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HEAT AND MOISTURE TRANSFER IN A TWO-LAYER BUILDING WALL AT SLOW HARMONIC CHANGES OF THERMO-HUMIDITIVE ENVIRONMENT PARAMETERS BASED ON THE THEORY OF THERMODIFFUSION IN SOLIDS

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In order to describe the thermo-humiditive processes occurring in external building walls, the theory of thermodiffusion in elastic solids is used. This theory allows to take into account the internal coupling between the processes of heat conduction and moisture transfer, as well as it makes it possible to determine the stresses during these processes in the wall material. The solutions of the equations describing the process of heat and moisture transfer in a two-layer (and, in general, n-layer) building wall at slow harmonic changes of thermo-humiditive environment parameters are presented. The solutions obtained are of importance for thermo-humiditive design of building walls; the mean month's values of external climate parameters are assumed to vary sinusoidally during the year. The method of solution, as well as some notions, are based on the theory of electric transmission lines.

1. INTRODUCTION

This paper offers a description of thermo-humiditive processes in building walls, based on the Podstrigač–Nowacki theory of thermodiffusion in deformable solids [1, 2]. The diffusing medium in building walls is the moisture, the humidity defines its concentration, and the role of chemical potential of diffusing medium plays the moisture potential, a quantity used in various theories of moisture exchange [3]. The description proposed enables a unified approach to the coupled processes of heat and moisture transfer in building walls and, moreover, within the same theory, it gives a possibility to calculate the stresses and strains which occur during the thermo-humiditive processes in the wall material.

Although the above processes are seen to be coupled, due to the difficulties encountered in mathematical calculations, the fields of temperature and the moisture potential are usually determined separately from two partial differential equations with variable coefficients of the Fourier's heat conduction type. The method of solving the one-dimensional problems of heat and moisture transfer in building walls presented in this paper follows the one used in the theory of electric transmission lines. That method allows to obtain the formulae for analytic solutions which are relatively simple in comparison with the solutions obtained by means of the traditional integral transformation methods used in the theory of thermodiffusion. The methods of the electric transmission lines theory are particularly useful in the analysis of one-dimensional systems with distributed parameters, subject to harmonic excitations.

Periodical changes of external climate parameters are a characteristic feature of the environment of building walls. Annual variation of mean month's values of external climate parameters are approximately sinusoidal.

The solutions obtained in [4] will be used in this paper to construct the solutions describing the heat and moisture transfer in a two-layer (and, in general, n-layer) building wall.

2. Heat and moisture transfer in a two-layer, external building wall during one year

Consider the process of heat and moisture transfer in an external building wall made of two material layers possessing different physical properties, under sinusoidally changing ($T = 1$ year) external climate parameters, i.e. mean month's values of external air temperature and humidity (instantaneous deviations of the parameter values compensate each other at the level of mean month's value [3]), Fig.1.

We assume that:

1. In the range of the considered values of the temperature, humidity and time, the system (i.e. the building wall) is linear and stationary;

2. In the region of each layer the building material is homogeneous and isotropic with respect to the thermal, humiditive and elastic

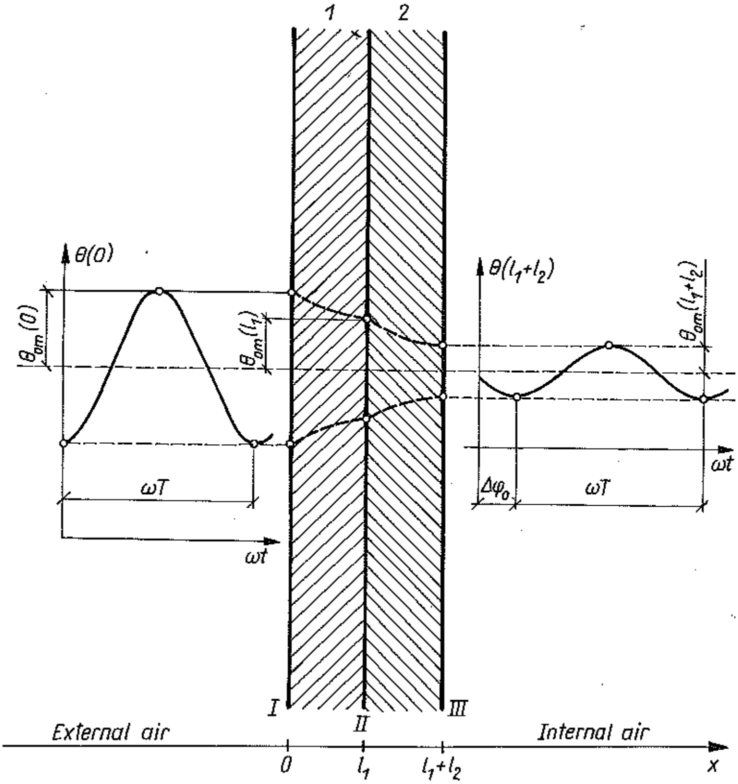

Fig. 1. Heat (and moisture) transfer through a two-layer building wall at slow harmonic state changes; $T = 1$ year, $\omega = 2\pi/T$ – pulsation, $\Theta(0)$ – temperature on external surface I, $\Theta(l_1 + l_2)$ – temperature on internal surface III, $\Delta\varphi_\Theta$ – temperature phase shift across the wall, II – dividing surface between the layers 1 and 2.

properties;

3. The temperature and humidity fields in the wall are one-dimensional;

4. The moisture termodiffusion process considered is slow and, therefore, the mechanical inertia of the building material may be neglected;

5. The effect of stress on the heat and moisture transfer is negligible.

The temperature and the moisture potential fields within the region of a two-layer building wall at slow harmonic changes will be determined from the equations of the Podstrigač – Nowacki theory of thermodiffusion in deformable solids [1, 2], using the solution for a one-layer building wall given in [4] and in Appendix 1.

One-dimensional, non-stationary process of moisture termodiffusion in an elastic material layer (the influence of stresses on the heat and moisture transfer being neglected) is described by the following equations [1,2,4]:

The complete set of equations of thermodiffusion theory

$$(2.1)\qquad \begin{aligned} (2\mu+\lambda)\frac{\partial^2 w}{\partial x^2} &= \gamma_T\frac{\partial \Theta}{\partial x}+\gamma_c\frac{\partial c}{\partial x},\\ k\frac{\partial^2\Theta}{\partial x^2}-c_{\varepsilon,c}\frac{\partial\Theta}{\partial t}+T_0 d\frac{\partial c}{\partial t} &= 0,\\ D_c\frac{\partial^2 c}{\partial x^2}-\frac{\partial c}{\partial t}+D_T\frac{\partial^2\Theta}{\partial x^2} &= 0, \end{aligned}$$

the constitutive relations

$$(2.2)\qquad \begin{aligned} \sigma_{11} &= (2\mu+\lambda)\varepsilon_{11}-\gamma_T\Theta-\gamma_c c,\\ S &= m\Theta-dc,\\ M &= d\Theta+nc, \end{aligned}$$

the laws of heat and moisture conduction

$$(2.3)\qquad q=-k\frac{\partial\Theta}{\partial x},\qquad \eta=-\kappa\frac{\partial M}{\partial x},$$

and the relations

$$(2.4)\qquad \dot{S}=\frac{1}{T_0}k\frac{\partial^2\Theta}{\partial x^2},\qquad \dot{c}=\kappa\frac{\partial^2 M}{\partial x^2},$$

$$(2.5)\qquad \eta=-D_c\frac{\partial c}{\partial x}-D_T\frac{\partial\Theta}{\partial x},\qquad D_c=\kappa\cdot n,\qquad D_T=\kappa\cdot d.$$

Here

$\Theta=T-T_0$	relative temperature,
T_0	initial state temperature,
S	entropy per unit volume,
q	heat flux,
M	moisture potential,
c	humidity (= concentration of diffusing medium),
η	moisture flux,
$\sigma_{11}(x)$	wall material stresses in x-direction,
$w(x)$	displacement vector component,

$\varepsilon_{11} = \partial w/\partial x$	strain in x-direction,
$\mu, \lambda, \gamma_T, \gamma_c, d, m, n$	basic material constants appearing in constitutive relations,
μ, λ	Lamé elastic constants,
k	heat conduction coefficient,
κ	moisture conduction coefficient.

We assume that the following quantities vary sinusoidally in time (the building wall considered is a linear and stationary system)

$$
\begin{aligned}
\sigma_{11}(x,t) &= \mathrm{Re}[\sigma_0(x)e^{j\omega t}], & \frac{\partial w}{\partial t} &:= v(x,t) = \mathrm{Re}[v_0(x)e^{j\omega t}],\\
\Theta(x,t) &= \mathrm{Re}[\Theta_0(x)e^{j\omega t}], & q(x,t) &= \mathrm{Re}[q_0(x)e^{j\omega t}],\\
M(x,t) &= \mathrm{Re}[M_0(x)e^{j\omega t}], & \eta(x,t) &= \mathrm{Re}[\eta_0(x)e^{j\omega t}],
\end{aligned}
\tag{2.6}
$$

where $j = \sqrt{-1}$ is the imaginary unit,

$$
\begin{aligned}
\sigma_0(x) &:= \sigma_{0m}(x)e^{j\varphi_\sigma(x)}, & v_0(x) &:= v_{0m}(x)e^{j\varphi_v(x)},\\
\Theta_0(x) &:= \Theta_{0m}(x)e^{j\varphi_\Theta(x)}, & q_0(x) &:= q_{0m}(x)e^{j\varphi_q(x)},\\
M_0(x) &:= M_{0m}(x)e^{j\varphi_M(x)}, & \eta_0(x) &:= \eta_{0m}(x)e^{j\varphi_\eta(x)}.
\end{aligned}
\tag{2.7}
$$

The quantities with with subscript 0 are the complex amplitudes; they are vectors in the Gaussian complex plane, and in Eqs.(2.7) they are expressed in terms of real amplitudes (the quantities with subscript $0m$) and of the appropriate phase shift angles $\varphi_{(\cdot)}(x)$. For example, $M_0(x)$ is the complex amplitude of moisture potential, and $\varphi_M(x)$ is the phase shift angle of the moisture potential.

The one-dimensional process of linear thermodiffusion of moisture through a two-layer building wall in direction x at slow harmonic state changes can be described, by analogy to Eqs.(1.11) and (1.12) (see Appendix 1), by means of the following equations:

$$
\frac{d}{dx}\begin{bmatrix} q_0(x) \\ \eta_0(x) \\ \Theta_0(x) \\ M_0(x) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \overset{(i)}{Z}_2 & -\overset{(i)}{Z}_{23} \\ 0 & 0 & -\overset{(i)}{Z}_{23} & \overset{(i)}{Z}_3 \\ \overset{(i)}{Y}_2 & 0 & 0 & 0 \\ 0 & \overset{(i)}{Y}_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_0(x) \\ \eta_0(x) \\ \Theta_0(x) \\ M_0(x) \end{bmatrix},
\tag{2.8}
$$

$$(2.9)\quad \frac{dv_0(x)}{dx} = \overset{(i)}{Z}_1\,\sigma_0(x) + \overset{(i)}{Z}_{12}\,\Theta_0(x) + \overset{(i)}{Z}_{13}\,M_0(x), \qquad \frac{d\sigma_0(x)}{dx} = 0,$$

where $i = 1, 2$; $i = 1$ for $x \in \langle 0, l_1\rangle$ the layer 1, $i = 2$ for $x \in \langle l_1, l_1 + l_2\rangle$ the layer 2,

$$(2.10)\quad \overset{(i)}{Z}_1 = j\omega\frac{1}{2\overset{(i)}{\mu} + \overset{(i)}{\lambda}}, \qquad \overset{(i)}{Z}_2 = j\omega\frac{\overset{(i)}{m}\overset{(i)}{n} + \overset{(i)}{d}}{\overset{(i)}{n}}, \qquad \overset{(i)}{Z}_3 = j\omega\frac{1}{\overset{(i)}{n}},$$

$$(2.11)\quad \overset{(i)}{Z}_{12} = j\omega\frac{\overset{(i)}{\gamma}_T\overset{(i)}{n} - \overset{(i)}{\gamma}_c\overset{(i)}{d}}{\overset{(i)}{n}(2\overset{(i)}{\mu} + \overset{(i)}{\lambda})}, \qquad \overset{(i)}{Z}_{13} = j\omega\frac{\overset{(i)}{\gamma}_c}{\overset{(i)}{n}(2\overset{(i)}{\mu} + \overset{(i)}{\lambda})}, \qquad \overset{(i)}{Z}_{23} = j\omega\frac{\overset{(i)}{d}}{\overset{(i)}{n}},$$

$$(2.12)\quad \overset{(i)}{Y}_2 = T_0\frac{1}{\overset{(i)}{k}}, \qquad \overset{(i)}{Y}_3 = \frac{1}{\overset{(i)}{\kappa}}.$$

The quantities (2.10) and (2.11), by analogy to the electrical impedances (1.13) (Appendix 1), may be called the thermo-humiditive impedances of the building wall layers.

We assume that at the plane dividing the layers 1 and 2 (surface II, Fig.1), the boundary conditions of kind IV are satisfied, i.e. the heat and the moisture fluxes are continuous,

$$(2.13)\quad \overset{(1)}{k}\left.\frac{d\Theta_0(x)}{dx}\right|_{x=l_1^-} = \overset{(2)}{k}\left.\frac{d\Theta_0(x)}{dx}\right|_{x=l_1^+},$$

$$(2.14)\quad \overset{(1)}{\kappa}\left.\frac{dM_0(x)}{dx}\right|_{x=l_1^-} = \overset{(2)}{\kappa}\left.\frac{dM_0(x)}{dx}\right|_{x=l_1^+},$$

the temperature moisture potential are also continuous,

$$(2.15)\quad \Theta_0(x = l_1^-) = \Theta_0(x = l_1^+),$$

$$(2.16)\quad M_0(x = l_1^-) = M_0(x = l_1^+).$$

At surface I of the wall (Fig.1) remaining in contact with external air at temperature Θ_z and moisture potential M_z, the following conditions are fulfilled:

heat exchange

$$(2.17)\quad \alpha_z(\Theta_z - \Theta_0(x = 0)) + r_z\Lambda_z = -\overset{(1)}{k}\left.\frac{d\Theta_0(x)}{dx}\right|_{x=0},$$

moisture exchange

$$\beta_z(M_z - M_0(x=0)) = -\overset{(1)}{\kappa}\,\frac{dM_0(x)}{dx}\bigg|_{x=0}; \tag{2.18}$$

here α_z – coefficient of surface heat exchange (due to heat convection and radiation), β_z – coefficient of moisture exchange between external air and the wall, Λ_z – intensity of the incident solar radiation at the external surface of the wall, r_z – coefficient of solar radiation absorption by the wall surface.

Similarly, at surface III of the building wall remaining in contact with the internal air at temperature Θ_w and moisture potential M_w, we have

$$\alpha_w(\Theta_w - \Theta_0(x = l_1 + l_2)) = -\overset{(2)}{k}\,\frac{d\Theta_0(x)}{dx}\bigg|_{x=l_1+l_2}, \tag{2.19}$$

$$\beta_w(M_w - M_0(x = l_1 + l_2)) = -\overset{(2)}{\kappa}\,\frac{dM_0(x)}{dx}\bigg|_{x=l_1+l_2}, \tag{2.20}$$

where α_w, β_w are, respectively, the coefficients of heat and moisture exchange between the wall and internal air; it was assumed that no internal radiation sources occur ($r_w\,\Lambda_w = 0$).

The boundary conditions (2.17)–(2.20) determine the relations between the temperature of wall bounding surfaces (external or internal) and the heat flux through the surfaces, and the relation between the moisture potential of the wall bounding surfaces (external or internal) and the moisture flux through the surfaces. Knowing (from measurements) the values of the temperature and the moisture potential for the external surface of the wall, we can determine, on the ground of conditions (2.17) and (2.18), the values of the heat and moisture fluxes through that surface.

In the following it will assumed that the values of $q_0(0)$, $\eta_0(0)$, $\Theta_0(0)$, $M_0(0)$ on the external surface of wall are known. We will be interested in the determination of the temperature and moisture potential fields within the two-layer building wall, and in the description of the thermo-humiditive insulation properties (or the transmission properties) of the wall, such as the amplitude attenuation factor of the temperature and that of the moisture potential.

Equation (2.8) is the so-called homogenous state equation; it can be written in the compact form

$$
(2.21) \qquad \frac{d\mathbf{S}(x)}{dx} = \overset{(i)}{\mathbf{A}}\mathbf{S}\,(x), \qquad \begin{array}{lll} i = 1, 2, & & \\ i = 1 & \text{if} & x \in \langle 0, l_1 \rangle, \\ i = 2 & \text{if} & x \in \langle l_1, l_1 + l_2 \rangle, \end{array}
$$

where

$$
(2.22) \quad \mathbf{S}(x) = \begin{bmatrix} q_0(x) \\ \eta_0(x) \\ \Theta_0(x) \\ M_0(x) \end{bmatrix}, \qquad \overset{(i)}{\mathbf{A}} = \begin{bmatrix} 0 & 0 & \overset{(i)}{Z}_2 & -\overset{(i)}{Z}_{23} \\ 0 & 0 & -\overset{(i)}{Z}_{23} & \overset{(i)}{Z}_3 \\ \overset{(i)}{Y}_2 & 0 & 0 & 0 \\ 0 & \overset{(i)}{Y}_3 & 0 & 0 \end{bmatrix}.
$$

(state vector) (matrices of the layers)

Equations (2.21) for the case $i = 1, 2, ..., n$, describe the heat and moisture transfer through a n-layer building wall. These equations may be solved in two manners: 1) step by step, i.e. by determining the fields of temperature, moisture potential, heat flux and moisture flux in the first layer, and then consecutively up to the layer i, or 2) in a selective manner (what is of importance in the case of multilayer building walls), i.e. by determining the fields $\Theta_0(x)$, $M_0(x)$, $q_0(x)$, $\eta_0(x)$ directly for the layer i, use being made of the concept of transmission matrix for the chain connection of the layers 1 to $i-1$, adopted from the electric four-terminal network theory (Appendix 1). Let us present both the ways.

The solution of the state equation (2.21) for layer 1 is the state vector $\mathbf{S}(x)$ represented by the following transmission equation (App. 1)

$$
(2.23) \qquad \mathbf{S}(x) = [e^{\overset{(1)}{\mathbf{A}}x}]\mathbf{S}(0), \qquad x \in \langle 0, l_1 \rangle,
$$

where $e^{\overset{(1)}{\mathbf{A}}x}$ is the transmission matrix of the layer 1 (App. 1), and $\mathbf{S}(0) = [q_0(0), \eta_0(0), \Theta_0(0), M_0(x)]$ is the state vector at the external surface of the building wall.

For layer i of the n-layer building wall the solution of the state equation is given by

$$
(2.24) \quad \mathbf{S}(x) = [e^{\overset{(i)}{\mathbf{A}}x}]\mathbf{S}(l_1 + ... + l_{i-1}), \quad x \in \langle l_1 + ... + l_{i-1}, l_1 + ... + l_i \rangle, \quad i = 1, 2, \ldots, n,
$$

where $e^{\overset{(i)}{\mathbf{A}}x}$ is the transmission matrix for the layer i, and

$$\mathbf{S}(l_1+...+l_{i-1}) = [q_0(l_1+...+l_{i-1}),\ \eta_0(l_1+...+l_{i-1}),\ \Theta_0(l_1+...+l_{i-1}),\ M_0(l_1+...+l_{i-1})]^T$$

is the state vector at the interface between the layers $i-1$ and i.

The transmission equation (2.24) makes it possible to determine the fields of temperature, moisture potential, heat flux and moisture flux in consecutive layers of the multilayer building wall, starting from its external layer.

Introduce now the concept of chain connection of the layers of a building wall.

The chain connection of the layers of a building wall is called a connection in the case when at the interfaces between the layers the boundary conditions of fourth kind are satisfied, i.e. the heat and moisture fluxes are continuous (see (2.13)–(2.16)).

The concept of chain connection of the layers corresponds to the concept of the chain connection of the electric four-terminal network (App. 1).

The fields of temperature, moisture potential, heat flux and moisture flux in the region of the layer i of a n-layer wall may be determined directly (i.e. without determining the fields in the preceding layers) from the following transmission equation:

$$\mathbf{S}(x) = [e^{\overset{(i)}{\mathbf{A}}x}][e^{\overset{i-1}{\mathbf{A}}l_{i-1}} \times ... \times e^{\overset{(1)}{\mathbf{A}}l_1}]\mathbf{S}(0), \qquad i = 1,2,...n, \tag{2.25}$$

$$x \in \langle l_1+...+l_{i-1}, l_1+...+l_i\rangle,$$

which is derived from Eq.(2.24).

The matrix

$$[e^{\overset{i-1}{\mathbf{A}}l_{i-1}} \times ... \times e^{\overset{(1)}{\mathbf{A}}l_1}] = [e^{\overset{i-1}{\mathbf{A}}l_{i-1}}][e^{\overset{i-2}{\mathbf{A}}l_{i-2}}] ... [e^{\overset{(1)}{\mathbf{A}}l_1}], \tag{2.26}$$

is the so-called transmission matrix of the chain connection of the building wall layers from 1 to $i-1$, associated with the following transmission equation:

$$\mathbf{S}(l_1+...+l_{i-1}) = [e^{\overset{i-1}{\mathbf{A}}l_{i-1}} \times ... \times e^{\overset{(1)}{\mathbf{A}}l_1}]\mathbf{S}(0). \tag{2.27}$$

The transmission equation (2.25) allows for a direct analysis of the influence of external climate conditions represented by the state vector $\mathbf{S}(0) = [q_0(0), \eta_0(0), \Theta_0(0), M_0(0)]^T$ acting at the external wall surface, on the distribution of temperature, moisture potential, heat flux and moisture flux within the region of the layer i of the wall. The transmission matrix (2.26) of the chain connection of the wall layers 1 to $i-1$ is equal to the product of the transmission matrices of those layers. Since the matrix product is, in general, noncommutative ($AB \neq BA$), therefore the sequence of transmission matrices in the product (2.26) is important: the transmission matrices of wall layers occur in the product (2.26) in the order inverse to that of the layers.

If we consider only the insulation properties of a multilayer building wall as a whole, it is useful to introduce the notion of an equivalent one-layer wall, which is equivalent to the wall considered as far as its thermo-humiditive properties are concerned [3]; the real distribution of temperature, moisture potential, heat flux and moisture flux within the region of the wall is here of secondary importance. Using the terminology proposed in this paper it is possible to describe the insulation thermo-humiditive properties of the wall equivalent to a n-layer building wall, by means of the following transmission equation

$$\mathbf{S}(l_1 + \ldots + l_n) = \overset{(n)}{\underset{(1)}{\mathbf{E}}}\, \mathbf{S}(0), \tag{2.28}$$

where the transmission matrix of the equivalent wall $\overset{(n)}{\underset{(1)}{\mathbf{E}}}$ is equal (on the basis of Eq.(2.27)) to the transmission matrix of the chain connection of layers 1 to n of the n-layer wall considered

$$\overset{(n)}{\underset{(1)}{\mathbf{E}}} = \left[e^{\overset{(n)}{\mathbf{A}} l_n} \times \ldots \times e^{\overset{(1)}{\mathbf{A}} l_1} \right]. \tag{2.29}$$

It is, of course, true under the assumption that at the interfaces between the component layers, the boundary conditions (2.13)–(2.16) are fulfilled.

For $n = 2$, i.e. in the case of a two-layer building wall, the distinction between the above two methods of solving the state equation (2.21) is of no importance.

The transmission equations for the two layers of the building wall presented in Fig.1 have the form

$$\mathbf{S}(x) = [e^{\overset{(1)}{\mathbf{A}} x}]\mathbf{S}(0), \quad \text{for} \quad x \in \langle 0, l_1 \rangle, \tag{2.30}$$

$$(2.31)\qquad \mathbf{S}(x) = [e^{\overset{(2)}{\mathbf{A}}(x-l_1)}]\mathbf{S}(l_1), \quad \text{for} \quad x \in \langle l_1, l_1+l_2\rangle,$$

where $\mathbf{S}(l_1)$ is the state vector at the interface of the wall.

Solution of the state equation (2.21), i.e. determination of the state vector $\mathbf{S}(x) = [q_0(x), \eta_0(x), \Theta_0(x), M_0(x)]$ reduces to the evaluation of the transmission matrix $e^{\overset{(i)}{\mathbf{A}}x}$. Applying the Cayley–Hamilton theorem to matrix $\overset{(i)}{\mathbf{A}}$ of order $k = 4$, it is possible to express the transmission matrix $e^{\overset{(i)}{\mathbf{A}}x}$ in the form (App.1)

$$(2.32)\qquad e^{\overset{(i)}{\mathbf{A}}x} = \overset{(i)}{g}_0(x)\mathbf{1} + \overset{(i)}{g}_1(x)\overset{(i)}{\mathbf{A}} + \overset{(i)}{g}_2(x)\overset{(i)}{\mathbf{A}}{}^2 + \overset{(i)}{g}_3(x)\overset{(i)}{\mathbf{A}}{}^3, \qquad i = 1,2,$$

where

$$(2.33)\qquad \begin{aligned}
\overset{(i)}{g}_0(x) &= \frac{\overset{(i)}{\gamma}{}_1^2 \operatorname{ch} \overset{(i)}{\gamma}_2 x - \overset{(i)}{\gamma}{}_2^2 \operatorname{ch} \overset{(i)}{\gamma}_1 x}{\overset{(i)}{\gamma}{}_1^2 - \overset{(i)}{\gamma}{}_2^2},\\
\overset{(i)}{g}_1(x) &= \frac{\overset{(i)}{\gamma}{}_1^2 \operatorname{sh} \overset{(i)}{\gamma}_2 x}{\overset{(i)}{\gamma}_2 (\overset{(i)}{\gamma}_1 - \overset{(i)}{\gamma}{}_2^2)} - \frac{\overset{(i)}{\gamma}{}_2^2 \operatorname{sh} \overset{(i)}{\gamma}_1 x}{\overset{(i)}{\gamma}_1 (\overset{(i)}{\gamma}{}_1^2 - \overset{(i)}{\gamma}{}_2^2)},\\
\overset{(i)}{g}_2(x) &= \frac{\operatorname{ch} \overset{(i)}{\gamma}_1 x - \operatorname{ch} \overset{(i)}{\gamma}_2 x}{\overset{(i)}{\gamma}{}_1^2 - \overset{(i)}{\gamma}{}_2^2},\\
\overset{(i)}{g}_3(x) &= \frac{\operatorname{sh} \overset{(i)}{\gamma}_1 x}{\overset{(i)}{\gamma}_1 (\overset{(i)}{\gamma}_1 - \overset{(i)}{\gamma}{}_2^2)} - \frac{\operatorname{sh} \overset{(i)}{\gamma}_2 x}{\overset{(i)}{\gamma}_2 (\overset{(i)}{\gamma}{}_1^2 - \overset{(i)}{\gamma}{}_2^2)},
\end{aligned}$$

and

$$(2.34)\qquad \overset{(i)}{\gamma}_1 = \sqrt{(\Sigma \overset{(i)}{P}) + \overset{(i)}{K}}, \qquad \overset{(i)}{\gamma}_2 = \sqrt{(\Sigma \overset{(i)}{P}) - \overset{(i)}{K}},$$

$$(2.35)\qquad \Sigma \overset{(i)}{P} = \frac{\overset{(i)}{P}_2 + \overset{(i)}{P}_3}{2}, \quad \overset{(i)}{K} = \sqrt{(\Delta \overset{(i)}{P}) + \overset{(i)}{Q}{}^2}, \quad \Delta \overset{(i)}{P} = \frac{\overset{(i)}{P}_2 - \overset{(i)}{P}_3}{2}, \quad \overset{(i)}{Q}{}^2 = \overset{(i)}{Q}_2 \overset{(i)}{Q}_3,$$

$$(2.36)\qquad \overset{(i)}{P}_2 = \overset{(i)}{Z}_2 \overset{(i)}{Y}_2, \qquad \overset{(i)}{P}_3 = \overset{(i)}{Z}_3 \overset{(i)}{Y}_3, \qquad \overset{(i)}{Q}_2 = \overset{(i)}{Y}_2 \overset{(i)}{Z}_{23}, \qquad \overset{(i)}{Q}_3 = \overset{(i)}{Y}_3 \overset{(i)}{Z}_{23}\,.$$

The quantities $\overset{(i)}{Z}_2$, $\overset{(i)}{Z}_3$, $\overset{(i)}{Z}_{23}$, $\overset{(i)}{Y}_2$, $\overset{(i)}{Y}_3$ are given by the formulae (2.10)–(2.12).

Components of the state vector $\mathbf{S}(x)$ constituting the solution of the state equation (2.21) for the layer 1 of the wall ($x \in \langle 0, l_1 \rangle$), i.e. the fields of heat flux $q_0(x)$, moisture flux $\eta_0(x)$, temperature $\Theta_0(x)$ and moisture potential $M_0(x)$ in layer 1, have the form:

$$
\begin{aligned}
(2.37)\qquad q_0(x) = {} & \frac{(\overset{(1)}{K} + \overset{(1)}{\Delta P})\eta_0(0) - \overset{(1)}{Q}_3\, q_0(0)}{2\overset{(1)}{K}} \operatorname{ch} \overset{(1)}{\gamma}_1 x \\
& + \frac{(\overset{(1)}{K} + \overset{(1)}{\Delta P})\, \overset{(1)}{B}_2(0) - \overset{(1)}{Q}_3 \overset{(1)}{B}_3(0)}{2\overset{(1)}{K}\overset{(1)}{\gamma}_1} \operatorname{sh} \overset{(1)}{\gamma}_1 x \\
& + \frac{(\overset{(1)}{K} - \overset{(1)}{\Delta P})\eta_0(0) + \overset{(1)}{Q}_3\, q_0(0)}{2\overset{(1)}{K}} \operatorname{ch} \overset{(1)}{\gamma}_2 x \\
& + \frac{(\overset{(1)}{K} - \overset{(1)}{\Delta P})\, \overset{(1)}{B}_2(0) + \overset{(1)}{Q}_3 \overset{(1)}{B}_3(0)}{2\overset{(1)}{K}\overset{(1)}{\gamma}_2} \operatorname{sh} \overset{(1)}{\gamma}_2 x,
\end{aligned}
$$

$$
\begin{aligned}
(2.38)\qquad \eta_0(x) = {} & \frac{(\overset{(1)}{K} - \overset{(1)}{\Delta P})q_0(0) - \overset{(1)}{Q}_2\, \eta_0(0)}{2\overset{(1)}{K}} \operatorname{ch} \overset{(1)}{\gamma}_1 x \\
& + \frac{(\overset{(1)}{K} - \overset{(1)}{\Delta P})\, \overset{(1)}{B}_3(0) - \overset{(1)}{Q}_2 \overset{(1)}{B}_2(0)}{2\overset{(1)}{K}\overset{(1)}{\gamma}_2} \operatorname{sh} \overset{(1)}{\gamma}_1 x \\
& + \frac{(\overset{(1)}{K} + \overset{(1)}{\Delta P})q_0(0) - \overset{(1)}{Q}_2\, \eta_0(0)}{2\overset{(1)}{K}} \operatorname{ch} \overset{(1)}{\gamma}_2 x \\
& + \frac{(\overset{(1)}{K} + \overset{(1)}{\Delta P})\, \overset{(1)}{Y}_2\, \eta_0(0) - \overset{(1)}{Q}_2 \overset{(1)}{Y}_3\, q_0(0)}{2\overset{(1)}{K}\overset{(1)}{\gamma}_2} \operatorname{sh} \overset{(1)}{\gamma}_2 x,
\end{aligned}
$$

$$
\begin{aligned}
(2.39)\qquad \Theta_0(x) = {} & \frac{(\overset{(1)}{K} + \overset{(1)}{\Delta P})\Theta_0(0) - \overset{(1)}{Q}_2\, M_0(0)}{2\overset{(1)}{K}} \operatorname{ch} \overset{(1)}{\gamma}_1 x \\
& + \frac{(\overset{(1)}{K} + \overset{(1)}{\Delta P})\, \overset{(1)}{Y}_2\, \eta_0(0) - \overset{(1)}{Q}_2 \overset{(1)}{Y}_3\, q_0(0)}{2\overset{(1)}{K}\overset{(1)}{\gamma}_1} \operatorname{sh} \overset{(1)}{\gamma}_1 x
\end{aligned}
$$

(2.39) [cont.]

$$+\frac{(\overset{(1)}{K}-\overset{(1)}{\Delta P})\Theta_0(0)+\overset{(1)}{Q}_2\,M_0(0)}{2\,\overset{(1)}{K}}\text{ch}\,\overset{(1)}{\gamma}_2\,x$$

$$+\frac{(\overset{(1)}{K}-\overset{(1)}{\Delta P})\,\overset{(1)}{Y}_2\,\eta_0(0)+\overset{(1)}{Q}_2\overset{(1)}{Y}_3\,q_0(0)}{2\,\overset{(1)}{K}\overset{(1)}{\gamma}_2}\text{sh}\,\overset{(1)}{\gamma}_2\,x,$$

(2.40)

$$M_0(x)=\frac{(\overset{(1)}{K}-\overset{(1)}{\Delta P})M_0(0)-\overset{(1)}{Q}_3\,\Theta_0(0)}{2\,\overset{(1)}{K}}\text{ch}\,\overset{(1)}{\gamma}_1\,x$$

$$+\frac{(\overset{(1)}{K}-\overset{(1)}{\Delta P})\,\overset{(1)}{Y}_3\,q_0(0)-\overset{(1)}{Q}_3\overset{(1)}{Y}_2\,\eta_0(0)}{2\,\overset{(1)}{K}\overset{(1)}{\gamma}_1}\text{sh}\,\overset{(1)}{\gamma}_1\,x$$

$$+\frac{(\overset{(1)}{K}+\overset{(1)}{\Delta P})M_0(0)+\overset{(1)}{Q}_3\,\Theta_0(0)}{2\,\overset{(1)}{K}}\text{ch}\,\overset{(1)}{\gamma}_2\,x$$

$$+\frac{(\overset{(1)}{K}+\overset{(1)}{\Delta P})\,\overset{(1)}{Y}_3\,q_0(0)+\overset{(1)}{Q}_3\overset{(1)}{Y}_2\,\eta_0(0)}{2\,\overset{(1)}{K}\overset{(1)}{\gamma}_2}\text{sh}\,\overset{(1)}{\gamma}_2\,x,$$

where $x\in<0,l_1>$ and

(2.41) $$\overset{(1)}{B}_2\,(0)=\overset{(1)}{Z}_2\,\Theta_0(0)-\overset{(1)}{Z}_{23}\,M_0(0),\quad \overset{(3)}{B}_3\,(0)=\overset{(1)}{Z}_3\,M_0(0)-\overset{(1)}{Z}_{23}\,\Theta_0(0).$$

If the state vector $\mathbf{S}(l_1)$ at the dividing surface between the wall layers 1 and 2 is known, then the fields of the heat flux $q_0(x-l_1)$, moisture flux $\eta_0(x-l_1)$, temperature $\Theta_0(x-l_1)$ and moisture potential $M_0(x-l_1)$ in layer 2 $x\in\langle l_1,l_1+l_2\rangle$, have (on the basis of Eq.(2.31)) the form

(2.42)

$$q_0(x-l_1)=\frac{(\overset{(2)}{K}+\overset{(2)}{\Delta P})\eta_0(l_1)-\overset{(2)}{Q}_3\,q_0(l_1)}{2\,\overset{(2)}{K}}\text{ch}\,\overset{(2)}{\gamma}_1\,(x-l_1)$$

$$+\frac{(\overset{(2)}{K}+\overset{(2)}{\Delta P})\,\overset{(2)}{B}_2\,(0)-\overset{(2)}{Q}_3\overset{(2)}{B}_3\,(0)}{2\,\overset{(2)}{K}\overset{(2)}{\gamma}_1}\text{sh}\,\overset{(2)}{\gamma}_1\,(x-l_1)$$

$$+\frac{(\overset{(2)}{K}-\overset{(2)}{\Delta P})\eta_0(l_1)+\overset{(2)}{Q}_3\,q_0(l_1)}{2\,\overset{(2)}{K}}\text{ch}\,\overset{(2)}{\gamma}_2\,(x-l_1)$$

(2.42) [cont.]
$$+\frac{(\overset{(2)}{K}-\overset{(2)}{\Delta}P)\,\overset{(2)}{B}_2(0)+\overset{(2)}{Q}_3\overset{(2)}{B}_3(0)}{2\,\overset{(2)}{K}\overset{(2)}{\gamma}_2}\operatorname{sh}\overset{(2)}{\gamma}_2(x-l_1),$$

(2.43)
$$\eta_0(x-l_1)=\frac{(\overset{(2)}{K}-\overset{(2)}{\Delta}P)q_0(l_1)-\overset{(2)}{Q}_2\,\eta_0(l_1)}{2\,\overset{(2)}{K}}\operatorname{ch}\overset{(2)}{\gamma}_1(x-l_1)$$
$$+\frac{(\overset{(2)}{K}-\overset{(2)}{\Delta}P)\,\overset{(2)}{B}_3(0)-\overset{(2)}{Q}_2\overset{(2)}{B}_2(0)}{2\,\overset{(2)}{K}\overset{(2)}{\gamma}_1}\operatorname{sh}\overset{(2)}{\gamma}_1(x-l_1)$$
$$+\frac{(\overset{(2)}{K}+\overset{(2)}{\Delta}P)q_0(l_1)-\overset{(2)}{Q}_2\,\eta_0(l_1)}{2\,\overset{(2)}{K}}\operatorname{ch}\overset{(2)}{\gamma}_2(x-l_1)$$
$$+\frac{(\overset{(2)}{K}+\overset{(2)}{\Delta}P)\,\overset{(2)}{Y}_2\,\eta_0(l_1)-\overset{(2)}{Q}_2\overset{(2)}{Y}_3\,q_0(0)}{2\,\overset{(2)}{K}\overset{(2)}{\gamma}_2}\operatorname{sh}\overset{(2)}{\gamma}_2(x-l_1),$$

(2.44)
$$\Theta_0(x-l_1)=\frac{(\overset{(2)}{K}+\overset{(2)}{\Delta}P)\Theta_0(l_1)-\overset{(2)}{Q}_2\,M_0(l_1)}{2\,\overset{(2)}{K}}\operatorname{ch}\overset{(2)}{\gamma}_1(x-l_1)$$
$$+\frac{(\overset{(2)}{K}+\overset{(2)}{\Delta}P)\,\overset{(2)}{Y}_2\,\eta_0(l_1)-\overset{(2)}{Q}_2\overset{(2)}{Y}_3\,q_0(l_1)}{2\,\overset{(2)}{K}\overset{(2)}{\gamma}_1}\operatorname{sh}\overset{(2)}{\gamma}_1(x-l_1)$$
$$+\frac{(\overset{(2)}{K}-\overset{(2)}{\Delta}P)\Theta_0(l_1)+\overset{(2)}{Q}_2\,M_0(l_1)}{2\,\overset{(2)}{K}}\operatorname{ch}\overset{(2)}{\gamma}_2(x-l_1)$$
$$+\frac{(\overset{(2)}{K}-\overset{(2)}{\Delta}P)\,\overset{(2)}{Y}_2\,\eta_0(l_1)+\overset{(2)}{Q}_2\overset{(2)}{Y}_3\,q_0(l_1)}{2\,\overset{(2)}{K}\overset{(2)}{\gamma}_2}\operatorname{sh}\overset{(2)}{\gamma}_2(x-l_1),$$

(2.45)
$$M_0(x-l_1)=\frac{(\overset{(2)}{K}-\overset{(2)}{\Delta}P)M_0(l_1)-\overset{(2)}{Q}_3\,\Theta_0(l_1)}{2\,\overset{(2)}{K}}\operatorname{ch}\overset{(2)}{\gamma}_1(x-l_1)$$
$$+\frac{(\overset{(2)}{K}-\overset{(2)}{\Delta}P)\,\overset{(2)}{Y}_3\,q_0(l_1)-\overset{(2)}{Q}_3\overset{(2)}{Y}_2\,\eta_0(l_1)}{2\,\overset{(2)}{K}\overset{(2)}{\gamma}_1}\operatorname{sh}\overset{(2)}{\gamma}_1(x-l_1)$$
$$+\frac{(\overset{(2)}{K}+\overset{(2)}{\Delta}P)M_0(l_1)+\overset{(2)}{Q}_3\,\Theta_0(l_1)}{2\,\overset{(2)}{K}}\operatorname{ch}\overset{(2)}{\gamma}_2(x-l_1)$$

$$+\frac{(\overset{(2)}{K}+\Delta\overset{(2)}{P})\overset{(2)}{Y}_3\, q_0(l_1)+\overset{(2)}{Q}_3\overset{(2)}{Y}_2\,\eta_0(l_1)}{2\,\overset{(2)(2)}{K\gamma_2}}\operatorname{sh}\overset{(2)}{\gamma}_2\,(x-l_1), \tag{2.45 [cont.]}$$

where $x \in \langle l_1, l_1+l_2\rangle$,

$$\begin{aligned}\overset{(2)}{B}_2\,(0) &= \overset{(2)}{Z}_2\,\Theta_0(l_1)-\overset{(2)}{Z}_{23}\,M_0(l_1),\\ \overset{(2)}{B}_3\,(0) &= \overset{(2)}{Z}_3\,M_0(l_1)-\overset{(2)}{Z}_{23}\,\Theta_0(l_1).\end{aligned} \tag{2.46}$$

The state vector $\mathbf{S}(l_1)=[q_0(l_1),\eta_0(l_1),\Theta_0(l_1),M_0(l_1)]^T$ may be calculated from Eq.(2.30) or from Eqs.(2.37)–(2.40) for $x=l_1$.

If we determine $\mathbf{S}(l_1)$ from Eq.(2.30) and substitute it into Eq.(2.31), we obtain

$$\mathbf{S}(x)=[e^{\overset{(2)}{\mathbf{A}}(x-l_1)}][e^{\overset{(1)}{\mathbf{A}}l_1}]\mathbf{S}(0),\qquad x\in\langle l_1,l_1+l_2\rangle. \tag{2.47}$$

The product of transmission matrices occurring in Eq.(2.47) may be derived as follows:

$$\begin{aligned}
[e^{\overset{(2)}{\mathbf{A}}(x-l_1)}][e^{\overset{(1)}{\mathbf{A}}l_1}] &= [\overset{(2)}{g_0}(x-l_1)\mathbf{1}+\overset{(2)}{g_1}(x-l_1)\overset{(2)}{\mathbf{A}}+\overset{(2)}{g_2}(x-l_1)\overset{(2)}{\mathbf{A}}{}^2\\
&+\overset{(2)}{g_3}(x-l_1)\overset{(2)}{\mathbf{A}}{}^3]\times[\overset{(1)}{g_0}(l_1)\mathbf{1}+\overset{(1)}{g_1}(l_1)\overset{(1)}{\mathbf{A}}+\overset{(1)}{g_2}(l_1)\overset{(1)}{\mathbf{A}}{}^2+\overset{(1)}{g_3}(l_1)\overset{(1)}{\mathbf{A}}{}^3]\\
&=\overset{(2)}{g_0}(x-l_1)\overset{(1)}{g_0}(l_1)\mathbf{1}+\overset{(2)}{g_0}(x-l_1)\overset{(1)}{g_1}(l_1)\overset{(1)}{\mathbf{A}}+\overset{(2)}{g_1}(x-l_1)\overset{(1)}{g_0}(l_1)\overset{(2)}{\mathbf{A}}\\
&+\overset{(2)}{g_0}(x-l_1)\overset{(1)}{\mathbf{A}}{}^2+\overset{(2)}{g_1}(x-l_1)\overset{(1)}{g_1}(l_1)\overset{(2)(1)}{\mathbf{A}\mathbf{A}}+\overset{(2)}{g_2}(x-l_1)\overset{(2)}{\mathbf{A}}{}^2\\
&+\overset{(2)}{g_0}(x-l_1)\overset{(1)}{g_3}(l_1)\overset{(1)}{\mathbf{A}}{}^3+\overset{(2)}{g_1}(x-l_1)\overset{(1)}{g_2}(l_1)\overset{(2)}{\mathbf{A}}\overset{(1)}{\mathbf{A}}{}^2\\
&+\overset{(2)}{g_2}(x-l_1)\overset{(1)}{g_1}(l_1)\overset{(2)}{\mathbf{A}}{}^2\overset{(1)}{\mathbf{A}}+\overset{(2)}{g_3}(x-l_1)\overset{(1)}{g_0}(l_1)\overset{(2)}{\mathbf{A}}{}^3\\
&+\overset{(2)}{g_1}(x-l_1)\overset{(1)}{g_3}(l_1)\overset{(2)}{\mathbf{A}}\overset{(1)}{\mathbf{A}}{}^3+\overset{(2)}{g_2}(x-l_1)\overset{(1)}{g_2}(l_1)\overset{(2)}{\mathbf{A}}{}^2\overset{(1)}{\mathbf{A}}{}^2\\
&+\overset{(2)}{g_3}(x-l_1)\overset{(1)}{g_1}(l_1)\overset{(2)}{\mathbf{A}}{}^3\overset{(1)}{\mathbf{A}}+\overset{(2)}{g_2}(x-l_1)\overset{(1)}{g_3}(l_1)\overset{(2)}{\mathbf{A}}{}^2\overset{(1)}{\mathbf{A}}{}^3\\
&+\overset{(2)}{g_3}(x-l_1)\overset{(1)}{g_2}(l_1)\overset{(2)}{\mathbf{A}}{}^3\overset{(1)}{\mathbf{A}}{}^2+\overset{(2)}{g_3}(x-l_1)\overset{(1)}{g_3}(l_1)\overset{(2)}{\mathbf{A}}{}^3\overset{(1)}{\mathbf{A}}{}^3,
\end{aligned} \tag{2.48}$$

where $\overset{(i)}{g_k}(\cdot)$, $i=1,2$, $k=0,1,2,3$, are given by Eq.(2.33). The transmission equation (2.47) makes it directly possible to analyse the influence of the external climate conditions, represented by the state vector $\mathbf{S}(0)$, on the distribution of the temperature, moisture potential, heat flux and moisture flux in the region of the internal layer of the building wall (layer 2).

The formulae (2.37)–(2.40) and (2.42)–(2.45) constitute the solution of the problem considered. They present the complex amplitudes of the fields of heat flux, moisture flux, temperature and moisture potential, describing the slow, linear and harmonic process of heat and moisture transfer in a two-layer building wall; the instantaneous values of those fields can be obtained on the basis of Eq.(2.6).

The solutions presented in the form of complex amplitudes allowed, owing to Eq.(2.7), for the determination of the following insulation properties of building wall:

the attenuation factor of the temperature amplitude

$$\xi_\Theta = \frac{\Theta_{0m}(l_1 + l_2)}{\Theta_{0m}(0)}; \tag{2.49}$$

the attenuation factor of the moisture potential amplitude

$$\xi_M = \frac{M_{0m}(l_1 + l_2)}{M_{0m}(0)}; \tag{2.50}$$

the temperature phase shift

$$\Delta\varphi_\Theta = \varphi_\Theta(l_1 + l_2) - \varphi_\Theta(0); \tag{2.51}$$

and the moisture potential phase shift

$$\Delta\varphi_M = \varphi_M(l_1 + l_2) - \varphi_M(0). \tag{2.52}$$

The insulation properties of the two-layer building wall depend on the thermo-humiditive impedances (2.10)–(2.12) of both the layers.

The stresses induced in the building wall material during the process of moisture thermodiffusion are the same in any direction, and, due to physical relations (2.2), they have the form

$$\sigma^{\Theta,c}(x,t) = -\gamma_T\Theta(x,t) - \gamma_c c(x,t), \tag{2.53}$$

where $\Theta(x,t)$ is obtained from Eqs.(2.39), (2.44) and (2.6). The instantaneous values of the field humidity $c(x,t)$ within the building wall may be obtained as follows:

From the balance equation of mass we have

$$\dot{c}(x,t) = -\operatorname{div}\eta(x,t) = -\frac{\partial}{\partial x}\eta(x,t), \tag{2.54}$$

thus

$$c(x,t) = \int \frac{\partial}{\partial x}[-\eta(x,t)]dt. \tag{2.55}$$

Using Eq.(2.6) we obtain

$$c(x,t) = \operatorname{Re}\left\{\frac{j}{\omega}\frac{d\eta_0(x)}{dx}e^{j\omega t}\right\}. \tag{2.56}$$

The derivative of the moisture flux complex amplitude $\eta_0(x)$ with respect to x occurring in Eq.(2.56) may be calculated from Eqs.(2.38) and (2.43), by taking into account that

$$\frac{d}{dx}[\operatorname{ch}z_0(x)] = \operatorname{sh}z_0(x), \qquad \frac{d}{dx}[\operatorname{sh}z_0(x)] = \operatorname{ch}z_0(x), \tag{2.57}$$

where $z_0(x)$ is an arbitrary complex function.

Since the stresses practically do not influence the process of heat and moisture transfer occurring in the material [3], the formulae for the temperature field $\Theta(x,t)$ and the humidity field $c(x,t)$ derived in the present paper allowed to determine the stresses induced during the process of moisture thermodiffusion in the wall (from Eq.(2.53)), independently of the stress state resulting from the loading imposed on the wall boundary. For example, if we assume for simplicity that the building wall material can be freely deformed only in direction x, whereas the possibility of wall deformation in directions perpendicular to x is strongly restricted (e.g. by the proper fastening of the wall to the building structure), then

$$\begin{aligned} \sigma_{11}(x,t) &= (2\mu+\lambda)\varepsilon_{11}(x,t)\sigma^{\Theta,c}(x,t), \\ \sigma_{12}(x,t) &= \sigma_{13}(x,t) = \sigma^{\Theta,c}(x,t), \end{aligned} \tag{2.58}$$

where $\sigma^{\Theta,c}(x,t)$ are given in Eq.(2.53), and the strain field $\varepsilon_{11}(x,t)$ may be determined from Eqs.(2.9) and (2.6):

$$\varepsilon_{11}(x,t) = \int \dot{\varepsilon}_{11}(x,t)dt = \int \frac{\partial v(x,t)}{\partial x}dt = \operatorname{Re}\left[\frac{1}{j\omega}\frac{dv_0(x)}{dx}e^{jwt}\right]. \tag{2.59}$$

3. Final remarks

In determining the thermo-humiditive characteristics of building walls, coupling between the processes of heat and moisture transfer in the wall material is usually not taken into account, and only the dependence of heat conduction coefficients on humidity of the material is accounted for [3]. The temperature and moisture potential fields in the walls are usually determined from two uncoupled partial differential equations of the type of Fourier's heat conduction equation with variable coefficients. This approach disregards the influence of the slow process of moisture exchange on the temperature field within the wall.

The equations of the Podstrigač – Nowacki theory of thermodiffusion in deformable solids, [1, 2] make it possible to treat both the coupled processes as one thermo-humiditive process of moisture thermodiffusion in the wall. In addition, equations of this theory make it possible to determine the stresses induced during the moisture thermodiffusion in the wall.

The solutions presented in this paper describe the process of moisture thermodiffusion in a two-layer building wall occurring under slow harmonic changes of thermo-humiditive environment parameters. Annual distribution of mean month's values of external climate parameters [3] is sinusoidal. The solutions have been presented in a manner which enables them to be used for determining the insulation properties (thermal and humiditive) in the general case of a n-layer building wall, under the assumption that on the surfaces dividing of the wall the boundary conditions of fourth kind are fulfilled (continuity of heat and moisture fluxes, of temperature and of moisture potentials).

The methods used here are taken from the theory of electric transmission lines and the theory of electric four-terminal (or $4n$-terminal) networks. They make it possible to obtain relatively simple formulae for the analytic solutions, simpler than those obtained from the theory of thermodiffusion by means of the traditional, integral transform methods.

The engineering application of the solutions presented in the paper depends on experimental determination of the numerical values of the basic material constants: d, m, n, κ, k for the moisture thermodiffusion process, and the constants μ, λ, γ_T, γ_c which are necessary for determining the stresses induced.

Appendix A

A.1. Electrical analogies

Slow, linear process of moisture thermodiffusion in a one-layer building wall described by the Eqs.(2.1)–(2.5), corresponds to the system of three magnetically coupled electric transmission lines, which is presented in Fig.2. Line 1 corresponds to the elastic deformation of the wall material, line 2 – to the heat conduction, and line 3 – to the moisture transfer.

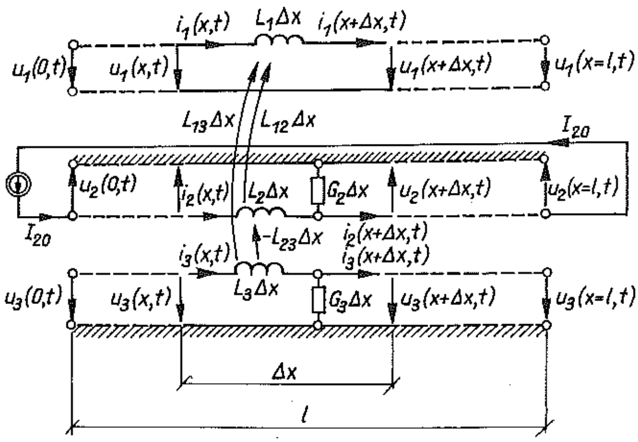

Fig. 2. System of three electric transmission lines coupled magnetically – the electrical analogue for a slow, one-dimensional process of thermodiffusion in elastic solids.

In Fig.2 u_1, u_2, u_3 are the line voltages, i_1, i_2, i_3 – the currents, L_1, L_2, L_3 – the self-inductances per unit length, L_{12}, L_{13}, L_{23} – the mutual inductances per unit length, G_2, G_3 – the self-conductances per unit length.

The system of electric transmission lines presented in Fig.2 is described by the following equations analogous to the Eqs.(2.1)–(2.5) of moisture thermodiffusion in building walls:

$$\text{(A.1)} \qquad i_1 = J_1 A = \frac{1}{L_1}\frac{\partial(-\psi_1)}{\partial x} - \frac{L_3 L_{12} + L_{13} L_{23}}{L_1 L_3} i_2 - \frac{L_{13}}{L_1 L_3}\frac{\partial(-\psi_3)}{\partial x},$$

$$\frac{\partial(-\psi_2)}{\partial x} = (L_2 - \frac{L_{23}^2}{L_3})i_2 - \frac{L_{23}}{L_3}\frac{\partial(-\psi_3)}{\partial x}, \tag{A.1 [cont.]}$$

$$i_3 = J_3 A = \frac{L_{23}}{L_3} i_2 + \frac{1}{L_3}\frac{\partial(-\psi_3)}{\partial x}.$$

Here J_k, $k = 1,2,3$, is the current density in line k, A is the cross-sectional area of each transmission line, ψ_k is the magnetic flux associated with the corresponding line.

The following relationship holds true between the line voltage and the associated magnetic flux:

$$u_k = \frac{\partial \psi_k}{\partial t}, \qquad k = 1,2,3. \tag{A.2}$$

Equations (A.1) are analogous to the physical relations of thermodiffusion (2.2).

On the ground of the Kirchhoff law [8] for the lines 2 and 3 we obtain the equations

$$u_2 = -\frac{1}{G_2}\frac{\partial i_2}{\partial x}, \qquad u_3 = -\frac{1}{G_3}\frac{\partial i_3}{\partial x}, \tag{A.3}$$

which are analogous to the Eqs.(2.3).

Substituting i_3 from Eq.(A.1)$_3$ into Eq.(A.3)$_2$, we obtain the equation

$$u_3 = -\frac{1}{L_3 G_3}\frac{\partial^2(-\psi_3)}{\partial x^2} - \frac{L_{23}}{L_3 G_3}\frac{\partial i_2}{\partial x}, \tag{A.4}$$

which corresponds to Eq.(2.5).

The complete set of differential equations of thermodiffusion (2.1) has the same form as the equations describing the electric system from Fig.2

$$\begin{aligned}
&\frac{1}{L_1}\frac{\partial^2(-\psi_1)}{\partial x^2} = \frac{L_3 L_{12} + L_{13} L_{23}}{L_1 L_3}\frac{\partial i_2}{\partial x} + \frac{L_{13}}{L_1 L_3}\frac{\partial^2(-\psi_2)}{\partial x^2},\\
&\frac{I_{20}}{G}\frac{\partial^2 i_2}{\partial x^2} - \frac{I_{20}(L_2 L_3 - L_{23}^2)}{L_3}\frac{\partial i_2}{\partial t} - \frac{I_{20} L_{23}}{L_3}\frac{\partial(-u_3)}{\partial t} = 0,\\
&\frac{1}{L_3 G_3}\frac{\partial^2}{\partial x^2}\frac{\partial(-\psi_3)}{\partial x} - \frac{\partial}{\partial t}\frac{\partial(-\psi_3)}{\partial x} + \frac{L_{23}}{L_3 G_3}\frac{\partial^2 i_2}{\partial x^2} = 0.
\end{aligned} \tag{A.5}$$

Comparison of the analogous sets of equations yields the following relations:

$$\sigma_{11} \leftrightarrow J_1, \qquad \varepsilon_{11} = \frac{\partial w}{\partial x} \leftrightarrow \frac{\partial(-\psi_1)}{\partial x}, \qquad v = \frac{\partial w}{\partial t} \leftrightarrow (-u_1), \tag{A.6}$$

(A.6) [cont.]
$$\Theta \leftrightarrow J_2, \qquad S \leftrightarrow \frac{\partial(-\psi_2)}{\partial x}, \qquad q \leftrightarrow (-u_2),$$
$$M \leftrightarrow J_3, \qquad c \leftrightarrow \frac{\partial(-\psi_3)}{\partial x}, \qquad \eta \leftrightarrow (-u_3),$$

which constitute the system of electro-elasto-thermodiffusive analogies [5].

A.2. Electric transmission lines under sinusoidal excitation

Consider the system shown in Fig.2 with the sinusoidal current excitation. The instantaneous values of the line voltages and currents are given by

(A.7)
$$\begin{aligned} u_k(x,t) &= \mathrm{Re}[U_k(x)e^{j\omega t}], \\ i_k(x,t) &= \mathrm{Re}[I_k(x)e^{j\omega t}], \qquad k=1,2,3. \end{aligned}$$

where $U_k(x)$, $I_k(x)$ are the complex amplitudes of voltages and currents, respectively, $j=\sqrt{-1}$ is the imaginary unit and ω is the angular frequency.

The system of electric transmission lines from Fig.2 is described by the following equations [8, 9]

$$\text{(A.8)} \quad \frac{\partial(-u_1)}{\partial x} = L_1\frac{\partial i_1}{\partial t} + L_{12}\frac{\partial i_2}{\partial t} + L_{13}\frac{\partial i_3}{\partial t}, \qquad \frac{\partial i_1}{\partial x} = 0,$$
$$\text{(A.9)} \quad \frac{\partial(-u_2)}{\partial x} = L_2\frac{\partial i_2}{\partial t} - L_{23}\frac{\partial i_3}{\partial t}, \qquad \frac{\partial i_2}{\partial x} = -G_2u_2,$$
$$\text{(A.10)} \quad \frac{\partial(-u_3)}{\partial x} = -L_{23}\frac{\partial i_2}{\partial t} + L_3\frac{\partial i_3}{\partial t}, \qquad \frac{\partial i_3}{\partial x} = -G_3u_3.$$

If we introduce (A.7) into Eqs.(A.8)–(A.10), we obtain the following set of equations for the complex impedances

$$\text{(A.11)} \qquad \frac{d(-U_1)}{dx} = Z_1^{(e)}I_1 + Z_{12}^{(e)}I_2 + Z_{13}^{(e)}I_3, \qquad \frac{dI_1}{dx} = 0,$$

$$\text{(A.12)} \quad \frac{d}{dx}\begin{bmatrix} -U_2(x) \\ -U_3(x) \\ I_2(x) \\ I_3(x) \end{bmatrix} = \begin{bmatrix} 0 & 0 & Z_2^{(e)} & -Z_{23}^{(e)} \\ 0 & 0 & -Z_{23}^{(e)} & Z_3^{(e)} \\ Y_2^{(e)} & 0 & 0 & 0 \\ 0 & Y_3^{(e)} & 0 & 0 \end{bmatrix} \begin{bmatrix} -U_2(x) \\ -U_3(x) \\ I_2(x) \\ I_3(x) \end{bmatrix},$$

where

$$\text{(A.13)}\qquad \begin{aligned} Z_1^{(e)} &= j\omega L_1, & Z_{12}^{(e)} &= j\omega L_{12}, & Z_{13}^{(e)} &= j\omega L_{13},\\ Z_2^{(e)} &= j\omega L_2, & Z_{23}^{(e)} &= j\omega L_{23}, & Y_2^{(e)} &= G_2,\\ Z_3^{(e)} &= j\omega L_3, & Y_3 &= G_3, \end{aligned}$$

are the complex impedances.

Multiplying both sides of equations (A.11) and (A.12) by $e^{j\omega t}$ and taking their real parts, we obtain equations (A.8)–(A.10) in which the instantaneous values of voltages and currents are given by (A.7).

Equation (A.12) is called the homogenous state equation and can be written in the form

$$\text{(A.14)}\qquad \frac{d\mathbf{S}(x)}{dx} = \mathbf{A}\mathbf{S}(x),$$

where $\mathbf{S}(x) = [-U_2(x), -U_3(x), I_2(x), I_3(x)]^T$ is the state vector, and $\mathbf{A}$ is the matrix of the system.

The solution of Eq.(A.14) is given by the following formula

$$\text{(A.15)}\qquad \mathbf{S}(x) = [e^{\mathbf{A}x}]\mathbf{S}(0),$$

where $\mathbf{S}(0)$ is the state vector on the input of the system, $e^{\mathbf{A}x}$ is the so-called transmission matrix (or transition matrix) [7-10].

Thus, solution of the state equation (A.14) is reduced to the determination of the transmission matrix $e^{\mathbf{A}x}$. That matrix can be defined by the following power series [7, 8]

$$\text{(A.16)}\qquad e^{\mathbf{A}x} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}x)^k}{k!}.$$

The series is absolutely convergent for every finite value of x [10].

Any convergent power series of a matrix of order m can be presented in a form of the uniquely determined polynomial of order $m-1$ of this matrix [10] (to prove this theorem the Cayley–Hamilton theorem must be used). Thus, for $m = 4$ we have

$$\text{(A.17)}\qquad e^{\mathbf{A}x} = g_0\mathbf{1} + g_1\mathbf{A} + g_2\mathbf{A}^2 + g_3\mathbf{A}^3,$$

where g_0, g_1, g_2, g_3 are the coefficients of the so-called generating polynomial $g(s)$ of a complex variable s

$$\text{(A.18)}\qquad g(s) = g_0 + g_1 s + g_2 s^2 + g_3 s^3.$$

The following conditions are fulfilled:

$$\text{(A.19)}\qquad \begin{aligned} g(\lambda_k) - e^{\lambda_k x} &= 0, \qquad k = 1,2,3,4, \\ g(\mathbf{A}) - e^{\mathbf{A}x} &= \mathbf{0}, \end{aligned}$$

where λ_k are the eigenvalues of matrix $\mathbf{A}$, and

$$\text{(A.20)}\qquad \lambda_{1,2} = \pm\gamma_1, \qquad \lambda_{3,4} = \pm\gamma_2,$$

where

$$\text{(A.21)}\qquad \begin{aligned} \gamma_1 &= \sqrt{\bar{P} + K}, \qquad \gamma_2 = \sqrt{\bar{P} - K}, \\ \bar{P} &= \frac{P_2 + P_3}{2}, \qquad K = \sqrt{(\Delta P)^2 + Q^2}, \\ \Delta P &= \frac{P_2 - P_3}{2}, \qquad Q^2 = Q_2 Q_3, \end{aligned}$$

$$P_2 = Z_2^{(e)} Y_2^{(e)}, \quad P_3 = Z_3^{(e)} Y_3^{(e)}, \quad Q_2 = Y_2^{(e)} Z_{23}^{(e)}, \quad Q_3 = Y_3^{(e)} Z_{23}^{(e)}.$$

The existence of solution of the set (A.19) requires the main determinant of the set to vanish,

$$\text{(A.22)}\qquad \begin{vmatrix} 1 & \gamma_1 & \gamma_1^2 & \gamma_1^3 & e^{\gamma_1 x} \\ 1 & -\gamma_1 & \gamma_1^2 & -\gamma_1^3 & e^{-\gamma_1 x} \\ 1 & \gamma_2 & \gamma_2^2 & \gamma_2^3 & e^{\gamma_2 x} \\ 1 & -\gamma_2 & \gamma_2^2 & -\gamma_2^3 & e^{-\gamma_2 x} \\ \mathbf{1} & \mathbf{A} & \mathbf{A}^2 & \mathbf{A}^3 & e^{\mathbf{A}x} \end{vmatrix} = 0.$$

Developing the determinant (A.22) with respect its last column, we obtain

$$\text{(A.23)}\quad \begin{aligned} e^{\mathbf{A}x} &= \left[\frac{\mathrm{sh}\gamma_1 x}{\gamma_1(\gamma_1^2 - \gamma_2^2)} - \frac{\mathrm{sh}\gamma_2 x}{\gamma_2(\gamma_1^2 - \gamma_2^2)}\right]\mathbf{A}^3 + \frac{\mathrm{ch}\gamma_1 x - \mathrm{ch}\gamma_2 x}{\gamma_1^2 - \gamma_2^2}\mathbf{A}^2 \\ &+ \left[\frac{\gamma_1^2 \mathrm{sh}\gamma_2 x}{\gamma_2(\gamma_1^2 - \gamma_2^2)} - \frac{\gamma_2^2 \mathrm{sh}\gamma_1 x}{\gamma_1(\gamma_1^2 - \gamma_2^2)}\right]\mathbf{A} + \frac{\gamma_1^2 \mathrm{ch}\gamma_2 x - \gamma_2^2 \mathrm{ch}\gamma_1 x}{\gamma_1^2 - \gamma_2^2}\mathbf{1}. \end{aligned}$$

Knowing the transmission matrix $e^{\mathbf{A}x}$ we can determine from Eq. (A.15) the components of the state vector $\mathbf{S}(x)$ representing the solution of equation (A.14).

If the analogies (A.6) are used to express the components of the state vector $\mathbf{S}(x)$ in terms of the quantities describing the process of moisture thermodiffusion, i.e.

$\mathbf{S}(x) = [q_0(x), \eta_0(x), \Theta_0(x), M_0(x)]^T$ (see Eq.(2.22)), and the matrix **A** in terms of the thermo-humiditive impedances (2.10)–(2.13), we obtain the solution of the state equation (2.8) in form of the formulae (2.37)–(2.40).

A.3. A multilayer building wall and the chain connection of four-terminal electric networks

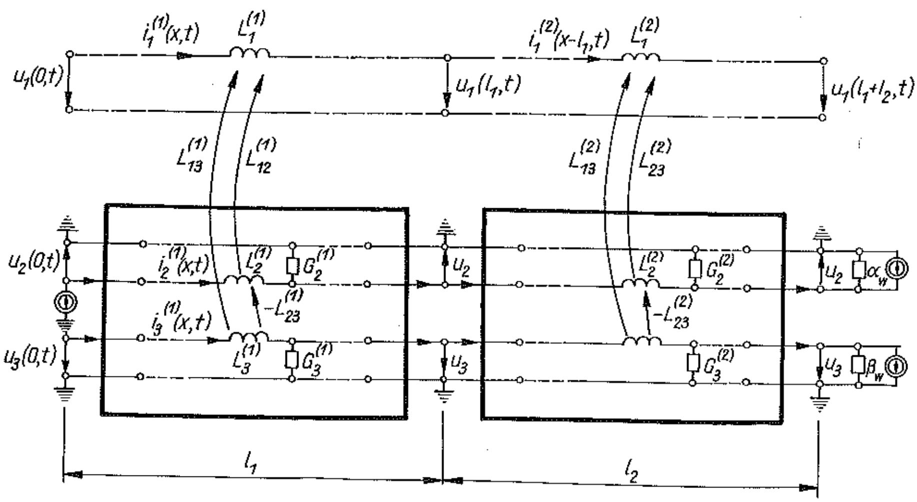

Fig. 3. Electrical model for heat and moisture transfer in a two-layer building wall – the chain connection of two systems of coupled electric transmission lines; l_1, l_2 – line length of 1 (or 2) system, corresponding to wall layers thicknesses.

In Fig.3 the electric analogue for a slow process of moisture thermodiffusion in a two-layer building wall is shown.

The heat conduction and moisture transfer in building walls are coupled processes and, therefore, the transmission lines 2 and 3 of two systems of transmission lines (Fig.3) are magnetically coupled, within each of these two systems. Lines 2 and 3 of the first system, as well as those of the second system, may be replaced by a 8-terminal network, i.e. by an element of four input terminals and four output terminals (see Fig.3).

The stress state of wall material does not practically influence the process of moisture thermodiffusion in the building wall [3]; thus, in the electric diagram in Fig.3, the influence of the electromagnetic field of line 1 on lines 2 and 3 is disregarded. On the other hand, the influence of electromagnetic fields of lines 2 and 3 on the line 1 is taken into account; it corresponds to the stress induction during the moisture thermodiffusion in the wall.

The 8-terminal electric network equivalent to the transmission lines 2 and 3 of each of the systems of transmission lines in Fig.3 is a particular case of a $4n$-terminal network [8, 9], Fig.4.

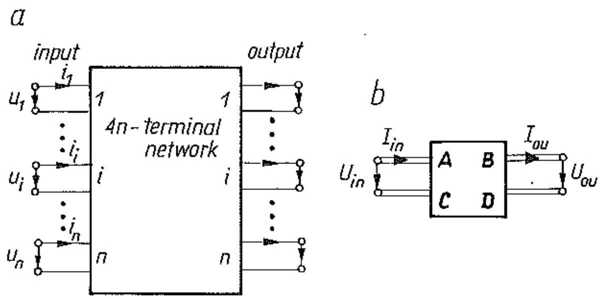

Fig. 4. $4n$-terminal network diagrams: a) general diagram with terminal pairs, b) equivalent four-terminal network with matrix parameters.

Since any $4n$-terminal can be replaced by a four-terminal network with matrix parameters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ (Fig.4b), we present here the results of the four-terminal electric networks theory, which are valid also in the general case of $4n$-terminal networks.

The chain connection of four-terminal networks is a connection in which the output terminals of one four-terminal network are connected to the input terminals of the next four-terminal network.

Consider the chain connection of three electric four-terminal networks described by the following transmission equations:

$$\begin{bmatrix} U_2 \\ I_2 \end{bmatrix} = \mathbf{a}_1 \begin{bmatrix} U_1 \\ I_1 \end{bmatrix}, \quad \begin{bmatrix} U_3 \\ I_3 \end{bmatrix} = \mathbf{a}_2 \begin{bmatrix} U_2 \\ I_2 \end{bmatrix}, \quad \begin{bmatrix} U_4 \\ I_4 \end{bmatrix} = \mathbf{a}_3 \begin{bmatrix} U_3 \\ I_3 \end{bmatrix}. \tag{A.24}$$

In the transmission equations (A.24) the output voltages and currents of each of the four-terminal networks are expressed in terms of the input voltages and currents, and by means of the so-called transmission matrices $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ representing the transmission properties of the four-terminal networks.

After substitutions we obtain from Eqs.(A.24)

$$\begin{bmatrix} U_4 \\ I_4 \end{bmatrix} = \mathbf{a}_3\,\mathbf{a}_2\,\mathbf{a}_1 \begin{bmatrix} U_1 \\ I_1 \end{bmatrix}. \tag{A.25}$$

The transmission equation of an equivalent four-terminal network replacing the chain connection considered is given by

$$\begin{bmatrix} U_4 \\ I_4 \end{bmatrix} = \mathbf{a} \begin{bmatrix} U_1 \\ I_1 \end{bmatrix}. \tag{A.26}$$

From the comparison of Eqs.(A.25) and (A.26) it follows that

$$\mathbf{a} = \mathbf{a}_3\,\mathbf{a}_2\,\mathbf{a}_1. \tag{A.27}$$

For a chain connection including n four-terminal networks characterized by the transmission matrices $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$, the transmission matrix of the equivalent four-terminal network is given by

$$\mathbf{a} = \mathbf{a}_n\,\mathbf{a}_{n-1} \dots \mathbf{a}_1. \tag{A.28}$$

It follows that the transmission matrix of the chain connection of a number of four-terminal networks is equal to the product of their transmission matrices. The sequence of transmission matrices in the product (A.28) is reversed as compared to the sequence of four-terminal networks in a chain connection, what is important since the matrix product is, in general, non-commutative $(\mathbf{a}_1\,\mathbf{a}_2 \neq \mathbf{a}_2\,\mathbf{a}_1)$.

References

1. W.Nowacki, *Thermodiffusion in solids,* in: Thermoelasticity, Ossolineum, Wroclaw 1972 [in Polish].

2. J.S.Podstrigač, *Differential equations for thermodiffusion problems in deformable isotropic solids* [in Russian], DAN USSR, 2, 1961.

3. W.N.Bogoslowski, *Thermal and humiditive processes in buildings* [transl. from Russian], Arkady, Warszawa 1985.

4. W.Dudziak, R.Uklejewski, *Heat and moisture transfer in a building wall material layer at slow harmonic changes of thermo-humiditive environment parameters, based on the theory of thermodiffusion in solids* [in Polish], Arch. Inż. Ląd., (to be published).

5. W.Dudziak, R.Uklejewski, *On the application of the electrical transmission lines theory in thermodiffusion problems in solids*, Bull. Acad. Polon. Sci., Série Sci. Techn., 34, 3-4, 1986.

6. Wł.Płoński, J.A.Pogorzelski, *Physics of building structures*, Arkady, Warszawa 1978 [in Polish].

7. L.Pipes,*Direct computation of transmission matrices of electrical transmission lines. Part I, II,* J.Frankl.Inst., 281, pp.275-292, 387-405, 1966.

8. T.Cholewicki, *Electrical transmission lines and nonuniform ladder networks*, PWN, Warszawa 1974 [in Polish].

9. T.Cholewicki, *Analysis of nonuniform electrical networks by means of the difference equations method*, PWN, Warszawa 1982 [in Polish].

10. K.Ogata, *State space methods in control theory*, WNT, Warszawa 1974 [transl. from English].

11. J.Wehr, *Electro-mechano-acoustical analogies for the systems with lumped and distributed parameters in the light of dimensional analysis* [in Polish], Arch. Akustyki, **2**, pp.211-226, 1967.

12. K.Kiessl, *Kapillaren und dampfformiger Feuchtetransport in mehrschichtigen Bauteilen*, Diss.Univ.Essen, Essen 1983.

13. R.Kohonen, *A method to analyse the transient hygrothermal behaviour of building materials and components*, Technical Research Center of Finland, Publ. 21, Espoo 1984.

Streszczenie

PRZENOSZENIE CIEPŁA I WILGOCI W DWUWARSTWOWEJ PRZEGRODZIE BUDOWLANEJ PRZY ZMIENIAJĄCYCH SIĘ SINUSOIDALNIE I QUASI-STATYCZNIE WARUNKACH TERMICZNYCH I WILGOTNOŚCIOWYCH OTOCZENIA, NA PODSTAWIE TEORII TERMODYFUZJI W CIAŁACH STAŁYCH

Do opisu procesów cieplno-wilgotnościowych w zewnętrznych przegrodach budowlanych zastosowano teorię termodyfuzji w ciałach stałych, odkształcalnych sprężyście, która umożliwia uwzględnienie wzajemnego sprzężenia procesów przepływu ciepła i przenikania wilgoci oraz wyznaczenie naprężeń powstających w wyniku tych procesów w przegrodzie. Przedstawiono rozwiązania równań opisujących liniowy, quasi-statyczny i sinusoidalny proces termodyfuzji wilgoci w dwuwarstwowej (i ogólnie w n–warstwowej) przegrodzie budowlanej. Otrzymane rozwiązania mają znaczenie dla obliczeń cieplno-wilgotnościowych przegród budowlanych, obejmujących okres 1 roku (średniomiesięczne wartości parametrów klimatu zewnętrznego rozkładają się w ciągu roku sinusoidalnie). Metoda rozwiązania równań oraz niektóre wprowadzone pojęcia pochodzą z teorii elektrycznych linii przesyłowych.

Резюме

ТЕПЛОПЕРЕНОС И ВЛГОПЕРЕНОС В ДВУХСЛОИСТОЙ СТРОИТЕЛЬНОЙ ПРЕГРАДЕ ПРИ ИЗМЕНЯЮЩИХСЯ СИНУСОИДАЛЬНО И КВАЗИСТАТИЧЕСКИ ТЕРМИЧЕСКИХ И ВЛАЖНОСТНЫХ УСЛОВИЯХ ОКРУЖАЮЩЕЙ СРЕДЫ НА ОСНОВЕ ТЕОРИИ ТЕРМОДИФФУЗИИ Б ТБЕРДЫХ ТЕЛАХ

Для описания термо-влажностных процессов, во внешных строительных преградах, применена теория термодиффузии в твердых, упруго деформируемых телах, которая дает возможность учета взаимного сопряжения процессов теплопереноса и проникания влажности, а также определения напряжений, возникающих в результате этих процессов в преграде.Представлены решения управнений, описывающих линейный, квазистатический и синусоидальный процесс термодыффызии влажности в двухслоистой (и общим образом в n-слоистой) строительной преграде. Полученные решения имеют значение для термо–влажностных расчетов строительных преград, охватывающих период одново года (среднемесячные значения параметров внешнего климата распределяются в течение года синусоидальным образом). Метод решения уравнений и некоторые веденые понятия происходят из теории электрических линий передачи.

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