$

$$\bar{K} = K_{ff} + K_{gg} + K_{fg}.$$

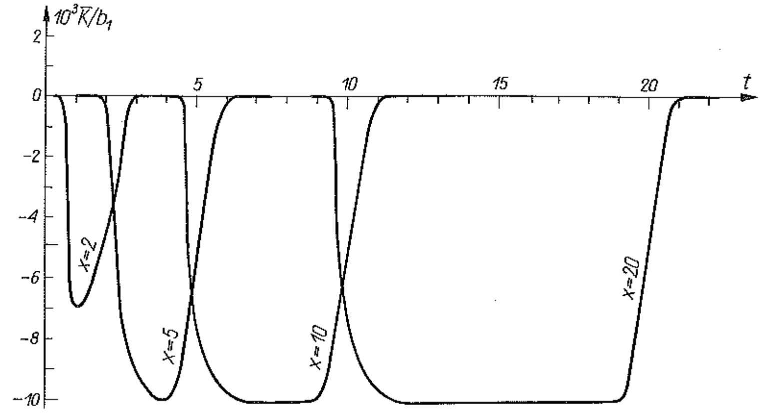

Fig. 7.

The remaining terms $K_{ff} + K_{gg}$ representing the correction terms for $\Phi = g(x - xt)$ and $\Phi = g(x + ct)$ taken separately were already considered. Several graphs of $K_{fg}(x, t)$ are shown in Fig.8. For large t

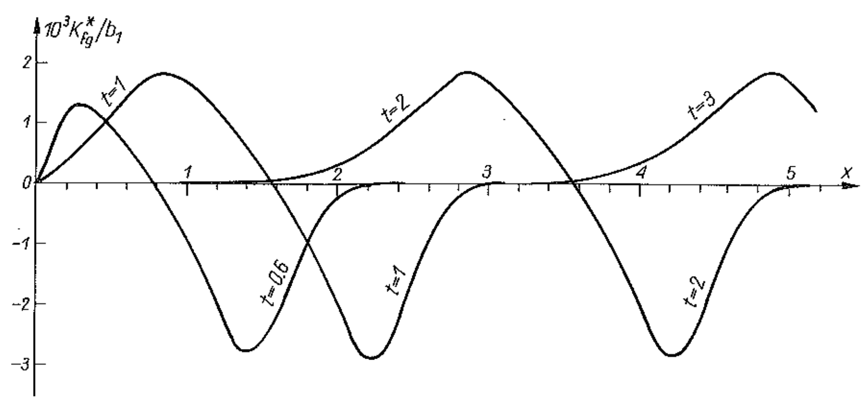

Fig. 8.

the profile moves to the right with speed a, and with good accuracy it coincides with the profile shown in Fig.8 for $t = 2$. Therefore, the profile is of the form $\varphi(x - at)$, and it satisfies the equation $K_{tt} - a^2 K_{xx} = 0$. For $x < 0$ we obtain

$$K_{fg}(-x, t) = -K_{fg}(x, t), \tag{3.8}$$

and the limiting profile has the form $-\varphi(x + at)$. We subtract now from

K the limiting profiles to obtain

$$K^*_{fg} = K_{fg} - \varphi(x - at) + \varphi(x + at). \tag{3.9}$$

The function $(K_{ff} + K_{gg} + K^*_{fg})$ satisfies Eq.(3.3) and represents the other choice of the correction terms for Eq.(3.1). The function K^*_{fg} for $t = 1, 2, ..., 8$ is given in Fig.9. For $t \geq 1$ and $t \leq .01$, K^*_{fg} is negligible.

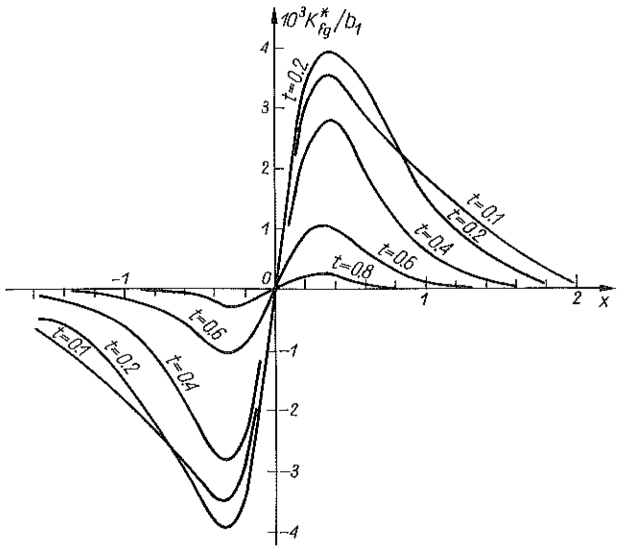

Fig. 9.

There exists a maximum of K^*_{fg} at about $x = 0.17$. The largest value (about 4) of this maximum is reached at $t = 0.265$.

References

1. A.E.Green and J.E.Adkins, *Large elastic deformations*, Oxford 1960.

2. W.Janowski, *Mathematics* [in Polish], t.II, PWN, Warszawa 1962.

3. F.R.N.Nabarro, *The force between two screw dislocations*, Phil. Mag., **A54**, 1986.

STRESZCZENIE

NIELINIOWE ODDZIAŁYWANIE POMIĘDZY FALAMI POPRZECZNYMI W MATERIALE SPRĘŻYSTYM

Podstawą rozważań jest teoria kolejnych przybliżeń dla materiału typu Murnaghana. Dwie jednowymiarowe fale poprzeczne o dowolnych profilach propagują się w przeciwnych kierunkach. Wyznacza się wyrazy korekcyjne dla każdej z nich. W celu wyznaczenia oddziaływania należy przeprowadzić jednokrotne całkowanie numeryczne. Tak wyrazy korekcyjne, jak i oddziaływanie są sumą fal poprzecznych i podłużnych. Przedstawia się odpowiednie rozwiązania dla fal pierwotnych w postaci funkcji $\exp(-x^2)$.

Резюме

ВЗАИЕМОДЕЙСТВИЕ НЕЛИНЕЙНЫХ ПОПЕРЕЧНЫХ ВОЛН В МАТЕРИАЛЕ

Основой рассуждений является теория последовательных приближений для материала типа Мурнагана.Две одномерные поперечные волны, с произвольными профилями, распространяются в противоположных направлениях. Определяются коррекционные члены для каждой из них. С целью определения взаимодействия следует провести однократное численное интегрирование. Так коррекционные члены, как и взаимодействие являются суммой поперечных и продольных волн. Показывается соответствующие решения для первичных волн в виде функции $\exp(-x^2)$.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received June 19, 1990.