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STRESSES AND STRAINS INDUCED IN A BUILDING WALL BY SLOW HARMONIC THERMO-HUMIDITIVE DIFFUSION

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The Podstrigač–Nowacki theory of thermodiffusion in deformable solids has been used to describe the coupled processes of heat and moisture transport through building walls. The analytic solutions for the stresses and strains induced in a wall material by slow harmonic thermo-humiditive diffusion process are obtained by means of the electric transmission line theory, owing to the electro-elasto-thermo-diffusive analogies.

1. Introduction

The heat and moisture transfer in building walls is described here by using the Podstrigač–Nowacki theory of thermodiffusion in deformable solids [1,2], developed in the monograph by Nowacki and Olesiak [3]. The description has an advantage that a unified approach to mutual couplings between the processes of heat and moisture transfer is possible; moreover, there is the possibility of calculating the stresses and strains which appear during thermo-humiditive processes in the wall material.

A system of electro-elasto-thermo-diffusive analogies has been given which enables the methods of the theory of electric transmission lines [4] to be employed in the analysis of a unidirectional process of thermodiffusion in solids. The system has been used to construct a general solution which describes the coupled process of heat and moisture transfer in a multi-layer building wall, induced by slow harmonic changes of the environmental thermo-humiditive parameters [5].

A measuring-computational procedure for determining the material constants of a building wall has been worked out for a model of moisture thermodiffusion in solids [6].

The present paper deals with determining the stresses and strains induced in an external building wall by the moisture thermodiffusion process.

2. Equations for heat and moisture transport in a building wall

The theoretical model adopted here assumes mutual interaction between the processes of heat and moisture flow and of wall deformation. Moisture stands for the diffusing medium, humidity represents concentration of the diffusing medium, and the role of a chemical potential of the diffusing medium is played by the moisture potential. The coupled system of parabolic-hyperbolic differential equations has the form [1]

$$
\begin{aligned}
\mu\nabla^2\mathbf{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{X} &= \rho\ddot{\mathbf{u}} + \gamma_T\operatorname{grad}\Theta + \gamma_c\operatorname{grad}c\,,\\
(\nabla^2 - \frac{1}{\kappa_1}\partial_t)\Theta - \eta_0\partial_t\operatorname{div}\mathbf{u} + \alpha\partial_t c &= -Q\,,\\
(\nabla^2 - \frac{1}{\kappa_2}\partial_t)c + \beta_0\nabla^2\Theta + \varepsilon_0\nabla^2\operatorname{div}\mathbf{u} &= -\sigma\,.
\end{aligned}
\tag{2.1}
$$

The physical relations between stresses σ_{ij}, strains ε_{ij}, temperature increment Θ and moisture increment c read

$$
\begin{aligned}
\sigma_{ij} &= 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma_T\Theta - \gamma_c c)\delta_{ij}\,,\\
S &= \gamma_T\varepsilon_{kk} - dc + m\Theta\,,\\
M &= -\gamma_c\varepsilon_{kk} + d\Theta + nc\,,
\end{aligned}
\tag{2.2}
$$

where $\mathbf{u}$ – displacement vector of a wall material particle, ε_{ij} – strain tensor, ε_{kk} – dilatation, $\Theta = T - T_0$ – relative temperature, T_0 – temperature in reference state, $c = C - C_0$ – relative humidity, C_0 – humidity in reference state, S – entropy per unit volume, M – moisture potential, $\mathbf{X}$ – body force vector, Q – heat sources, σ – moisture sources, μ, λ – Lamé elastic constants, ρ – material density, $\gamma_T, \gamma_c, \eta_0, \kappa_1, \kappa_2, \alpha_0, \beta_0, \varepsilon_0, d, m, n$ – basic material constants.

We will consider the heat and moisture transfer in an external building wall, induced by sinusoidal changes of the thermo-humiditive parameters of the external climate in a one-year period.

In a certain range of variations of the temperature, humidity and stress state, and within the time period under consideration, the following assumptions are adopted: the wall constitutes a linear and stationary system; the wall material is homogeneous and isotropic with respect to the thermal, humiditive and elastic properties; the process of heat and moisture transfer is slow so that mechanical inertia of the wall material may be neglected.

Distribution of the temperature, moisture potential and stress state within the building wall under slow harmonic variations of the state will be determined by using the solution given in the paper [5] and in Appendix I. In the commonly used theoretical models of heat and moisture flow in walls it is assumed that the wall material is non-deformable e.g. [7,8].

A one-dimensional, nonstationary process of linear thermodiffusion of moisture in an elastic layer is described by the following equations [5,6]:

$$
\begin{aligned}
(2\mu+\lambda)\frac{\partial^2 u}{\partial x^2} &= \gamma_T\frac{\partial \Theta}{\partial x}+\gamma_c\frac{\partial c}{\partial x}\,,\\
k\frac{\partial^2\Theta}{\partial x^2}-c_{\varepsilon,c}\frac{\partial \Theta}{\partial t}+T_0 d\frac{\partial c}{\partial t} &= 0,\\
D_c\frac{\partial^2 c}{\partial x^2}-\frac{\partial c}{\partial t}+D_T\frac{\partial^2\Theta}{\partial x^2} &= 0,
\end{aligned}
\tag{2.3}
$$

the physical relations

$$
\begin{aligned}
\sigma_{11} &= (2\mu+\lambda)\varepsilon_{11}-\gamma_T\Theta-\gamma_c c\,,\\
S &= m\Theta-dc\,,\\
M &= d\Theta+nc\,,
\end{aligned}
\tag{2.4}
$$

the laws of heat and moisture transfer

$$
q=-k\frac{\partial\Theta}{\partial x}\,,\qquad \eta=-\kappa\frac{\partial M}{\partial x}\,,
\tag{2.5}
$$

and the other relationships

$$
\dot S = \frac{1}{T_0}k\frac{\partial^2\Theta}{\partial x^2}\,,\qquad \dot c=\kappa\frac{\partial^2 M}{\partial x^2}\,,
\tag{2.6}
$$

$$
\eta = -D_c\frac{\partial c}{\partial x}-D_T\frac{\partial\Theta}{\partial x}\,,\qquad D_c=\kappa n,\quad D_T=\kappa d,
\tag{2.7}
$$

where q, η are heat and moisture fluxes, k, κ are coefficients of heat and moisture conduction, and D_T, D_c are coefficients of thermodiffusion of heat and moisture, respectively.

We assume that the following quantities are the harmonic functions of time t:

$$
\begin{aligned}
\sigma_{11}(x,t) &= \operatorname{Re}\left[\sigma_0(x)e^{j\omega t}\right], & \frac{\partial u}{\partial t}:=v(x,t) &= \operatorname{Re}\left[v_0(x)e^{j\omega t}\right],\\
\Theta(x,t) &= \operatorname{Re}\left[\Theta_0(x)e^{j\omega t}\right], & q(x,t) &= \operatorname{Re}\left[q_0(x)e^{j\omega t}\right],\\
M(x,t) &= \operatorname{Re}\left[M_0(x)e^{j\omega t}\right], & \eta(x,t) &= \operatorname{Re}\left[\eta_0(x)e^{j\omega t}\right],
\end{aligned}
\tag{2.8}
$$

where $j = \sqrt{-1}$ is the imaginary unit, and

$$(2.9)\qquad \begin{aligned} \sigma_0(x) &:= \sigma_{0m}(x)e^{j\varphi_\sigma(x)}, & v_0(x) &:= v_{0m}(x)e^{j\varphi_v(x)},\\ \Theta_0(x) &:= \Theta_{0m}(x)e^{j\varphi_\Theta(x)}, & q_0(x) &:= q_{0m}(x)e^{j\varphi_q(x)},\\ M_0(x) &:= M_{0m}(x)e^{j\varphi_M(x)}, & \eta_0(x) &:= \eta_{0m}(x)e^{j\varphi_\eta(x)}. \end{aligned}$$

The quantities with a subscript "0" are so-called complex ampitudes (they are vectors in the complex Gaussian plane) of the exponential form (2.9). The complex amplitudes are expressed in terms of real amplitudes (quantities with a subscript "$0m$") and of the respective phase shift angles $\varphi_{(\cdot)}(x)$. For instance, $M_0(x)$ is the complex amplitude of the moisture potential, $M_{0m}(x)$ is the real amplitude of the moisture potential, and $\varphi_M(x)$ is the phase shift angle of the moisture potential.

The boundary conditions on the wall bounding surfaces (external and internal) are formulated in the form of relationships between the temperature at a wall bounding surface and the heat flux through that surface, and between the moisture potential at a wall bounding surface and the moisture flux through that surface [7].

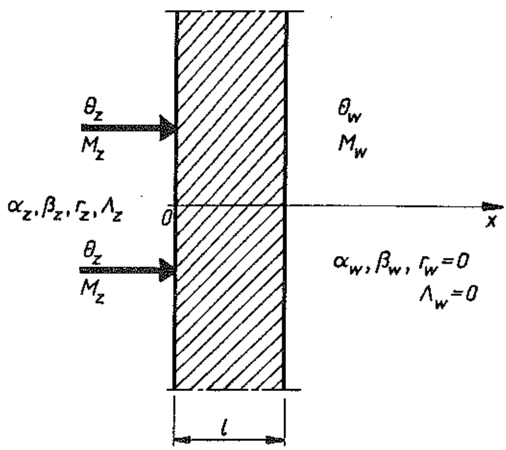

FIG. 1. Diagram of heat and moisture transfer through an external building wall.

On the external surface of the wall (Fig. 1) remaining in contact with air of temperature Θ_z and of moisture potential M_z, the following conditions of heat exchange

$$(2.10)\qquad \alpha_z[\Theta_z - \Theta_0(x=0)] + r_z\Lambda_z = -k\frac{d\Theta_0(x)}{dx}\bigg|_{x=0},$$

and of moisture exchange

$$\beta_z[M_z - M_0(x=0)] = -\kappa \frac{dM_0(x)}{dx}\bigg|_{x=0}, \tag{2.11}$$

are satisfied, where α_z – coefficient of heat exchange (due to convection and radiation), β_z – coefficient of moisture exchange with external air, Λ_z – intensity of incident solar radiation at the external surface of the wall, r_z – coefficient of absorption of solar radiation by the wall surface.

On the internal surface of the wall being in contact with internal air of temperature Θ_w and of moisture potential M_w, we analogously have

$$\alpha_w[\Theta_w - \Theta_0(x=L)] = k\frac{d\Theta_0(x)}{dx}\bigg|_{x=L}, \tag{2.12}$$

$$\beta_w[M_w - M_0(x=L)] = \kappa\frac{dM_0(x)}{dx}\bigg|_{x=L}, \tag{2.13}$$

where α_w, β_w denote coefficients of heat and moisture exchange, respectively, with internal air. It is assumed that there are no internal radiation sources in the room (r_w, $\Lambda_w = 0$).

The boundary conditions (2.10)–(2.13) determine the relations between the temperature at a wall bounding surface (external or internal) and the heat flux through that surface, and between the moisture potential at a wall bounding surface and the moisture flux through that surface.

The linear process of heat and moisture transfer through a building wall along a single axis x under slow harmonic changes of the state can be described, in analogy to Eqs.(I.11), (I.12) in Appendix I, by the following equations

$$\frac{d}{dx}\begin{bmatrix} q_0(x) \\ \eta_0(x) \\ \Theta_0(x) \\ M_0(x) \end{bmatrix} = \begin{bmatrix} 0 & 0 & Z_2 & -Z_{23} \\ 0 & 0 & -Z_{23} & Z_3 \\ Y_2 & 0 & 0 & 0 \\ 0 & Y_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_0(x) \\ \eta_0(x) \\ \Theta_0(x) \\ M_0(x) \end{bmatrix}, \tag{2.14}$$

$$\frac{dV_0(x)}{dx} = Z_1\sigma_0(x) + Z_{12}\Theta_0(x) + Z_{13}M_0(x), \tag{2.15}$$

$$\frac{d\sigma_0(x)}{dx} = 0 \longrightarrow \sigma_0(x) = \sigma_0(0) = \text{const}, \tag{2.16}$$

where

$$\begin{aligned} Z_1 &= j\omega\frac{1}{2\mu+\lambda}, & Z_2 &= j\omega\frac{mn+d}{n}, & Z_3 &= j\omega\frac{1}{n}, \\ Z_{12} &= j\omega\frac{\gamma_T n - \gamma_c d}{n(2\mu+\lambda)}, & Z_{13} &= j\omega\frac{\gamma_c}{n(2\mu+\lambda)}, & Z_{23} &= j\omega\frac{d}{n}, \end{aligned} \tag{2.17}$$

$$Y_2 = T_0 \frac{1}{k}, \qquad Y_3 = \frac{1}{\kappa}. \tag{2.18}$$

The quantities (2.17), by analogy to electrical impedances (I.13) (Appendix I), may be called thermodiffusional impedances (or thermo-humiditive impedances) of a building wall.

In the following considerations we shall assume that the values of $q_0(0)$, $\eta_0(0)$, $\Theta_0(0)$, $C_0(0)$, $M_0(0)$, $\sigma_0(0)$ at the external surface of the wall are known. We will be concerned with determining the distribution of the temperature, moisture potential and stress state inside the wall.

Equation (2.14) represents the so-called homogeneous state equation; it can be written in a compact form as

$$\frac{d\mathbf{S}(x)}{dx} = \mathbf{A}\mathbf{S}(x), \qquad \mathbf{S}(x) = [q_0(x), \eta_0(x), \Theta_0(x), M_0(x)]. \tag{2.19}$$

From equation (2.19) the quantities which describe heat and moisture flow through the wall can be determined.

The solution of the state equation is given by the state vector $\mathbf{S}(x)$ represented by the transmission equation (Appendix I):

$$\mathbf{S}(x) = \left[e^{\mathbf{A}x}\right] S(0), \qquad x \in <0, L>, \tag{2.20}$$

where $e^{\mathbf{A}x}$ is the so-called transmission matrix, and $\mathbf{S}(0) = [q_0(0), \eta_0(0), Q_0(0), M_0(0)]$ is the state vector at the external surface of the wall.

The problem of finding the solution $\mathbf{S}(x)$ which satisfies the state equation (2.19) reduces to determining the transmission matrix $e^{\mathbf{A}x}$. On using the Cayley–Hamilton theorem for matrix $\mathbf{A}$, the transmission matrix can be expressed in the form (Appendix I):

$$e^{\mathbf{A}x} = g_0(x)\mathbf{1} + g_1(x)\mathbf{A} + g_2(x)\mathbf{A}^2 + g_3(x)\mathbf{A}^3, \tag{2.21}$$

where

$$\begin{aligned} g_0(x) &= \frac{\gamma_1^2 \mathrm{ch}\gamma_2 x - \gamma_2^2 \mathrm{ch}\gamma_1 x}{\gamma_1^2 - \gamma_2^2}, \\ g_1(x) &= \frac{\gamma_1^2 \mathrm{sh}\gamma_2 x}{\gamma_2(\gamma_1^2 - \gamma_2^2)} - \frac{\gamma_2^2 \mathrm{sh}\gamma_1 x}{\gamma_1(\gamma_1^2 - \gamma_2^2)}, \\ g_2(x) &= \frac{\mathrm{ch}\gamma_1 x - \mathrm{ch}\gamma_2 x}{\gamma_1^2 - \gamma_2^2}, \\ g_3(x) &= \frac{\mathrm{sh}\gamma_1 x}{\gamma_2(\gamma_1^2 - \gamma_2^2)} - \frac{\mathrm{sh}\gamma_2 x}{\gamma_1(\gamma_1^2 - \gamma_2^2)}, \end{aligned} \tag{2.22}$$

and

$$
\begin{aligned}
&\gamma_1 = \left(\sum P + K\right)^{1/2}, \qquad \gamma_2 = \left(\sum P - K\right)^{1/2},\\
&\sum P = \frac{P_2 + P_3}{2}, \quad K = (\Delta P^2 + Q^2)^{1/2}, \quad \Delta P = \frac{P_2 - P_3}{2},\\
&Q^2 = Q_2 Q_3, \qquad P_2 = Z_2 Y_2, \qquad P_3 = Z_3 Y_3,\\
&Q_2 = Y_2 Z_{23}, \qquad Q_3 = Y_3 Z_{23}.
\end{aligned}
\tag{2.23}
$$

The components of the state vector $\mathbf{S}(x)$ being a solution of the state equation (2.19), i.e. the fields of heat flux $q_0(x)$, moisture flux $\eta_0(x)$, temperature $\Theta_0(x)$ and moisture potential $M_0(x)$, have the form:

$$
\begin{aligned}
q_0(x) = {} & \frac{(K+\Delta P)\eta_0(0) - Q_3 q_0(0)}{2K}\operatorname{ch}\gamma_1 x\\
&+ \frac{(K+\Delta P)B_2(0) - Q_3 B_3(0)}{2K\gamma_1}\operatorname{sh}\gamma_1 x\\
&+ \frac{(K-\Delta P)\eta_0(0) + Q_3 q_0(0)}{2K}\operatorname{ch}\gamma_2 x\\
&+ \frac{(K-\Delta P)B_2(0) + Q_3 B_3(0)}{2K\gamma_2}\operatorname{sh}\gamma_2 x,
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
\eta_0(x) = {} & \frac{(K-\Delta P)q_0(0) - Q_2 \eta_0(0)}{2K}\operatorname{ch}\gamma_1 x\\
&+ \frac{(K-\Delta P)B_3(0) - Q_2 B_2(0)}{2K\gamma_1}\operatorname{sh}\gamma_1 x\\
&+ \frac{(K+\Delta P)q_0(0) - Q_2 \eta_0(0)}{2K}\operatorname{ch}\gamma_2 x\\
&+ \frac{(K+\Delta P)Y_2\eta_0(0) - Q_2 Y_3 q_0(0)}{2K\gamma_2}\operatorname{sh}\gamma_2 x,
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
\Theta_0(x) = {} & \frac{(K+\Delta P)\Theta_0(0) - Q_2 M_0(0)}{2K}\operatorname{ch}\gamma_1 x\\
&+ \frac{(K+\Delta P)Y_2\eta_0(0) - Q_2 Y_3 q_0(0)}{2K\gamma_2}\operatorname{sh}\gamma_1 x\\
&+ \frac{(K-\Delta P)\Theta_0(0) + Q_2 M_0(0)}{2K}\operatorname{ch}\gamma_2 x\\
&+ \frac{(K-\Delta P)Y_2\eta_0(0) + Q_2 Y_3 q_0(0)}{2K\gamma_2}\operatorname{sh}\gamma_2 x,
\end{aligned}
\tag{2.26}
$$

$$M_0(x) = \frac{(K-\Delta P)M_0(0) - Q_3\Theta_0(0)}{2K}\text{ch}\gamma_1 x + \frac{(K-\Delta P)Y_3 q_0(0) - Q_3 Y_2 \eta_0(0)}{2K\gamma_1}\text{sh}\gamma_1 x + \frac{(K+\Delta P)M_0(0) + Q_3\Theta_0(0)}{2K}\text{ch}\gamma_2 x + \frac{(K+\Delta P)Y_3 q_0(0) + Q_3 Y_2 \eta_0(0)}{2K\gamma_2}\text{sh}\gamma_2 x, \tag{2.27}$$

where

$$\begin{aligned} B_2(0) &= Z_2\Theta_0(0) - Z_{23}M_0(0), \\ B_3(0) &= Z_3 M_0(0) - Z_{23}\Theta_0(0). \end{aligned} \tag{2.28}$$

Formulae (2.24)–(2.27) constitute the searched solution of the state equation (2.19) and represent the complex amplitudes of the fields of heat flux, moisture flux, temperature and moisture potential. Instantaneous values of these fields can be obtained from Eqs.(2.8).

The purpose of the present paper is to derive the formulae from which the state of strain and stress arising within the wall during heat and moisture flow can be determined. The stresses induced in the wall material by the process of moisture thermodiffusion $\sigma^{\Theta,c}(x,t)$ are the same in any direction and, according to the physical relations (2.4), have the form

$$\sigma^{\Theta,c}(x,t) = -\gamma_T\Theta(x,t) - \gamma_c c(x,t). \tag{2.29}$$

The temperature field $\Theta(x,t)$ can be found from the solution (2.26) and relationship in (2.8) as

$$\Theta(x,t) = \text{Re}\left\{\left[\frac{(K+\Delta P)\Theta_0(0) - Q_2 M_0(0)}{2K}\text{ch}\gamma_1 x + \frac{(K+\Delta P)Y_3\eta_0(0) - Q_2 Y_3 q_0(0)}{2K\gamma_1}\text{sh}\gamma_1 x + \frac{(K-\Delta P)\Theta_0(0) + Q_2 M_0(0)}{2K}\text{ch}\gamma_2 x + \frac{(K-\Delta P)Y_2\eta_0(0) + Q_2 Y_3 q_0(0)}{2K\gamma_2}\text{sh}\gamma_2 x\right] e^{j\omega t}\right\}. \tag{2.30}$$

The moisture concentration $c(x,t)$ within the wall can be determined as follows. From the balance equation of mass we have

$$\dot{c}(x,t) = -\text{div}\boldsymbol{\eta}(x,t) = -\frac{\partial}{\partial x}\eta(x,t), \tag{2.31}$$

and thus

$$c(x,t) = \int \frac{\partial}{\partial x}[-\eta(x,t)]dt, \tag{2.32}$$

By using the relationships (2.8), we obtain

$$c(x,t) = \mathrm{Re}\left[c_0(x)e^{j\omega t}\right] = \mathrm{Re}\left[\frac{j}{\omega}\frac{d\eta_0(x)}{dx}e^{j\omega t}\right].$$

On usig Eq.(2.25) and performing the differentiation, we arrive at the following form of the field of moisture concentration within the wall:

$$\begin{aligned} c(x,t) = \mathrm{Re}\Bigg\{\frac{j}{\omega}\Bigg[&\frac{\gamma_1((K-\Delta P)q_0(0)-Q_2\eta_0(0))}{2K}\mathrm{sh}\gamma_1 x \\ &+\frac{(K-\Delta P)B_3(0)-Q_2B_2(0)}{2K}\mathrm{ch}\gamma_1 x \\ &+\frac{\gamma_2((K+\Delta P)q_0(0)-Q_2\eta_0(0))}{2K}\mathrm{sh}\gamma_2 x \\ &+\frac{(K+\Delta P)Y_2\eta_0(0)-Q_2Y_3q_0(0)}{2K}\mathrm{ch}\gamma_2 x\Bigg]e^{j\omega t}\Bigg\}. \end{aligned} \tag{2.33}$$

Since the state of stress in a building material practically does not influence the process of heat and moisture transfer in it [7], by using the formulae for the temperature field $\Theta(x,t)$ and the moisture field $c(x,t)$, obtained in this paper, the stresses induced by the process of moisture thermodiffusion in the wall can be determined independently of the stress state which could result from various possible boundary conditions of elasticity. From Eqs. (2.29), (2.30), (2.33) we have

$$\begin{aligned} \sigma^{\Theta,c}(x,t) = -\gamma_t\mathrm{Re}\Bigg\{\Bigg[&\frac{(K+\Delta P)\Theta_0(0)-Q_2M_0(0)}{2K}\mathrm{ch}\gamma_1 x \\ &-\frac{(K+\Delta P)Y_2\eta_0(0)-Q_2Y_3q_0(0)}{2K\gamma_1}\mathrm{sh}\gamma_1 x \\ &+\frac{(K-\Delta P)\Theta_0(0)+Q_2M_0(0)}{2K}\mathrm{ch}\gamma_2 x \\ &+\frac{(K-\Delta P)Y_2\eta_0(0)+Q_2Y_3q_0(0)}{2K\gamma_2}\mathrm{sh}\gamma_2 x\Bigg]e^{j\omega t}\Bigg\} \\ -\gamma_c\mathrm{Re}\Bigg\{\frac{j}{\omega}\Bigg[&\frac{\gamma_1((K-\Delta P)q_0(0)-Q_2\eta_0(0))}{2K}\mathrm{sh}\gamma_1 x \\ &+\frac{(K-\Delta P)B_3(0)-Q_2B_2(0)}{2K}\mathrm{ch}\gamma_1 x \\ &+\frac{\gamma_2((K+\Delta P)q_0(0)-Q_2\eta_0(0))}{2K}\mathrm{sh}\gamma_2 x \\ &+\frac{(K+\Delta P)Y_2\eta_0(0)-Q_2Y_3q_0(0)}{2K}\mathrm{ch}\gamma_2 x\Bigg]e^{j\omega t}\Bigg\}. \end{aligned} \tag{2.34}$$

If we assume for simplicity that the wall material can be freely deformed only in the direction of x-axis, while the possibility of wall deformations in directions orthogonal to x-axis is strongly limited ($\varepsilon_{12} \cong \varepsilon_{13} \cong 0$) (e.g. the wall is fixed in the building structure), then

$$\begin{aligned} \sigma_{11}(x,t) &= (2\mu+\lambda)\varepsilon_{11}(x,t) + \sigma^{\Theta,c}(x,t), \\ \sigma_{12}(x,t) &= \sigma_{13}(x,t) = \sigma^{\Theta,c}(x,t). \end{aligned} \tag{2.35}$$

The strain field $\varepsilon_{11}(x,t)$ within the wall can be determined as follows: By using Eq.(2.8) we obtain

$$\varepsilon_{11}(x,t) = \int \dot{\varepsilon}_{11}(x,t)dt = \int \frac{\partial V(x,t)}{\partial x}\,dt = \mathrm{Re}\left[\frac{1}{j\omega}\,\frac{dV_0(x)}{dx}e^{j\omega t}\right]. \tag{2.36}$$

and then the use of the relationships (2.15) and (2.16) yields

$$\varepsilon_{11}(x,t) = \mathrm{Re}\left\{\frac{1}{j\omega}\left[Z_1\sigma_0(0) + Z_{12}\Theta_0(x) + Z_{13}M_0(x)\right]e^{j\omega t}\right\}, \tag{2.37}$$

$$\begin{aligned} \varepsilon_{11}(x,t) = \mathrm{Re}\Bigg\{\frac{1}{j\omega}\Bigg[Z_1\sigma_0(0) + Z_{12}\Bigg[&\frac{(K+\Delta P)\Theta_0(0) - Q_2M_0(0)}{2K}\mathrm{ch}\gamma_1 x \\ &- \frac{(K+\Delta P)Y_2\eta_0(0) - Q_2Y_3q_0(0)}{2K\gamma_1}\mathrm{sh}\gamma_1 x \\ &+ \frac{(K-\Delta P)\Theta_0(0) + Q_2M_0(0)}{2K}\mathrm{ch}\gamma_2 x \\ &+ \frac{(K-\Delta P)Y_2\eta_0(0) + Q_2Y_3q_0(0)}{2K\gamma_2}\mathrm{sh}\gamma_2 x\Bigg] \\ +Z_{13}\Bigg[&\frac{(K-\Delta P)M_0(0) - Q_3\Theta_0(0)}{2K}\mathrm{ch}\gamma_1 x \\ &+ \frac{(K-\Delta P)Y_3q_0(0) - Q_3Y_2\eta_0(0)}{2K\gamma_1}\mathrm{sh}\gamma_1 x \\ &+ \frac{(K+\Delta P)M_0(0) + Q_3\Theta_0(0)}{2K}\mathrm{ch}\gamma_2 x \\ &+ \frac{(K+\Delta P)Y_3q_0(0) + Q_3Y_3\eta_0(0)}{2K\gamma_2}\mathrm{sh}\gamma_2 x\Bigg]\Bigg]e^{j\omega t}\Bigg\}. \end{aligned} \tag{2.38}$$

The component of the stress tensor in x-direction takes finally the form

$$
\begin{aligned}
(2.39)\qquad \sigma_{11}(x,t) = (2\mu+\lambda)\mathrm{Re}\Bigg\{\frac{1}{j\omega}\Big[Z_1\sigma_0(0) \\
&+Z_{12}\Bigg[\frac{(K+\Delta P)\Theta_0(0)-Q_2M_0(0)}{2K}\mathrm{ch}\gamma_1 x \\
&+\frac{(K+\Delta P)Y_2\eta_0(0)-Q_2Y_3q_0(0)}{2K\gamma_1}\mathrm{sh}\gamma_1 x \\
&+\frac{(K-\Delta P)\Theta_0(0)+Q_2M_0(0)}{2K}\mathrm{ch}\gamma_2 x \\
&+\frac{(K-\Delta P)Y_2\eta_0(0)+Q_2Y_3q_0(0)}{2K\gamma_2}\mathrm{sh}\gamma_2 x\Bigg] \\
&+Z_{13}\Bigg[\frac{(K-\Delta P)M_0(0)-Q_3\Theta_0(0)}{2K}\mathrm{ch}\gamma_1 x \\
&+\frac{(K-\Delta P)Y_3q_0(0)-Q_3Y_2\eta_0(0)}{2K\gamma_1}\mathrm{sh}\gamma_1 x \\
&+\frac{(K+\Delta P)M_0(0)+Q_3\Theta_0(0)}{2K}\mathrm{ch}\gamma_2 x \\
&+\frac{(K+\Delta P)Y_3q_0(0)+Q_3Y_2\eta_0(0)}{2K\gamma_2}\mathrm{sh}\gamma_2 x\Bigg]\Big]\, e^{j\omega t}\Bigg\} \\
&-\gamma_T\mathrm{Re}\Bigg\{\Bigg[\frac{(K+\Delta P)\Theta_0(0)-Q_2M_0(0)}{2K}\mathrm{ch}\gamma_1 x \\
&+\frac{(K+\Delta P)Y_2\eta_0(0)-Q_2Y_3q_0(0)}{2K\gamma_1}\mathrm{sh}\gamma_1 x \\
&+\frac{(K-\Delta P)\Theta_0(0)+Q_2M_0(0)}{2K}\mathrm{ch}\gamma_2 x \\
&+\frac{(K-\Delta P)Y_2\eta_0(0)+Q_2Y_3q_0(0)}{2K\gamma_2}\mathrm{sh}\gamma_2 x\Bigg]\, e^{j\omega t}\Bigg\} \\
&-\gamma_c\mathrm{Re}\Bigg\{\frac{j}{\omega}\Bigg[\frac{\gamma_1((K-\Delta P)q_0(0)-Q_2\eta_0(0))}{2K}\mathrm{sh}\gamma_1 x \\
&+\frac{(K-\Delta P)B_3(0)-Q_2B_2(0)}{2K}\mathrm{ch}\gamma_1 x \\
&+\frac{\gamma_2((K+\Delta P)q_0(0)-Q_2\eta_0(0))}{2K}\mathrm{sh}\gamma_2 x \\
&+\frac{(K+\Delta P)Y_2\eta_0(0)-Q_3Y_3q_0(0)}{2K}\mathrm{ch}\gamma_2 x\Bigg]\, e^{j\omega t}\Bigg\}.
\end{aligned}
$$

The formulae (2.34), (2.39) and (2.38) represent the state of stress and strain within the building wall, induced by the unidirectional process of moisture thermodiffusion.

3. Final remarks

The results obtained in this paper constitute the last part of our model study of the process of moisture thermodiffusion in building walls, aimed at determining the state of strain and stress in the wall material. The results have the form of relatively simple analytic solutions. In the previous papers, the Podstrigač–Nowacki theory of thermodiffusion has been used to describe the process of heat and moisture flow through building walls [4,5], and a procedure has been proposed for measuring the material constants occurring in the model [6].

The use of the solutions obtained in this paper in engineering practice and in a qualitative analysis of the physical phenomena occurring in building walls will be possible after determining by a theoretical-experimental method the values of the basic material constants in the process of moisture thermodiffusion.

Appendix I

I.1. Electrical analogies

A linear, quasi-static process of moisture thermodiffusion in a one-layer building wall, described by Eqs. (2.3)–(2.7), corresponds to the system of three magnetically coupled electric transmission lines, presented in Fig.2. Line 1 corresponds to the process of elastic deformation of the material, line 2 – to the process of heat conduction, line 3 – to the process of moisture transfer.

In Fig. 2, u_1, u_2, u_3 denote the line voltages, i_1, i_2, i_3 – the currents, L_1, L_2, L_3 – the self-inductances per unit length, L_{12}, L_{13}, L_{23} – the mutual inductances per unit length, G_2, G_3 – the self-conductances per unit length.

The system of electric transmission lines in Fig. 2 is described by the following equations, analogous to Eqs. (2.3)–(2.7) of moisture thermodiffusion in a building wall:

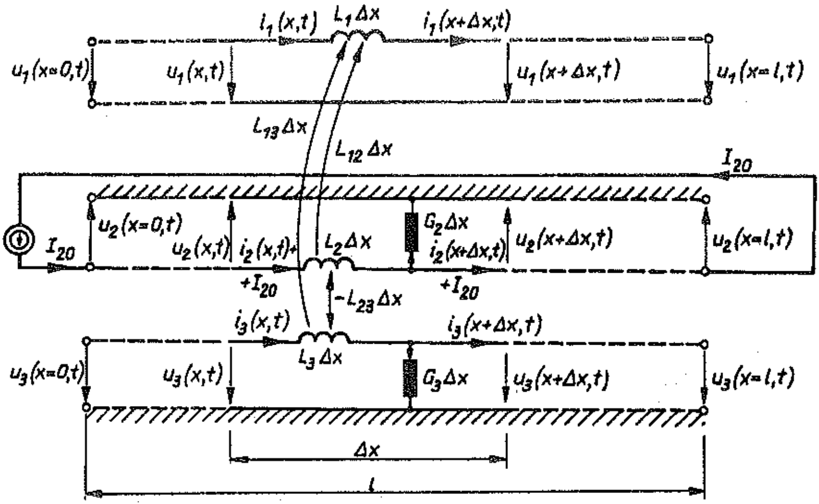

FIG. 2. System of three magnetically coupled electric transmission lines – the electrical analogue of a one-dimensional, quasi-static process of moisture thermodiffusion in a solid.

$$
\text{(I.1)} \qquad
\begin{aligned}
i_1 = J_1 A &= \frac{1}{L_1}\frac{\partial(-\Psi_1)}{\partial x} - \frac{L_3 L_{12} + L_{13} L_{23}}{L_1 L_3} i_2 - \frac{L_{13}}{L_1 L_3}\frac{\partial(-\Psi_3)}{\partial x}, \\
\frac{\partial(-\Psi_2)}{\partial x} &= \left(L_2 - \frac{L_{23}^2}{L_3}\right) i_2 - \frac{L_{23}}{L_3}\frac{\partial(-\Psi_3)}{\partial x}, \\
i_3 = J_3 A &= \frac{L_{23}}{L_3} i_2 + \frac{1}{L_3}\frac{\partial(-\Psi_3)}{\partial x},
\end{aligned}
$$

where J_k, $k = 1, 2, 3$ is the current density in line k, A is the cross-sectional area of each transmission line conductor, Ψ_k is the magnetic flux associated with the respective transmission line.

The following relationship exists between the line voltage and the associated magnetic flux:

$$
\text{(I.2)} \qquad u_k = \frac{\partial \Psi_k}{\partial t}, \qquad k = 1, 2, 3.
$$

Equations (I.1) are analogous to the physical relations (2.4) of thermodiffusion.

From the first Kirchhoff law [9] for the lines 2 and 3 we obtain the equations

$$
\text{(I.3)} \qquad u_2 = -\frac{1}{G_2}\frac{\partial i_2}{\partial x}, \qquad u_3 = -\frac{1}{G_3}\frac{\partial i_3}{\partial x},
$$

analogous to Eqs.(2.5).

On substituting i_3 from Eq.(I.1)$_3$ into Eq.(I.3)$_2$, we obtain the equation

$$u_3 = -\frac{1}{L_3G_3}\frac{\partial^2(-\Psi_3)}{\partial x^2} - \frac{L_{23}}{L_3G_3}\frac{\partial i_2}{\partial x}, \tag{I.4}$$

analogous to Eq. (2.7).

The set of thermodiffusion equations (2.3) corresponds to the equations for the electrical system,

$$\frac{1}{L_1}\frac{\partial^2(-\Psi_1)}{\partial x^2} = \frac{L_3L_{12}+L_{13}L_{23}}{L_1L_3}\frac{\partial i_2}{\partial x} + \frac{L_{13}}{L_1L_3} + \frac{\partial^2(-\Psi_2)}{\partial x^2},$$

$$\frac{I_{20}}{G}\frac{\partial^2 i_2}{\partial x^2} - \frac{I_{20}(L_2L_3 - L_{23}^2)}{L_3}\frac{\partial i_2}{\partial t} - \frac{I_{20}L_{23}}{L_3}\frac{\partial(-u_3)}{\partial t} = 0, \tag{I.5}$$

$$\frac{1}{L_3G_3}\frac{\partial^2}{\partial x^2}\frac{\partial(-\Psi_3)}{\partial x} - \frac{\partial}{\partial t}\frac{\partial(-\Psi_3)}{\partial x} + \frac{L_{23}}{L_3G_3}\frac{\partial^2 i_2}{\partial x^2} = 0.$$

Comparison of the equations leads to the following correspondences:

$$\begin{aligned}
&\sigma_{11} \longleftrightarrow J_1, \quad \varepsilon_{11} = \frac{\partial w}{\partial x} \longleftrightarrow \frac{\partial(-\Psi_1)}{\partial x}, \quad v = \frac{\partial w}{\partial t} \longleftrightarrow (-u_1),\\
&\Theta \longleftrightarrow J_2, \qquad S \longleftrightarrow \frac{\partial(-\Psi_2)}{\partial x}, \qquad q \longleftrightarrow (-u_2),\\
&M \longleftrightarrow J_3, \qquad c \longleftrightarrow \frac{\partial(-\Psi_3)}{\partial x}, \qquad \eta \longleftrightarrow (-u_3),
\end{aligned} \tag{I.6}$$

which constitute the system of electro-elasto-thermo-diffusive analogies [4].

I.2. Electric transmission lines under sinusoidal current excitation

Let in the system from Fig.2,

$$\begin{aligned}
u_k(x,t) &= \mathrm{Re}\left[U_k(x)e^{j\omega t}\right],\\
i_k(x,t) &= \mathrm{Re}\left[I_k(x)e^{j\omega t}\right], \qquad k = 1,2,3,
\end{aligned} \tag{I.7}$$

where $u_k(x,t)$, $i_k(x,t)$ are the instantaneous values and $U_k(x)$, $I_k(x)$ are the complex amplitudes of the voltages and currents, respectively, j is the imaginary unit and ω is the angular frequency.

The system of electric transmission lines from Fig.2 is described by the equations [9,10]:

$$\text{(I.8)}\qquad \frac{\partial(-u_1)}{\partial x} = L_1\frac{\partial i_1}{\partial t} + L_{12}\frac{\partial i_2}{\partial t} + L_{13}\frac{\partial i_3}{\partial t}, \qquad \frac{\partial i_1}{\partial x} = 0,$$

$$\text{(I.9)}\qquad \frac{\partial(-u_2)}{\partial x} = L_2\frac{\partial i_2}{\partial t} - L_{23}\frac{\partial i_3}{\partial t}, \qquad \frac{\partial i_2}{\partial x} = -G_2u_2,$$

$$\text{(I.10)}\qquad \frac{\partial(-u_3)}{\partial x} = -L_{23}\frac{\partial i_2}{\partial t} + L_3\frac{\partial i_3}{\partial t}, \qquad \frac{\partial i_3}{\partial x} = -G_3u_3.$$

Equations (I.8)–(I.10) correspond, after substituting Eq.(I.7), to the following system of equations for complex amplitudes:

$$\text{(I.11)}\qquad \frac{d(-U_1)}{dx} = Z_1^{(e)}I_1 + Z_{12}^{(e)}I_2 + Z_{13}^{(e)}I_3, \qquad \frac{dI_1}{dx} = 0,$$

$$\text{(I.12)}\qquad \frac{d}{dx}\begin{bmatrix} -U_2(x) \\ -U_3(x) \\ I_2(x) \\ I_3(x) \end{bmatrix} = \begin{bmatrix} 0 & 0 & Z_2^{(e)} & -Z_{23}^{(e)} \\ 0 & 0 & -Z_{23}^{(e)} & Z_3^{(e)} \\ Y_2^{(e)} & 0 & 0 & 0 \\ 0 & Y_3^{(e)} & 0 & 0 \end{bmatrix} \begin{bmatrix} -U_2(x) \\ -U_3(x) \\ I_2(x) \\ I_3(x) \end{bmatrix},$$

where

$$\text{(I.13)}\qquad \begin{aligned} Z_1^{(e)} &= j\omega L_1, & Z_{12}^{(e)} &= j\omega L_{12}, & Z_{13}^{(e)} &= j\omega L_{13}, \\ Z_2^{(e)} &= j\omega L_2, & Z_{23}^{(e)} &= j\omega L_{23}, & Y_2^{(e)} &= G_2, \\ Z_3^{(e)} &= j\omega L_3, & Y_3^{(e)} &= G_3, \end{aligned}$$

are the electric complex impedances.

By multiplying both sides of Eqs.(I.11), (I.12) by $e^{j\omega t}$ and taking their real parts, we obtain the equations (I.8)–(I.10) in which Eq.(I.7) has been taken into account.

Equation (I.12) represents the so-called homogeneous state equation; it is usually written in the compact form

$$\text{(I.14)}\qquad \frac{d\mathbf{S}(x)}{dx} = \mathbf{A}\mathbf{S}(x),$$

where $\mathbf{S}(x) = [-U_2(x), -U_3(x), I_2(x), I_3(x)]$ is the so-called state vector and $\mathbf{A}$ denotes the matrix of the system.

The solution of Eq.(I.14) is given by the state vector represented by the following transmission equation

$$\text{(I.15)}\qquad \mathbf{S}(x) = \left[e^{\mathbf{A}x}\right]\mathbf{S}(0),$$

where $\mathbf{S}(0)$ is the state vector on the input of the system, and $e^{\mathbf{A}x}$ is the so-called transmission matrix (or transition matrix).

Solution of the state equation (I.14) is thus reduced to determining the transmission matrix $e^{\mathbf{A}x}$. It is defined by the power series [9,11]

$$e^{\mathbf{A}x} := \sum_{k=0}^{\infty} \frac{(\mathbf{A}x)^k}{k!} . \tag{I.16}$$

This series is absolutely convergent for every finite value of x [12].

A convergent power series of a matrix of order m can be expressed in the form of a uniquely defined polynomial of order $m-1$ of that matrix [12] (in the proof of this theorem the Cayley–Hamilton theorem is used). For $m = 4$ we thus have

$$e^{\mathbf{A}x} = g_0\mathbf{1} + g_1\mathbf{A} + g_2\mathbf{A}^2 + g_3\mathbf{A}^3 , \tag{I.17}$$

where g_0, g_1, g_2, g_3 are the coefficients of the so-called generating polynomial $g(s)$ of a complex variable s

$$g(s) = g_0 + g_1 s + g_2 s^2 + g_3 s^3 . \tag{I.18}$$

The following equations are satisfied:

$$\begin{aligned} g(\lambda_k) - e^{\lambda_k x} &= 0, \qquad k = 1,2,3,4. \\ g(\mathbf{A}) - e^{\mathbf{A}x} &= 0, \end{aligned} \tag{I.19}$$

where λ_k are the eigenvalues of matrix $\mathbf{A}$

$$\lambda_{1,2} = \pm\gamma_1, \qquad \lambda_{3,4} = \pm\gamma_2 \tag{I.20}$$

and

$$\begin{aligned} \gamma_1 &= \sqrt{\bar{P}+k}, & \gamma_2 &= \sqrt{\bar{P}-k}, \\ \bar{P} &= \frac{P_2+P_3}{2}, & k &= \sqrt{(\Delta P)^2 + Q^2}, \\ \Delta P &= \frac{P_2-P_3}{2}, & Q^2 &= Q_2 Q_3 , \\ P_2 &= Z_2^{(e)} Y_2^{(e)}, & P_3 &= Z_3^{(e)} Y_3^{(e)}, \\ Q_2 &= Y_2^{(e)} Z_{23}^{(e)}, & Q_3 &= Y_3^{(e)} Z_{23}^{(e)}. \end{aligned} \tag{I.21}$$

For the system of equations (I.19) to have a solution, the principal determinant must vanish:

$$\text{(I.22)}\qquad \begin{vmatrix} 1 & \gamma_1 & \gamma_1^2 & \gamma_1^3 & e^{\gamma_1 x} \\ 1 & -\gamma_1 & \gamma_1^2 & -\gamma_1^3 & e^{-\gamma_1 x} \\ 1 & \gamma_2 & \gamma_2^2 & \gamma_2^3 & e^{\gamma_2 x} \\ 1 & -\gamma_2 & \gamma_2^2 & -\gamma_2^3 & e^{-\gamma_2 x} \\ 1 & \mathbf{A} & \mathbf{A}^2 & \mathbf{A}^3 & e^{\mathbf{A}x} \end{vmatrix} = 0$$

By expanding the determinant (I.22) with respect to the last column, we obtain

$$\text{(I.23)}\qquad e^{\mathbf{A}x} = \left[\frac{\mathrm{sh}\gamma_1 x}{\gamma_1(\gamma_1^2-\gamma_2^2)} - \frac{\mathrm{sh}\gamma_2 x}{\gamma_2(\gamma_1^2-\gamma_2^2)}\right]\mathbf{A}^3 + \frac{\mathrm{ch}\gamma_1 x - \mathrm{ch}\gamma_2 x}{\gamma_1^2-\gamma_2^2}\mathbf{A}^2 + \left[\frac{\gamma_1^2\mathrm{sh}\gamma_2 x}{\gamma_2(\gamma_1^2-\gamma_2^2)} - \frac{\gamma^2\mathrm{sh}\gamma_1 x}{\gamma_1(\gamma_1^2-\gamma_2^2)}\right]\mathbf{A} + \frac{\gamma_1^2\mathrm{ch}\gamma_2 x - \gamma_2^2\mathrm{ch}\gamma_1 x}{\gamma_1^2-\gamma_2^2}\mathbf{1}\,.$$

Knowing the transmission matrix $e^{\mathbf{A}x}$, from Eqs.(I.15) we can determine the components of the state vector $\mathbf{S}(x)$ which represents the solution of the state equation (I.14).

If the analogies (I.6) are used to express the components of the state vector $\mathbf{S}(x)$ in terms of the quantities which describe the process of moisture thermodiffusion in a building wall, i.e. $\mathbf{S}(x) = [q_0(x), \eta_0(x), \Theta_0(x), M_0(x)]$ (cf. Eq.(2.19)), and the matrix $\mathbf{A}$ in terms of the thermo-humiditive impedances (2.17) and (2.18), then we obtain the solution of the state equation (2.14) in the form given by Eqs.(2.24)–(2.27).

References

1. W.Nowacki, *Thermodiffusion in solids*, in: Thermoelasticity [in Polish], Ossolineum, Wroclaw 1977.
2. J.S.Podstrigač, *Differential equations for the thermodiffusion problem in deformable solids* [in Russian], DAN USSR, 2, 1961.
3. W.Nowacki and Z.S.Olesiak, *Thermodiffusion in solids* [in Polish], PWN, Warsaw 1991.
4. W.Dudziak, R. Uklejewski, *On the application of the electrical transmission line theory in thermodiffusion problems in solids*, Bull. Acad. Pol. Sci., Techn., **34**, 3-4, 1986.

5. W.Dudziak and R.Uklejewski, *Heat and moisture transfer in a two-layer building wall at slow harmonic changes of thermo-humiditive environment parameters, based on the theory of thermodiffusion in solids*, Eng. Trans., **38**, 3, 1990.

6. W. Dudziak and R.Uklejewski, *On the determination of material coefficients of the theory of thermodiffusion in deformable solids for the heat and moisture transfer processes in building walls*, Arch. Mech., **44**, 2, 1992.

7. W.N.Bogosłowski, *Thermal and humiditive processes in buildings* [in Polish, transl. from Russian], Arkady, Warsaw 1985.

8. D.Gawin and P.Klemm, *A mathematical model of coupled transport of heat and moisture in capillary-porous media* [in Polish], Wrocław University of Technology, 1991 in issue: Processes of heat and mass transfer in building walls [in Polish].

9. T.Cholewicki, *Electrical transmission lines and nonuniform ladder networks* [in Polish], PWN, Warsaw 1974.

10. T.Cholewicki, *Analysis of nonuniform electrical networks by the method of difference equations*, [in Polish], PWN, Warsaw 1982.

11. L.Pipes, *Direct computation of transmission matrices of electrical transmission lines*, J. Frankl. Inst., 281, 1966.

12. K.Ogata, *State space methods in control theory*, [in Polish], WNT, Warsaw 1974.

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