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NONLINEAR SOLUTION METHODS FOR FEM ANALYSIS OF CONCRETE STRUCTURES

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The essential objective of the nonlinear FEM analysis is the solution of a system of nonlinear equations. Due to strain-softening nature of fracture processes, concrete structures need a special attention in applying a suitable nonlinear solution procedure. New, sophisticated material models and the need to predict limit points and post-peak behaviour of a structure require methods capable of converging in the vicinity of such limit points. This paper reviews several nonlinear solution methods with particular reference to the usefulness in the nonlinear analysis of concrete structures exhibiting strain-softening characteristics.

1. Introduction

The effective application of the numerical model of concrete requires suitable nonlinear solution techniques to trace the equilibrium path of a structure. Due to numerical problems encountered during fracture processes computations, the use of the true incremental tangent stiffness matrix is often undesirable. Thus the solution schemes have to rely heavily on iterative procedures.

Historically, the most popular and widely used nonlinear solution procedures have been the Newton-Raphson and the modified Newton – Raphson methods [1, 2]. These methods, however, have proved to be inadequate when the more sophisticated material models were introduced.

In the last decade considerable research effort has been expanded in developing more efficient and stable solution procedures capable of following the nonlinear processes introduced by the fracturing of concrete.

The requirements of a solution procedure for the analysis of structures with such a strain-softening material differ significantly from those, for instance, appropriate for analysis of structures featuring geometric nonlinearity, where the tangent stiffness matrix remains continuously variable.

In this paper, a few advanced solution procedures are reviewed with particular reference to their usefulness in the analysis of concrete structures. The attention was focused on numerical solution techniques in which iterations are carried out in displacement as well as load space.

2. Methods based on modified Newton–Raphson iterations

One of the first successful attempts at a variable load level iteration is that of Pian and Tong [3]. Their method, however, requires some initial modification of the global stiffness matrix. The method was simplified by Batoz and Dhatt [4]. In their displacement control method they used the original global stiffness matrix.

In Batoz and Dhatt's approach, the displacement increment vector is composed of two parts:

$$\Delta \mathbf{U}_i = \Delta \bar{\mathbf{U}}_i + \Delta\lambda_i \Delta \mathbf{U}_T, \tag{2.1}$$

where $\Delta\lambda_i$ is the load parameter to be found from some constraint equation. The components of the displacement increment vector are expressed as:

$$\Delta \bar{\mathbf{U}}_i = K_i^{-1} \Delta \mathbf{p}_i, \tag{2.2}$$

$$\Delta \mathbf{U}_T = K_i^{-1} \mathbf{q}, \tag{2.3}$$

where $\Delta \mathbf{p}_i$ is the out-of-balance force vector in iteration i, and $\mathbf{q}$ is some reference external load vector, and K_i is the tangent stiffness matrix. The load parameter $\Delta\lambda_i$ is obtained from the displacement control constraint. If the j-th degree of freedom is to be constrained, then in the first iteration $\Delta\lambda_1$ is found as:

$$\Delta\lambda_1 = \frac{\Delta \mathbf{U}_0^j - \Delta \bar{\mathbf{U}}_i^j}{\Delta U_T^j}, \tag{2.4}$$

where $\Delta \mathbf{U}_0^j$ is the prescribed displacement for the j-th degree of freedom. In further iterations, the load parameter $\Delta\lambda_i$ is calculated from the condition that in subsequent iterations the displacement increment for j-th degree of freedom is zero. Then,

$$\Delta\lambda = -\frac{\Delta\bar{\mathbf{U}}_i^j}{\Delta\mathbf{U}_T^j}. \tag{2.5}$$

The performance of the method is demonstrated graphically in Fig.1. Due to the constant displacement constraint, the method is not capable

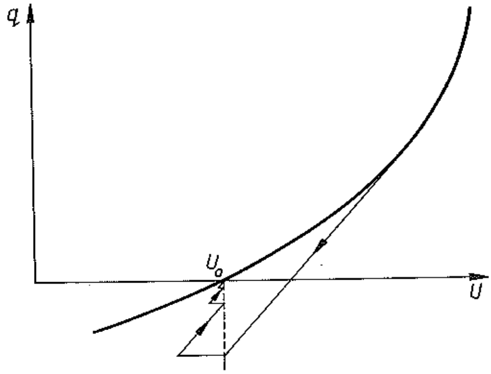

Fig. 1. Direct displacement control method.

of converging in snap-back problems. The method was used successfully by the author to trace the load-deflection curve in a numerical simulation of the three-point-bent test and in the analysis of a notched concrete beam in mixed mode fracture, both described in detail in Ref.[20].

A very successful method was proposed by CRISFIELD [5-8]. The method uses the displacement increment vector decomposition of Eq.(2.1), with the components of the vector calculated according to Eqs. (2.2) and (2.3). For the constraint equation, Crisfield used a modification of a generalized arc length formulation. In its general form, the spherical arc length constraint was proposed independently by RIKS [9] and WEMPNER [10] in the following manner:

$$\mathbf{U}^T\mathbf{U} + \Delta\lambda^2\mathbf{q}^T\mathbf{q} = \Delta l^2, \tag{2.6}$$

where $\mathbf{U}$ is the incremental displacement vector, $\mathbf{q}$ is the total applied loading vector and Δl is the prescribed scalar controlling the length of the displacement increment.

Crisfield discarded the loading term of Eq.(2.6) and used the following, simplified constraint equation:

$$\mathbf{U}^T\mathbf{U} = \Delta l^2. \tag{2.7}$$

The incremental vector in iteration i can be expressed as:

$$\mathbf{U}_i = \mathbf{U}_{i-1} + \Delta\bar{\mathbf{U}}_i + \Delta\lambda_i \Delta\mathbf{U}_T. \tag{2.8}$$

From Eqs.(2.7) and (2.8), the step length adjustment, $\Delta\lambda_i$, is obtained from the following quadratic equation:

$$A_1 \Delta\lambda_i^2 + A_2 \Delta\lambda_i + A_3 = 0, \tag{2.9}$$

where

$$A_1 = \Delta\mathbf{U}_T^T \Delta\mathbf{U}_T, \tag{2.10}$$

$$A_2 = 2(\mathbf{U}_{i-1} + \Delta\bar{\mathbf{U}}_i)^T \Delta\mathbf{U}_T, \tag{2.11}$$

$$A_3 = (\mathbf{U}_{i-1} + \Delta\bar{\mathbf{U}}_i)^T (\mathbf{U}_{i-1} + \Delta\bar{\mathbf{U}}_i) - \Delta l^2. \tag{2.12}$$

Normally, Eq.(2.9) has two roots. The suitable root is the one that guarantees a positive angle betwen vectors $\mathbf{U}_i$ and $\mathbf{U}_{i-1}$, calculated as the dot product of the two vectors. If both roots yield positive angles, the root closer to the linear solution $\Delta\lambda_i = -A_1/A_2$ is selected. After $\Delta\lambda_i$ is selected, the load level is adjusted as:

$$\mathbf{q}_i^n = \mathbf{q}_i^0 (1 + \Delta\lambda_i), \tag{2.13}$$

where q_i^0 and q_i^n are the load vectors at the beginning and at the end of iteration i, respectively. The performance of the method is shown schematically in Fig.2.

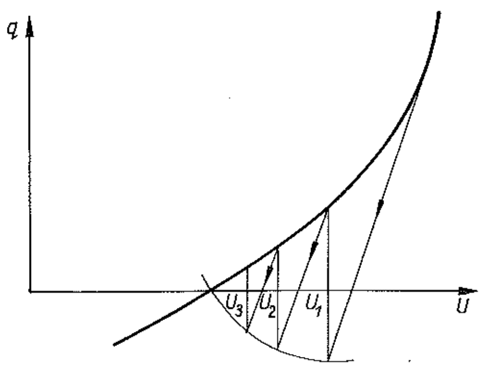

Fig. 2. Arc-length method.

It is interesting to note that for one-dimensional problems the discriminant of Eq.(2.9)

$$\Delta = A_2^2 - 4A_1 A_3 = 4\Delta\mathbf{U}_T^T \Delta\mathbf{U}_T \Delta l^2 \tag{2.14}$$

is always positive for any non-zero Δl. This indicates that for any displacement increment vector $\Delta\bar{\mathbf{U}}_i$, a suitable value of $\Delta\lambda_i$ can be guaranteed. Therefore, as long as the tangent stiffness function and its derivative are both finite and non-zero [11], that is they satisfy the general requirements for the convergence of the Newton–Raphson type iteration process, the solution of the nonlinear equations can always be obtained with the desired accuracy. This conclusion, however, does not hold for multidimensional systems of equations since the non-negative value of the discriminant of Eq.(2.9) cannot be assured.

The arc-length method has proved efficient in tracing load-deflection curves in structures exhibiting, primarily, geometrical nonlinearity. The method does not perform equally well in the analyses of concrete structures. The main difficulty is due to the constraint equation (Eq.(2.7)) which often fails to predict the real root and thus it seems too restrictive in the analysis of cracking concrete. Crisfield obtained an improvement in the performance of the method when analysing concrete structures by combining the arc-length constrained iterations with the line search technique [12].

Also very suitable for the analysis of concrete structures is the linear constraint equation proposed by RAMM [13]. Ramm's equation is a modification of RIKS'S and WEMPNER'S spherical arc-length formula [9,10] and is expressed as:

$$\Delta\mathbf{U}_1^T\Delta\mathbf{U}_i + \Delta\lambda_1\Delta\lambda_i\mathbf{q}^T\mathbf{q} = 0, \tag{2.15}$$

in which $\Delta\mathbf{U}_1$ and $\Delta\mathbf{U}_i$ are the displacement increment vectors, $\Delta\lambda_1$ and $\Delta\lambda_i$ are the load step parameters in iteration 1 and i, respectively, $\mathbf{q}$ is the total load vector. This equation constrains the iterations to follow the plane normal to the tangent direction in the particular iteration cycle. The displacement increment vector is decomposed according to Eqs.(2.1)-(2.3). From Eqs.(2.1) and (2.15), the value of the load parameter, $\Delta\lambda_i$, is calculated as:

$$\Delta\lambda_i = -\frac{\Delta\mathbf{U}_1^T\Delta\bar{\mathbf{U}}_i}{\Delta\mathbf{U}_1^T\Delta\mathbf{U}_T + \Delta\lambda_1\mathbf{q}^T\mathbf{q}}. \tag{2.16}$$

Two modifications can be introduced to the formula in Eq.(2.16). First, it was noticed that the term $\Delta\lambda_1\mathbf{q}^T\mathbf{q}$ in the denominator can be suppressed since in multi-degree-of-freedom systems, its influence on the value $\Delta\lambda_i$ is negligible. Secondly, the vector $\Delta \mathbf{U}_1$ can be replaced by the vector of accumulated displacements U_{i-1} . This is possible because all the iterations are performed in the same plane and thus the direction of the vector $\mathbf{U}_{i-1}$ is the same as the direction of all its components. Finally, the following formula for the calculation of parameter $\Delta\lambda_i$ is obtained:

$$\Delta\lambda_i = -\frac{\mathbf{U}_{i-1}^T \Delta\bar{\mathbf{U}}_i}{\mathbf{U}_{i-1}^T \Delta\mathbf{U}_T}. \tag{2.17}$$

In one-dimensional problems, the convergence of Ramm's normal path method can be demonstrated if the stiffness function complies with the conditions which guarantee the convergence of Newton–Raphson type solution procedures. The value of the load parameter $\Delta\lambda_i$ is obtained at any iteration due to the linear nature of the constraint formulation. Ocasionally, the method may fail in real, multi-degree-of-freedom problems. De Borst [14] suggested that the failure of constrained iterative procedures is caused by employing global displacement vectors to calculate load parameters $\Delta\lambda$. It is known that in porous and cementitious materials like concrete, material damage tends to localize in a certain regions of a structure. Due to localization, certain degrees of freedom should have more influence on controlling the load during iterations than others. Therefore, the vectors composed only of the dominant nodal values should enter constraint equation. The major difficulty of this so-called indirect displacement control method is that it is usually not known which degrees of freedom should be controlled. Furthermore, it may be necessary to change the composition of the dominant degrees of freedom being constrained as damage to the structure progresses. The method may become atractive for materially nonlinear problems if some practical guidelines on selecting the nodal values to be controlled are given.

For sufficiently small load increments, the Ramm's normal path method does not differ from the arc-length approach. The simplicity and flexibility of the proposed constraint makes the method very useful in the analysis of concrete and reinforced concrete structures. The performance of the method in one DOF problem is demonstrated in Fig.3.

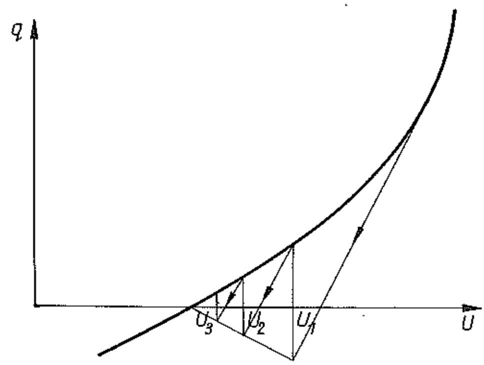

Fig. 3. Orthogonal path method (Ramm's method).

3. Methods based on quasi-Newton iterations

Load level variability during iterations in the quasi-Newton method is achieved by the decomposition of the displacement increment vector described by Eqs.(2.1)-(2.3). In the quasi-Newton approach, the inverse of the stiffness matrix K_i^{-1}, used in Eqs(2.2) and (2.3), is replaced by a modified matrix obtained by a suitable quasi-Newton update procedure, such as Broyden's approach [15] or BFGS approach [16].

Alternatively, the original tangent stiffness matrix can be used throughout the iteration cycle and modifications can be carried out on the components of the displacement increment vector. This approach was successfully pursued by de Borst in Ref.[17]. Using Broyden's updating procedure, vectors $\Delta\bar{\mathbf{U}}_{i+1}$ and $\Delta\mathbf{U}_{Ti+1}$ are calculated as:

$$\Delta\bar{\mathbf{U}}_{i+1} = (I + \alpha_i a_i \Delta\bar{\mathbf{U}}_i^T)(a_i + \Delta\lambda_i \Delta\mathbf{U}_{Ti}), \tag{3.1}$$

$$\Delta\mathbf{U}_{Ti+1} = \Delta\mathbf{U}_{Ti} + \alpha_i \Delta\bar{\mathbf{U}}_i^T \Delta\mathbf{U}_{Ti} a_i, \tag{3.2}$$

where

$$\alpha_i = [\Delta\bar{\mathbf{U}}_i^T(\Delta\bar{\mathbf{U}}_i - a_i)]^{-1}, \tag{3.3}$$

$$a_i = \Delta\lambda_i \Delta\mathbf{U}_{Ti} + \prod_{j=1}^{i-1}(I + \alpha_j a_j \Delta\bar{\mathbf{U}}_j^T) K_0^{-1} \Delta\mathbf{p}_i; \tag{3.4}$$

K_0^{-1} is the stiffness matrix at the beginning of the iteration cycle. The first iteration is the Newton–Raphson iteration. After the values $\Delta\bar{\mathbf{U}}_1$, $\Delta\mathbf{U}_{T1}$ and $\Delta\lambda_1$ are found, the procedure follows the quasi-Newton regime. The loading parameter $\Delta\lambda_{i+1}$ is calculated on the basis of vectors

$\Delta\bar{\mathbf{U}}_{i+1}$ and $\Delta\mathbf{U}_{Ti+1}$ from the normal path or the arc-length constraint. It was demonstrated in Ref.[17] that the method gives satisfactory results when employed in materially nonlinear problems.

4. Methods based on secant-Newton formulation

Secant-Newton methods are the variations of the secant method [11] where true tangent iteration is approximated by some secant relationship. Crisfield [18, 19] showed that secant-Newton methods, which he derived from quasi-Newton and conjugate gradient methods, can be efficiently employed to improve the performance of the modified Newton-Raphson technique.

The simplest secant-Newton method is obtained by applying a one-step line search to the modified Newton-Raphson iteration. The line search based step length parameter, A_i , calculated in iteration $i-1$, can be used to obtain a better approximation of the new displacement vector:

$$\mathbf{U}_i = \mathbf{U}_{i-1} + A_i \Delta \mathbf{U}_i, \tag{4.1}$$

where

$$A_i = \frac{\Delta \mathbf{U}_{i-1}^T \Delta \mathbf{p}_{i-1}}{\Delta \mathbf{U}_{i-1}^T (\Delta \mathbf{p}_{i-1} - \Delta \mathbf{p}_i)}; \tag{4.2}$$

$\mathbf{U}_{i-1}$ and $\mathbf{U}_i$ are the displacement vectors, $\Delta\mathbf{p}_{i-1}$ and $\Delta\mathbf{p}_i$ are the out-of-balance force vectors in iteration $i-1$ and i, respectively. As seen from Eq.(4.2), this first order secant-Newton approach coincides with the one step approximation line search.

A more stable, second order secant-Newton procedure approximates the secant displacement increment in iteration i by means of a modified Newton–Raphson displacement increment vector $\Delta\bar{\mathbf{U}}_i$ and the secant increment vector $\Delta\mathbf{U}_{i-1}$ obtained in iteration $i-1$. The displacement vector U_i in iteration i is approximated as:

$$\mathbf{U}_i = \mathbf{U}_{i-1} + A_i \Delta \bar{\mathbf{U}}_i + B_i \Delta \mathbf{U}_{i-1}, \tag{4.3}$$

where A_i is the line search parameter obtained in iteration $i-1$ according to Eq.(4.2). The following secant relationship holds for the incremental vectors shown in Fig.4:

$$\frac{\Delta \mathbf{U}_i}{\Delta \mathbf{U}_{i-1}} = -\frac{\Delta \mathbf{p}_i}{\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1}}. \tag{4.4}$$

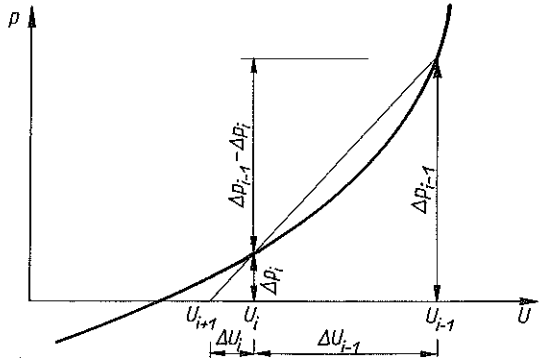

Fig. 4. Secant-relationship.

From Eqs.(4.2), (4.3) and (4.4), the value of parameter B_i is derived as:

$$B_i = A_i \left(1 - \frac{\Delta \bar{\mathbf{U}}_i^T (\Delta \mathbf{p}_i \Delta \mathbf{p}_{i-1})}{\Delta \mathbf{U}_{i-1}^T (\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1})}\right) - 1. \tag{4.5}$$

Occasionally, the above second order secant-Newton method may produce more iterations than the modified Newton–Raphson method. This can happen whenever the value of parameter B_i becomes large compared with A_i . CRISFIELD [18] recommends the use of standard modified Newton–Raphson iteration whenever the following holds:

$$-\frac{1}{2}\,\mathrm{STOL} < \frac{B_i}{A_i} < \mathrm{STOL} \quad or \quad \frac{1}{\mathrm{STOL1}} < A_i < \mathrm{STOL1},$$

where STOL1 is usually taken around 0.5 and STOL is assumed between 2 and 3.

The second order secant-Newton method is used with the implicit line search introduced by parameter A_i (Eq.(4.2)). However, explicit line searches can be introduced to establish a more exact value of A_i which may be necessary for problems involving concrete cracking.

The constrained formulation of the secant-Newton methods can be obtained by changing the out-of-balance force vector to allow for variation of the load level. In the iteration i, the following adjustment to the out-of-balance force vector has to be made [19]:

$$\Delta \mathbf{p}_i = \Delta \bar{\mathbf{p}}_i - \Delta \lambda_i \mathbf{q}, \tag{4.6}$$

where $\Delta \bar{\mathbf{p}}_i$ is the out-of-balance force vector at the beginning of iteration i and $\Delta \lambda_i$ is the load adjustment parameter.

Assuming the first order secant-Newton scheme described by Eq.(4.2), the parameter A_i can be expressed as:

$$A_i = 1 - \frac{\Delta \mathbf{U}_{i-1}^T \Delta \mathbf{p}_i}{\Delta \mathbf{U}_{i-1}^T (\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1})}. \tag{4.7}$$

Replacing vector $\Delta \mathbf{p}_i$ in the numerator of Eq.(4.7) by the vector given in Eq.(4.6), and noting that the variation of the load level in iteration i does not affect the length of the vector $\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1}$ (see Fig.4), the following form of the displacement increment in iteration i is obtained:

$$\Delta \mathbf{U}_i = A_i \Delta \bar{\mathbf{U}}_i + \Delta \lambda_i \frac{\Delta \mathbf{U}_{i-1}^T \mathbf{q}}{\Delta \mathbf{U}_{i-1}^T (\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1})} \Delta \bar{\mathbf{U}}_i. \tag{4.8}$$

Introducing the decomposition of Eq.(2.1) into Eq.(4.8) and ignoring higher order terms in $\Delta \lambda_i$, the final form of the displacement increment is derived as:

$$\Delta \mathbf{U}_i = A_i \Delta \bar{\mathbf{U}}_i + \Delta \lambda_i (A_i \Delta \mathbf{U}_T + C_i \Delta \bar{\mathbf{U}}_i), \tag{4.9}$$

in which

$$C_i = \frac{\Delta \mathbf{U}_{i-1}^T \mathbf{q}}{\Delta \bar{\mathbf{U}}_{i-1}^T (\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1})}. \tag{4.10}$$

The second order constrained secant-Newton procedure described in Eqs.(4.3)-(4.5) is derived in an identical way to the first order procedure. The first parameter required in the method is given by Eqs.(4.9) and (4.10), the second parameter is calculated from Eq.(4.5) with the aid of Eqs.(2.1) and (4.6).

Therefore, ignoring higher order terms in $\Delta \lambda_i$:

$$(A_i + \Delta \lambda_i C_i) \left(1 - \frac{(\Delta \bar{\mathbf{U}}_i + \Delta \lambda_i \Delta \mathbf{U}_T)^T (\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1})}{\Delta \mathbf{U}_{i-1}^T (\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1})} \right) - 1 \tag{4.11}$$

$$= B_i - \Delta \lambda_i A_i D_i + \Delta \lambda_i C_i - \Delta \lambda_i C_i E_i,$$

where

$$D_i = \frac{\Delta \mathbf{U}_T^T (\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1})}{\Delta \mathbf{U}_{i-1}^T (\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1})}, \tag{4.12}$$

$$E_i = \frac{\Delta \bar{\mathbf{U}}_i^T (\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1})}{\Delta \mathbf{U}_{i-1}^T (\Delta \mathbf{p}_i - \Delta \mathbf{p}_{i-1})}. \tag{4.13}$$

From Eqs.(4.9) and (4.11), the final form of the constrained second-order secant-Newton method displacement increment vector $\Delta \mathbf{U}_i$ is obtained as:

$$\Delta \mathbf{U}_i = \Delta \mathbf{U}_i^{(0)} + \Delta\lambda_i \Delta \mathbf{U}_1^{(1)}, \tag{4.14}$$

where

$$\Delta \mathbf{U}^{(0)} = A_i \Delta \bar{\mathbf{U}}_i + B_i \Delta \mathbf{U}_{i-1}, \tag{4.15}$$

$$\Delta \mathbf{U}_i^{(1)} = A_i \Delta \mathbf{U}_T + C_i \Delta \bar{\mathbf{U}}_i - [A_i D_i - (1 - E_i) C_i] \Delta \mathbf{U}_{i-1}. \tag{4.16}$$

The parameters A_i and B_i used in formulae (4.11), (4.15) and (4.16) are the second order secant-Newton parameters for the fixed load level iterations. The load adjustment parameter, $\Delta\lambda_i$, can be calculated from a suitable constraint equation obtained from the normal path (Eq.(2.17)) or the arc-length (Eq.(2.9)) control.

For difficult iterations, it may be necessary to change from secant iterations to modified Newton–Raphson iterations. For automatic control of the iteration process, cut-out criteria are introduced, similar to those described earlier. For the variable load iterations, these criteria take the following form:

$$\frac{1}{\mathrm{STOL1}} < A_i + \Delta\lambda_i C_i < \mathrm{STOL1},$$

$$-\frac{1}{2}\mathrm{STOL} < \frac{B_i - \Delta\lambda_1 [A_i D_i - (1 - E_i) C_i]}{A_i + \Delta\lambda_i C_i} < \mathrm{STOL}.$$

5. Further accelerations

The highly nonlinear behaviour of structures entering strain-softening regions requires special treatment of the system of equations obtained in the incremental finite element method. On many occasions, the solution procedure was more stable if the modified stiffness matrix based on the secant material stiffness relationship for the damaged material was used. This, however, tended to slow down the convergence rate if modified Newton-Raphson iterations were employed. The following method helps to overcome convergence problems in load control regime.

5.1. Accelerated secant-Newton method

For problems requiring a load controlled solution, the second-order secant-Newton method was implemented. As seen from Eqs.(4.2) and (5.4), the secant-Newton method, compared with the modified Newton-Raphson method, requires two additional vectors $\Delta\mathbf{U}_{i-1}$ and $\Delta\mathbf{p}_{i-1}$ to be stored in order to calculate parameters A_i and B_i. The method presented earlier was further accelerated by applying the explicit line search parameter η_i to Eq.(4.3):

$$\mathbf{U}_i = \mathbf{U}_{i-1} + \eta_i(A_i\Delta\bar{\mathbf{U}}_i + B_i\Delta\mathbf{U}_{i-1}), \tag{5.1}$$

where the step length parameter η_i is selected according to the procedure described in Ref.[18].

The line search procedure employed is the three-step search. If the tolerance criteria for the line search are not satisfied within the three steps, the search is terminated. The method was used in load controlled examples described in Ref.[20, 21] and proved to be very efficient. As with the secant-Newton method of Eq.(4.3), the accelerated method requires cut-out criteria. The following limits on A_i and B_i were placed:

$$A_i < \text{STOL}, \qquad \text{STOL1} < \frac{B_i}{A_i} < \text{STOL2},$$

where, depending on the problem analysed, STOL was taken between 1.5 and 3.0, STOL1 was between -0.20 and -0.40, and STOL2 was between 0.30 and 0.50.

5.2. Modified Ramm's method

To trace full load-deflection curves, the variable load level iterative method with Ramm's normal path constraint can be successfully employed. Since, in its original form, Ramm's method was based on modified Newton–Raphson iterations, the method was modified by including the line search concept.

The consistent use of the line search with a variable load level iterative method requires the use of an internal loop to determine compatible values of the line search based step length parameter, η_i , and the load adjustment, $\Delta\lambda_i$. The introduction of the step length parameter η_i into

Eq.(2.1) results in the change of the value of $\Delta\lambda_i$ calculated in Eq.(2.17). The new value of $\Delta\lambda_i$ alters, in turn, the value of the dot product of the out-of-balance force vector and the displacement increment vector.

To avoid the need to implement an internal loop, an approximate method is proposed. This alternative method is a two-step approach. In the first step, the load level is adjusted by using the constraint of Eq.(2.17) as:

$$\lambda_{i+1} = \lambda_i + \Delta\lambda_i, \tag{5.2}$$

where λ_i and λ_{i+1} are the load multipliers at the beginning of iteration i and $i+1$, respectively. After the load level is altered, the line search on the new load level is carried out as:

$$\Delta\mathbf{U}_i = \eta_i(\Delta\bar{\mathbf{U}}_i + \Delta\lambda_i\Delta\mathbf{U}_T). \tag{5.3}$$

This variable load level iterative method was successfully employed in the numerical examples requiring displacement controlled loading presented in detail in Refs [21-23].

6. Conclusions

One of the most important aspects of nonlinear finite element analysis is the solution procedure used for the system of nonlinear equations. This paper has presented several important solution algorithms which can be used effectively in nonlinear problems. The emphasis has been placed on numerical procedures capable of converging in problems where severe material nonlinearity occurs. It is the author's belief that the introduction of some form of line search, preferably the explicit procedure, may greatly improve the performance of the solution technique in problems involving strain-softening of the material. This is of great importance in strategies where different material stiffness laws are used to assemble the global stiffness matrix and to integrate element stresses. The use of procedures iterating simultaneously in load and displacement spaces opens possibility of detecting instabilities in the behaviour of a structure. Such instabilities are easily missed if load controlled process is used.

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Streszczenie

ANALIZA NIELINIOWYCH ROZWIĄZAŃ METODĄ ELEMENTÓW SKOŃCZONYCH DLA KONSTRUKCJI BETONOWYCH

Podstawowym celem nieliniowej metody elementów skończonych jest rozwiązanie układu nieliniowych równań. Ze względu na odkształceniowo-osłabieniową naturę procesów zniszczenia konstrukcje betonowe wymagają specjalnej ostrożności w stosowaniu wygodnej nieliniowej procedury rozwiązującej. Nowe, złożone modele materiału i potrzeba określenia punktów granicznych oraz pokrytycznego zachowania się konstrukcji wymaga zastosowania metod zdolnych do uzyskania rozwiązań zbieżnych w otoczeniu takich punktów granicznych. Praca podaje przegląd kilku nieliniowych metod rozwiązań ze szczególnym zwróceniem uwagi na ich użyteczność w nieliniowej analizie konstrukcji betonowych wykazujących efekty odkształceniowego osłabienia.

Резюме

АНАЛИЗ НЕЛИНЕЙНЫХ РЕШЕНИЙ МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ БЕТОННЫХ КОНСТРУКЦИЙ

Основной целью нелинейного метода конечных элементов является решение системы нелинейных управнений. Из-за деформационной-ослабленной природы процессов разрушения, бетонные конструкции требуют специальной осторожности, при применени выгодной нелинейной решающей процедуры. Новые, сложные модели матеряла и необходимость определения предельных точек, а также послекритического поведения конструкции требуют применения методов способных для получения решений сходящихся в окрестности таких предельных точек. Настоящая работа приводит обозрение нескольких нелинейных методов решений, с особенным обращением внимания на их полезность в нелинейном анализе бетонных конструкций, обладающих эффектами деформационного ослабления.

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