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IMPACT OF A CYLINDER AGAINST A RIGID TARGET PART I. COMPATIBILITY CONDITIONS AND THE VISCOPLASTIC REGION EVOLUTION

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In the paper the impact test of a short deformable cylinder against a rigid target is analysed in terms of the rigid viscoplastic model [1]. The one-dimensional analysis of that experimental configuration has been carried out by numerous authors, e.g., [2, 3, 4, 5, 7]. In this paper the axi-symmetric geometry, radial inertia, and finite deformations are taken into account. The equations of motion together with the constitutive model lead to an initial moving boundary-value problem. The analysis of the field regularity on the moving rigid-viscoplastic boundary is the crucial point of the considerations. It leads to the conclusion that the initial discontinuous condition does not belong to the solution. The relationship describing the displacement velocity of the boundary is derived as a function of the process fields.

1. Introduction

The paper is related to the series of publications (cf.[2]–[7]) devoted to the analysis of the dynamic test carried out for the first time by WHIFFIN [6], widely known as the "Taylor's experimental configuration" [7]. In those papers, however, the $1D$ (one-dimensional) rigid viscoplastic formulation of the test has been presented. Moreover, a complicated elastic-plastic wave analysis has been omitted by disregarding the elastic response of materials.

In the present paper the $3D$ formulation of space variables is discussed under the cylindrical symmetry conditions. As far as the constitutive model is concerned, the rigidity before the plastic limit is assumed first and rate-dependent viscoplastic behaviour afterwards. For the discussed test of the impact of a cylinder against a rigid target, the equations of motion together with the constitutive model adopted in the paper lead to an initial moving boundary-value problem.

The moving boundary bounds the plastic region propagating towards the rear end. The axi-symmetric formulation of this problem demands the analysis of the properties of fields on the moving surface bounding the plastic region.

The statement of the initial conditions for the system of equations is a crucial point. The phenomenon of the impact of a cylinder against a rigid target leads in the mathematical model to the initial discontinuity of the velocity on the cylinder-target contact plane. Therefore, the analysis of the regularity degree carried out in the paper is very important for the problem statement correctness. The continuity and compatibility conditions on the separating surface for the fields describing the deformation process have been investigated. Moreover, the relationships determining the shape and the velocity of the surface during the process for the proposed parametrization have been derived. The proof of the velocity field continuity presented in the paper shows that the finite jumps of discontinuites are not admitted for the system of equations governing the problem and, consequently, a discontinuous initial-boundary condition can not be included in the solution of the problem. Thus, the equations of the viscoplastic process have to be considered in a left-sided open interval $(t_0, t_k]$, where t_k is the process final time.

2. Formulation of the initial-boundary problem

A short and stress-free cylindrical specimen strikes perpendicularly on a rigid target with the velocity v_0 (Fig.1). The velocity v_0 belongs to the interval from ca. 50m/s to 500m/s. The interval is related to the structural dynamic problems for a viscoplastic material model. Hence, the velocity does not exceed the value of 600 m/s when the complete penetration phenomenon may occur (according to empirical tests) (1).

It is assumed that thermal effects, body forces and friction forces between the target and the specimen are neglected. Radial inertia and

(1)The critical velocity determined empirically when the complete penetration occurs with the 50% probability is 850 m/s. Some considerations on that topic are to be found, e.g., in [8], [9].

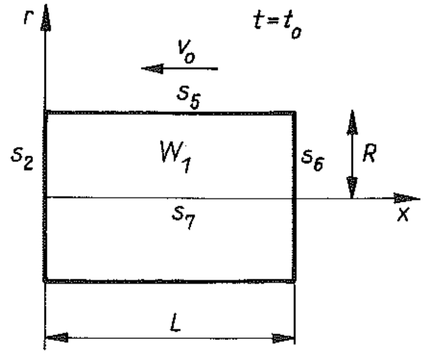

Fig. 1.

axi-symmetric state of stress only are taken into account. The axial-symmetry condition allows to restrict considerations to fields determined for two space Euler variables x, r and the time t. Hence, instead of $3D$ regions W_1 and W_2, it is sufficient to consider their projections on the plane $x-r$ (subsets of a semiplane lying above the x axis). Then, instead of the boundary surfaces S_1, S_2, S_4, S_5 and S_6, it is sufficient to consider boundary curves.

The behaviour of the material is described by the rigid-perfectly plastic constitutive equations (2.1), (2.2) proposed by PERZYNA [1].

$$\mathbf{d} = \gamma\left(\frac{\sqrt{J_2}}{\kappa}-1\right)^3\frac{\mathbf{s}}{\sqrt{J_2}} \quad \text{for} \quad \frac{\sqrt{J_2}}{\kappa}-1>0, \tag{2.1}$$

$$\mathbf{d} = 0 \quad \text{for} \quad \frac{\sqrt{J_2}}{\kappa}-1\leq 0, \tag{2.2}$$

where $\mathbf{d}$ is the stretching tensor, $\mathbf{s}$ the deviatoric Cauchy stress tensor, J_2 the second invariant of $\mathbf{s}$, γ the viscosity coefficient, and κ the yield limit in shearing. Hence, equation $\sqrt{J_2}=\kappa$ represents the yield condition. In the problem under consideration, the tensor $\mathbf{d}$ is equal to its deviatoric part because of the incompressibility constraints $\operatorname{tr}\mathbf{d}=0$.

In the case when the elastic response of the material is taken into account, the wave propagation phenomena must be considered. The introducing of the constitutive relations (2.1), (2.2) is equivalent to the assumption that the elastic wave speed caused by dynamic loading is infinite.

From the physical point of view, after the impact the elastic loading and unloading processes in the specimen occur in the final time interval. For the rigid-viscoplastic material model, however, the loading process occurs in an infinitly short time, because the wave propagation process has been neglected. In what follows, for the process model under consideration, the time t_0 refers to the instant at which the stress-free cylinder just touches the target and the condition $v_x = v_0$ on S_2 is fulfilled. If the initial striking velocity is high enough, the yield condition

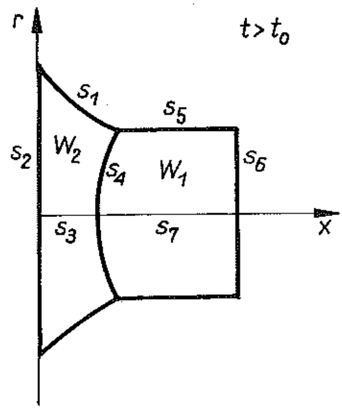

Fig. 2.

is exceeded ($\sqrt{J_2} - \kappa > 0$) and the plastic deformation in the region W_2 occurs. On S_4 (Fig.2), where the surface separates two parts of the specimen, i.e., the plastic one W_2 and the rigid one W_1 (where the condition $\sqrt{J_2} - \kappa \leq 0$ is valid), the relation $\sqrt{J_2} - \kappa = 0$ holds. The velocity $\mathbf{c}$ of the surface S_4 (see, e.g., [11], [12]) is one of unknowns to be determined as a part of the solution.

The rigid specimen end moves parallelly to the x direction with the velocity $\mathbf{v}$. Hence, the physical components of $\mathbf{v}$ are $v_x = v(t)$, $v_r = 0$, $v_\theta = 0$, with the normal in the x direction.

The viscoplastic flow process for $t > t_0$, when $W_2 \neq \emptyset$, is governed by the system of Eqs.(2.3)–(2.12) formulated in the physical components of the cylindrical coordinate system

$$(2.3) \qquad \frac{\partial(s_{xx} + \sigma)}{\partial x} + \frac{\partial s_{xr}}{\partial r} + \frac{s_{xr}}{r} = \rho \dot{v}_x,$$

$$(2.4) \qquad \frac{\partial s_{xr}}{\partial x} + \frac{\partial(s_{rr} + \sigma)}{\partial r} + \frac{2s_{rr} + s_{xx}}{r} = \rho \dot{v}_r,$$

$$d_{xx} = \gamma \left(\frac{\sqrt{J_2}}{\kappa} - 1 \right)^3 \frac{s_{xx}}{\sqrt{J_2}} \quad \text{for} \quad \frac{\sqrt{J_2}}{\kappa} - 1 > 0, \tag{2.5}$$

$$d_{rr} = \gamma \left(\frac{\sqrt{J_2}}{\kappa} - 1 \right)^3 \frac{s_{rr}}{\sqrt{J_2}} \quad \text{for} \quad \frac{\sqrt{J_2}}{\kappa} - 1 > 0, \tag{2.6}$$

$$d_{xr} = \gamma \left(\frac{\sqrt{J_2}}{\kappa} - 1 \right)^3 \frac{s_{xr}}{\sqrt{J_2}} \quad \text{for} \quad \frac{\sqrt{J_2}}{\kappa} - 1 > 0, \tag{2.7}$$

$$d_{xx} = d_{rr} = d_{xr} = 0 \qquad \text{for} \quad \frac{\sqrt{J_2}}{\kappa} - 1 \leq 0, \tag{2.8}$$

$$d_{xx} = \frac{\partial v_x}{\partial x}, \qquad d_{rr} = \frac{\partial v_r}{\partial r}, \qquad d_{\theta\theta} = \frac{v_r}{r}, \tag{2.9}$$

$$dxr = \frac{1}{2} \left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x} \right), \tag{2.10}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0. \tag{2.11}$$

The equations are nontrivial to be satisfied only in region W_2 . In the rigid (time-dependent) region W_1 for each $t \in (t_0, t_k)$, due to vanishing of $\mathbf{d}$ and v_r, the only nontrivial equation is the global momentum balance

$$\rho \frac{d}{dt} \left(v(t) \int_{W_1} dv \right) = \int_{S_4} (\sigma n_x + s_{xx} n_x + s_{xr} n_r)\, ds + \int_{\bar{S}_5 \cup \bar{S}_6} (\sigma n_x + s_{xx} n_x + s_{xr} n_r)\, ds. \tag{2.12}$$

The system of equations (2.3)–(2.12) is accompanied by boundary and initial conditions for $t > t_0$ (Fig.2) in the following form

$$\begin{aligned} (s_{xx} + \sigma) n_x + s_{xr} n_r &= 0, \\ s_{xr} n_x + (s_{rr} + \sigma) n_r &= 0, \end{aligned} \quad \text{on} \quad \bar{S}_1 \cup \bar{S}_5 \cup \bar{S}_6, \tag{2.13}$$

$$v_x = 0, \qquad s_{xr} = 0, \qquad \text{on} \quad \bar{S}_2, \tag{2.14}$$

$$v_r = 0, \qquad s_{xr} = 0, \qquad \text{on} \quad \bar{S}_3 \cup \bar{S}_7. \tag{2.15}$$

The spherical part of the stress tensor is denoted by σ, while ρ is the specimen mass density, n_x and n_r are components of the unit normal to the surface S_4, and $\dot{v}_x, \dot{v}_r$ denote components of the velocity material derivatives.

The physical initial conditions at $t = t_0$ (Fig.1) are as follows

$$\chi = \xi_0, \tag{2.16}$$

$$v_x = v_0, \qquad v_r = 0, \qquad \text{on} \quad W_1 \cup \bar{S}_2, \tag{2.17}$$

where χ is the motion function on $W_2 \cup \bar{S}_2$, χ_0 describes the initial specimen configuration, while conditions $(2.17)_1$, $(2.17)_2$ are the consequence of the rigid specimen behaviour (Fig.1).

In the time interval $(t_0,\ t_k]$ the boundary conditions (2.13)–(2.15) have been satisfied. When the boundary condition (2.13) is taken into account, the last integral in Eq.(2.12) disappears. The form of Eq.(2.12) simplified in such a way will be used in further considerations.

To compare the present formulation with the $1D$ formulation discussed for example by TING [4], let us notice that the latter formulation resolves itself into the solution of the nonlinear parabolic equation

$$\frac{\partial}{\partial x}\left(\alpha \frac{\partial v}{\partial x}\right)^{1/p} - \frac{\partial v}{\partial t} = 0, \qquad 0 \le x \le \zeta(t), \tag{2.18}$$

with a moving boundary $\zeta(t)$. Here $\zeta(t)$ denotes the relative displacement of the boundary with respect to the target, p is the exponent of the constitutive function, α is the equivalent Saint-Venant parameter (see [2]), whereas v and x are nondimensional quantities denoting the velocity field and the Lagrangian coordinate, respectively. The existence of the moving boundary in the Ting's solution is expressed by the following scalar differential equation

$$\left.\frac{\partial v}{\partial t}\right|_{x=\zeta} = -\frac{1}{1-\zeta}, \tag{2.19}$$

which completes the basic equation (2.18) determining the character of the problem considered.

For the case considered in the present paper the $3D$ axi-symmetric state permits the displacement velocity of the boundary S_4 (Fig.2) to be radially variable. Moreover, the governing system of equations (2.3)–(2.11) can not be simplified to one scalar equation for the velocity component v; we are left with three unknown field components v_x, v_r and σ not mentioning the position of the moving boundary. All these variables are functions of two space variables x, r and time t. By a process of resubstitution it is possible, however, to end up with two equations of motion of the second order in the spatial derivatives for v_x and v_r, and one equation of the first order in v_x, v_r and σ. In the former two equations the first order time derivatives of v_x and v_r appear, whereas the latter equation is time independent. Consequently, the resulting system of equations formally unclassified can be split in two equations of a parabolic type and one first order differential equation in spatial variables, provided that σ is regarded as known (in the first two equations). For the local problem formulation, the local equation relating this velocity with other process fields has to be found. The velocity of the displacement of the viscoplastic region will be defined in Sect.3. Since the boundary between the viscoplastically deformed region denoted by W_2 and the rigid one denoted by W_1 may be a carrier of discontinuities in some field variables, it is necessary to check whether the velocity and acceleration fields preserve their continuity on the surface separating the both regions (similarly to the $1D$ case), or whether they are loosing it. If the latter case happens, it will be necessary to add to the system of equations the relations imposing constraints on existing jumps, i.e., the so-called compatibility conditions. The study of that problem has already been initiated in paper [10].

3. Compatibility conditions on the moving surface

In the problem considered the moving surface S_4 separates the specimen regions possessing different material properties. It may happen that on S_4 fields describing the process lose their continuity. The values of discontinuity jumps are not unrestricted, however, so some constraints are imposed on them resulting from the balance laws and the kinematic compatibility conditions. These constraints, known as the generalized Rankine-Hugoniot (dynamic and kinematic) compatibility conditions (see, e.g., [11], [12][(2)]) for points lying on the surface, supplement the basic system of equations (2.3)–(2.12). They will form for us a kind of tool in analyzing the continuity of the fields describing the process in the whole region $\bar{W}_1 \cup \bar{W}_2$.

Let us consider any tensor function f continuous and differentiable in the region $\bar{W}_1 \cup \bar{W}_2$ excluding the surface S_4. A jump of the function

[(2)]The position [12] is an extended English version of [11].

f on the surface S_4 is defined as

$$[\![f]\!] \equiv f_2 - f_1, \tag{3.1}$$

where f_2 and f_1 are the limit values of the function f when any point on S_4 is reached from W_2 or W_1, respectively. If $[\![f]\!] \neq 0$, then the surface S_4 is called the discontinuity surface of the function f. We assume that at any time from $(t_0, t_k]$ of the process, the component of the unit normal n_x is different from zero. The stretching tensor field in the rigid material region W_1 is equal zero. On the other side of the boundary S_4 from the constitutive properties of the region W_2 it results that for points belonging to S_4 and reached from the side of that region equality (2.2) holds. Consequently, it follows that on the boundary S_4 separating these two regions the field of stretching tensor $\mathbf{d}$ is equal to zero and does not lose the continuity, i.e.,

$$\mathbf{d}_1 = \mathbf{d}_2 = 0. \tag{3.2}$$

The mass balance equation restricted to the surface S_4 is following

$$[\![\rho(v_n - u_n)]\!] = 0, \tag{3.3}$$

where $v_n = \mathbf{v} \cdot \mathbf{n}$ ($\mathbf{n}$ is the unit normal to S_4) is the normal component of the particle velocity on S_4 and u_n is the normal speed of displacement of the surface (see e.g., [11], [12]), i.e., $u_n = \mathbf{c} \cdot \mathbf{n}$, where $\mathbf{c}$ is the velocity of displacement of S_4.

The mass balance law for incompressible continuum (**div v**=0, see Eq.(2.11)) having a continuous initial density distribution is satisfied by the density field ρ, uniform and stationary in the whole process. However, the incompressibility condition is valid for both the rigid and plastic regions. The field $\rho = \rho_1 = \rho_2$ (see denotation (3.1)) is the constant parameter of the process. From relationship (3.3) it results that the normal component of the particle velocity on the moving boundary S_4 preserves its continuity for continuum with the isochoric motion condition,

$$[\![\mathbf{v}]\!] \cdot \mathbf{n} = [\![v_n]\!] = 0, \tag{3.4}$$

where $\mathbf{n} = [n_x, n_r, 0]$.

In the $1D$ case (see [4]), when the only component of the particle velocity $\mathbf{v}$ is v_n, the condition (3.4) denotes the continuity of the velocity field $\mathbf{v}$. In the axi-symmetric situation the continuity of the field results from the constitutive properties of the material. Moreover, the relationship $(2.9)_3$ due to the continuity and vanishing of $\mathbf{d}$ on either side of S_4 leads to

$$v_{2r} = 0, \tag{3.5}$$

whereby, taking into account condition (3.4) and the kinematic restriction on the region W_1 ($v_{1r} = 0$) we get, provided that $n_x \neq 0$,

$$[\![\mathbf{v}]\!] = 0. \tag{3.6}$$

Let us examine whether the assumption about the continuous distribution of the stress tensor field on the surface S_4 can be accepted. For the deviatoric part of the stress tensor the inverse transformation to the constitutive equation (2.1) gives

$$\mathbf{s} = \kappa \left[\left(\frac{\sqrt{J_d}}{\gamma} \right)^{\frac{1}{3}} + 1 \right] \frac{\mathbf{d}}{\sqrt{J_d}} \quad \text{for} \quad \mathbf{d} \neq \mathbf{0}. \tag{3.7}$$

If $\mathbf{d}$ tends to zero, then the deviatoric stress $\mathbf{s}$ reaches its limit value

$$\lim_{\mathbf{d}\to 0} \mathbf{s} = \lim_{\mathbf{d}\to 0} \kappa \frac{\mathbf{d}}{\sqrt{J_d}}. \tag{3.8}$$

Due to the continuity of $\mathbf{d}$, this limit value(3) can be assigned to the deviatoric stress field for points lying on the surface S_4 and reached from the region W_2. Then, the deviatoric stress tensor could be made continuous in the whole region $\bar{W}_1 \cup \bar{W}_2$. However it is not possible till now to determine the exact value of the deviatoric stress tensor on S_4 only from the relation (3.8), because the expression on the right side will take in the limit the form of 0/0. On the contrary, the invariant J_2 has the determined value κ^2 on the surface S_4 .

To examine the spherical part of the stress tensor let us utilize the first order dynamical condition of compatibility expressed by the relation

$$[\![\rho(v_n - u_n)\mathbf{v} + (\mathbf{s} + \sigma\mathbf{1}) \cdot \mathbf{n}]\!] = 0. \tag{3.9}$$

Making use of (3.3) and the conclusion (3.6), the equation (3.9) can be simplified to

$$[\![\mathbf{s}]\!] \cdot \mathbf{n} + [\![\sigma\mathbf{1}]\!] \cdot \mathbf{n} = 0. \tag{3.10}$$

(3)It should be noted that this limit exists for each (x,t), because the series $\{\mathbf{s}\}_{\mathbf{d}\to 0}$ is norm convergent and the tensor space is finite-dimensional, hence it is reflexive (and each sphere is weakly compact).

The continuity of the deviatoric stress $\mathbf{s}$ on S_4 would imply that the first term in (3.10) disappears; this would lead to the continuity of the pressure σ.

Let us analyse the continuity of the acceleration field. Because of the continuity of the velocity field on S_4 one can employ the Maxwell theorem (see, e.g., [11], [12]) for this field expressed by the following relations

$$[\![\mathrm{grad}\,\mathbf{v}]\!] = [\![\frac{\partial \mathbf{v}}{\partial n}]\!] \otimes \mathbf{n}, \qquad [\![\frac{\partial \mathbf{v}}{\partial t}]\!] = -u_n [\![\frac{\partial \mathbf{v}}{\partial n}]\!], \tag{3.11}$$

where "$\otimes$" denotes the tensor product. Recall that in the process considered we exclude such configuration of the boundary S_4 for which at some point $n_x = 0$. Let us carry out the componentwise analysis of the tensor relationships $(3.11)_1$ and $(3.11)_2$:

$$[\![\frac{\partial v_x}{\partial x}]\!] = [\![\frac{\partial v_x}{\partial n}]\!] n_x. \tag{3.12}$$

On the left side of (3.12) we have the component d_{xx} of the stretching tensor $\mathbf{d}$ which is continuous on the boundary S_4, hence for the first component of the velocity gradient on the normal direction we have condition

$$[\![\frac{\partial v_x}{\partial n}]\!] = 0. \tag{3.13}$$

Exploiting it for the next component (3.11)

$$[\![\frac{\partial v_x}{\partial r}]\!] = [\![\frac{\partial v_x}{\partial n}]\!] n_r \tag{3.14}$$

gives us the continuity of the mixed components of the velocity gradient

$$[\![\frac{\partial v_x}{\partial r}]\!] = 0, \qquad [\![\frac{\partial v_r}{\partial x}]\!] = 0, \tag{3.15}$$

where $(3.15)_2$ can be obtained from (3.2) for mixed components of the tensor $\mathbf{d}$

$$\frac{1}{2}[\![\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}]\!] = 0. \tag{3.16}$$

Let us analyse the subsequent componentwise consequence of Eq.$(3.11)_1$

$$[\![\frac{\partial v_r}{\partial x}]\!] = [\![\frac{\partial v_r}{\partial n}]\!] n_x. \tag{3.17}$$

From the above relationship one can see that condition $(3.15)_2$ implies the continuity of the component $\partial v_r / \partial n$.

The above considerations lead to the conclusion that the both normal components and whole velocity gradient preserve the continuity on the moving boundary S_4 of the spreading plastic region. The conclusion can be formulated in the tensor notation

$$(3.18) \qquad [\![\frac{\partial \mathbf{v}}{\partial n}]\!] = 0, \qquad [\![\mathbf{grad}\ \mathbf{v}]\!] = 0.$$

Note that from the rigidity of the region W_1 characterized by the disappearing of the velocity gradient in the whole region W_1 it results that $(\mathbf{grad}\ \mathbf{v})_2 = 0$, because of Eq.$(3.18)_2$. Utilizing relationship $(3.18)_1$, we can prove easily the continuity of the velocity field partial derivative. The zero value of the jump

$$(3.19) \qquad [\![\frac{\partial \mathbf{v}}{\partial t}]\!] = 0$$

results directly from taking into account $(3.18)_1$ in relationship $(3.11)_2$, if the boundary S_4 is not stationary, i.e., $u_n \neq 0$.

Currently we have all conditions needed to state whether on S_4 can lay points of the first order discontinuity for the acceleration field. The acceleration of the material points of the continuum is given by relationship

$$(3.20) \qquad \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{grad}\ \mathbf{v} \cdot \mathbf{v},$$

which formulated in jumps takes a form

$$(3.21) \qquad [\![\dot{\mathbf{v}}]\!] = [\![\frac{\partial \mathbf{v}}{\partial t}]\!] + [\![\mathbf{grad}\ \mathbf{v} \cdot \mathbf{v}]\!].$$

Disappearing of the jumps at the right side of Eq.(3.21) (see Eqs.(3.6), $(3.18)_2$ and (3.19)) preserves the continuity of the acceleration field on S_4

$$(3.22) \qquad [\![\dot{\mathbf{v}}]\!] = 0,$$

and in the whole region $\bar{W}_1 \cup \bar{W}_2$ as well.

Note that the local time derivative $\partial \mathbf{v}/\partial t$ does not need to disappear in the rigid region, so it is not neccesary that

$$(3.23) \qquad \left(\frac{\partial \mathbf{v}}{\partial t}\right)_2 = 0$$

should hold.

Let us check the material derivative of the acceleration field. Assuming that S_4 is a third order singular surface of the motion function χ, it is possible to derive (using the Maxwell theorem for second derivatives of the velocity field) following relationships

$$(3.24)\qquad [\![\mathbf{grad\ grad\ v}]\!] = [\![\frac{\partial^2 \mathbf{v}}{\partial n^2}]\!]\mathbf{n}\otimes\mathbf{n},$$

$$(3.25)\qquad [\![\mathbf{grad}\ \frac{\partial \mathbf{v}}{\partial t}]\!] = -u_n[\![\frac{\partial^2 \mathbf{v}}{\partial n^2}]\!]\otimes\mathbf{n},$$

$$(3.26)\qquad [\![\frac{\partial^2 \mathbf{v}}{\partial t^2}]\!] = u_n^2[\![\frac{\partial^2 \mathbf{v}}{\partial n^2}]\!].$$

After contraction of Eq.(3.25) we get

$$(3.27)\qquad [\![\mathbf{div}\frac{\partial \mathbf{v}}{\partial t}]\!] = -u_n[\![\frac{\partial^2 \mathbf{v}}{\partial n^2}]\!]\cdot\mathbf{n}.$$

Recall that the condition of isochoric motion $\mathbf{div\ v} = 0$ written after differentiation

$$(3.28)\qquad \frac{\partial}{\partial t}\mathbf{div\ v} = \mathbf{div}\ \frac{\partial \mathbf{v}}{\partial t} = 0,$$

should be fulfilled on both sides of the boundary S_4. Thus, it can be written as follows

$$(3.29)\qquad [\![\mathbf{div\ v}]\!] = 0 \quad \text{and} \quad [\![\mathbf{div}\frac{\partial \mathbf{v}}{\partial t}]\!] = 0.$$

Comparing Eq.(3.27) with Eqs.$(3.29)_2$ we can see that, if the boundary S_4 is not stationary, i.e., $u_n \neq 0$, then the normal projection of the jump of the second derivative of $\mathbf{v}$ must be zero. If this jump is denoted by

$$(3.30)\qquad [\![\frac{\partial^2 \mathbf{v}}{\partial n^2}]\!] =: \mathbf{a},$$

then for the nonstationary boundary the condition (3.27) takes the form

$$(3.31)\qquad \mathbf{a}\cdot\mathbf{n} = 0.$$

The expression (3.32) for the jump acceleration material derivative is obtained using the definition of this field written in jumps

$$(3.32)\qquad [\![\dot{\mathbf{v}}]\!] = [\![\frac{\partial \dot{\mathbf{v}}}{\partial t}]\!] + [\![\mathbf{grad}\ \dot{\mathbf{v}} \cdot \mathbf{v}]\!] = \\ = [\![\frac{\partial^2 \mathbf{v}}{\partial t^2} + \frac{\partial}{\partial t}(\mathbf{grad}\ \mathbf{v} \cdot \mathbf{v}) + \mathbf{grad}\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{grad}\ \mathbf{v} \cdot \mathbf{v}\right) \cdot \mathbf{v}]\!].$$

Differentiating and making use of the continuity conditions $(3.18)_2$ and (3.19) for the first time and space derivatives of the velocity, we get

$$(3.32)\qquad [\![\dot{\mathbf{v}}]\!] = [\![\frac{\partial^2 \mathbf{v}}{\partial t^2} + 2\mathbf{grad}\ \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{v} + \mathbf{grad}\ \mathbf{grad}\ \mathbf{v} \cdot \mathbf{v} \otimes \mathbf{v}]\!].$$

Taking into account the Maxwell theorem (Eqs.(3.24)–(3.26)) and the denotation (3.30), we obtain an expression for the jump of the acceleration material derivative in the following form

$$(3.33)\qquad [\![\dot{\mathbf{v}}]\!] = (u_n - v_n)^2 \mathbf{a}.$$

If we employ relationship (3.31) in the analysis of the equation written above, then as a result we conclude that the tangent component of the acceleration field only can suffer a jump on the surface S_4 . The continuity of both the components of that derivative takes place only in the case, when the surface S_4 is the material one, i.e., $u_n = v_n$.

The results we have obtained indicate that the moving boundary is at least the second order surface for the velocity function $\mathbf{v}$, i.e., the first derivatives of the velocity function are continuous. Closing these considerations, let us turn to the condition (3.24) that, combined with the rigidity condition for the region W_1, leads to the relationship

$$(3.34)\qquad (\Delta \mathbf{v})_2 = \mathbf{a},$$

where Δ denotes Laplacian.

4. Description of the moving boundary in the process

In the $1D$ case [4] the description of movement of the rigid-plastic boundary is based on Eq.(2.19) which gives the motion of the point at which two subintervals, i.e., rigid and plastically deformed, meet. For the combined state of stress, however, the projection of the boundary surface on the $x-r$-plane is a curve, moving in the x direction. Hence, its velocity of displacement $\mathbf{c}$ will depend on r; after performing the time differentation in Eq.(2.12) the velocity $\mathbf{c}$ will appear on its left hand side. It is worthwile to notice that (2.19) corresponds to the differentiated form of (2.12) which expresses the global balance law of linear momentum for the rigid part W_1.

For the global formulation of the initial-value problem with the moving boundary separating two $2D$ regions, a local relationship determinig the velocity $\mathbf{c}$ of this boundary could be of the great help. The surface S_4 (the curve, exactly) moving in the continuum can be described either by the implicit form

$$g(x, r, t) = 0 \tag{4.1}$$

or by the parametric representation

$$x = \varphi(l, t); \qquad r = l, \tag{4.2}$$

where l is a parameter of the curve S_4 and g is a C^2 -class function with the nonvanishing gradient. The velocity of displacement of S_4 is given by (see, e.g., [11], [12])

$$\mathbf{c} = \frac{\partial}{\partial t}(\varphi, r) = \left(\frac{\partial \varphi}{\partial t}, 0\right). \tag{4.3}$$

The main problem concerns the form of the function $g(x, r, t)$. For this purpose we employ the yield condition that has to hold on the boundary S_4 in every configuration during the deformation process. Hence, the moving boundary S_4 is given by

$$\{(x, r, t) : g(x, r, t) = 0\}, \quad \text{where} \quad g(x, r, t) \equiv 1 - \sqrt{J_2}/\kappa. \tag{4.4}$$

The total time differentation of Eqs.(4.4) gives on the boundary S_4 the following relation

$$\frac{\partial g}{\partial t} + \mathbf{grad}\, g \cdot \mathbf{c} = 0, \tag{4.5}$$

where

$$\mathbf{n} := \frac{\mathbf{grad}\, g}{\|\mathbf{grad}\, g\|} \tag{4.6}$$

represents the normal vector to the boundary S_4 and

$$u_n := -\frac{\dfrac{\partial g}{\partial t}}{\|\mathbf{grad}\, g\|} \tag{4.7}$$

gives the normal speed of displacement of the surface, i.e., $u_n = \mathbf{c} \cdot \mathbf{n}$ the normal component of the velocity $\mathbf{c}$. The velocity vector $\mathbf{c}$ has two components normal and tangential to S_4. The latter one depends on the assumed parametrization. We have proposed the parametric representation of the moving boundary in the form (4.2). It results from the axi-symmetric geometry that permits to consider a curve S_4 instead of the surface S_4 .

In the general $3D$ case the third component θ should occur with the refering parametric equation $\theta = \theta$. Then, we have two vectors tangent to the surface $S_4 : \mathbf{e}_1 = (\varphi_{,r}, 1, 0)$ and $\mathbf{e}_2 = (0, 0, 1)$; hence we get

$$\mathbf{n} = (1, -\varphi_{,r}, 0) \;/\; \sqrt{1 + \varphi_{,r}^2}. \tag{4.8}$$

For the parametrization Eq.(4.2) the paths of the points l = constans lying on the surface S_4 are straight lines parallel to the x axis. This parametrization comes in the natural way from the geometric character of the phenomenon and simplifies the analysis of the boundary kinematics. From the form of Eq.(4.2) one can see that for this parametrization the c_r component of the velocity disappears

$$c_r = 0. \tag{4.9}$$

What we have to determine yet is the c_x component that is to be obtained from Eq.(4.5).

Using the definition of the scalar product and taking into account Eq.(4.9) we get from Eq.(4.5)

$$\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} c_x + \frac{\partial g}{\partial r} c_r = 0, \tag{4.10}$$

$$c_x = -\left(\frac{\partial g}{\partial t}\right) \;/\; \left(\frac{\partial g}{\partial x}\right). \tag{4.11}$$

If for the representation (4.2) the normal versor takes the form (4.8), then the normal velocity component of the moving curve will be given by

$$u_n = -\varphi_{,t} / \sqrt{1 + \varphi_{,r}^2}, \tag{4.12}$$

where $\varphi_{,t}$ denotes the partial time derivative of φ.

Due to the form $g(x, r, t)$ given by Eq.(4.4), the relation (4.10) yields

$$c_x = -\left(\frac{\partial J_2}{\partial t}\right) \;/\; \left(\frac{\partial J_2}{\partial x}\right) \tag{4.13}$$

$$u_n = \left(\frac{\partial J_2}{\partial t}\right) / \sqrt{\left(\frac{\partial J_2}{\partial x}\right)^2 + \left(\frac{\partial J_2}{\partial r}\right)^2}. \tag{4.14}$$

An important feature of this formulation is that the invariant J_2 is given on the boundary S_4 by the yield condition

$$J_2 = \kappa^2, \tag{4.15}$$

whereas the deviatoric part of the Cauchy's stress tensor is undetermined on this boundary (see constitutive equations (2.1) and (2.2)).

The relationships (4.9) and (4.13) give the explicit representation of the velocity vector $\mathbf{c} = [c_x, 0]$ as well as its normal component by the field derivatives of the second invariant of the deviatoric Cauchy's stress.

5. Conclusions

From the analysis of the compatibility conditions on the surface bounding the viscoplastic region on can conclude that the equations describing the problem do not permit the first order discontinuity of the velocity field. It means that, taking into account the discontinuity of the physical initial conditions, one should look for the solution of this problem in the one-sided open interval $(t_0, t_k]$. Hence, in order to solve numerically the initial boundary-value problem in the whole interval $[t_0, t_k]$ an extra treatement of the initial condition is necessary. It will be the subject of the next part of this paper [13], where the rigidity property of the material of specimen up to the yield limit will be weakened by introducing non-vanishing initial time increment Δt after which an initially advanced plastic flow will proceed.

To conclude this part one should notice that, in spite of the lack of deviatoric stress tensor determination on the moving boundary (according with material model), the local relationship has been proposed in the previous section that determines the velocity of the bounding curve. For this purpose the field of the deviatoric invariant is used, for it is well defined in the whole viscoplastic region W_2 including S_4.

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Streszczenie

PROBLEM UDERZENIA WALCA O PRZEGRODĘ
Cz. I. WARUNKI ZGODNOŚCI I EWOLUCJA STREFY LEPKOSPRĘŻYSTEJ

W pracy podano analizę próby dynamicznej zderzenia krótkiej odkształcalnej próbki cylindrycznej ze sztywną przegrodą dla sztywno-lepkoplastycznego modelu materiału zaproponowanego przez Perzynę w pracy [1]. Problem ten rozważany był w ramach teorii jednoosiowej w wielu pracach, np. [2, 3, 4, 5, 7]. Przedstawione w niniejszej pracy sformułowanie uwzględnia osiowo-symetryczny charakter problemu, inercję radialną i skończone deformacje. Równanie ruchu wraz ze związkami konstytutywnymi opisującymi przyjęty model materiału prowadzą do problemu początkowo-brzegowego z ruchomą granicą. Istotną część pracy stanowi analiza regularności pól procesu na ruchomej, sztywno-lepkoplastycznej granicy. Z analizy tej wynika, że nieciągły warunek początkowy nie należy do rozwiązania. W pracy sformułowano ponadto związek wyrażający prędkość przemieszczania ruchomej granicy w funkcji pól procesu.

Резюме

ПРОБЛЕМА УДАРА ЦИЛИНДРА О ПРЕГРАДУ
Ч. I. УСЛОВИЯ СОВМЕСТНОСТИ И ЭВОЛЮЦИЯ ВЯЗКОУПРУГОЙ ЗОНЫ

В работе представлен анализ динамического испытания столкновения короткого деформируемого цилиндрического образца с жесткой преградой для жестко-вязкопластической модели материала, предложенной Пэжина в работе [1]. Эта проблема рассматривалась в рамках одноосной теории в многих работах, например, [2, 3, 4, 5, 7]. Представленная в настоящей работе формулировка учитывает осесимметричный характер проблемы, радиальную инерцию и конечные деформации. Управнение дви жения, совместно с определяющими соотношениями описывающими принятую модель материала, приводят к начально-краевой задаче с подвижной границей. Существенную часть работы составляет анализ регулярности полей процесса на подвижной, жестко-вязкопластической границе. Из этого анализа следует, что разривное начальное условие не принадлежит к решению. Кроме этого в работе сформулировано соотношение, выражающее скорость перемещения подвижной границы в функции полей процесса.

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