ENGINEERING TRANSACTIONS • *Engng. Trans.* • **59**, 1, 31–63, 2011
Polish Academy of Sciences • Institute of Fundamental Technological Research (IPPT PAN)
National Engineering School of Metz (ENIM)

ELASTIC ENERGY DECOMPOSITION AND LIMIT CRITERIA

J. Rychlewski

In memoriam Wacław Olszak

Translated from the original paper in Russian from Advances in Mechanics, **7***, 51–80 (1984)*
by Andrzej Blinowski, scientific editor Ryszard B. Pęcherski.
The notes from the editor are added in brackets [...]

1. A bit of history

Exactly 80 years ago[1)], in one little known engineering journal, appeared a paper by a young scientist – "*Właściwa praca odkształcenia jako miara wytężenia materiału*" [*Specific work of strain as a measure of material effort* M.T. Huber [1]]. The author's destiny was to become in future a founder of the Polish school of solid deformable bodies mechanics. The article contained one of the classic assertions of contemporary mathematical theory of plasticity – the limit condition for isotropic bodies, which we nowadays use to express as

$$\mathbf{s} \cdot \mathbf{s} \leq 2k^2, \tag{1.1}$$

where $\mathbf{s}$ is a deviatoric part of the stress tensor $\boldsymbol{\sigma}$, k denotes the limit value of pure shear stress (the notation is specified in Appendix 1).

We should notice that condition (1.1), gained popularity in scientific environment only ten years later being rediscovered by R. von Mises [2], and subsequently, additionally explained by H. Hencky [3]. This story has been discussed in 1924 at the I-st International Congress of Applied Mechanics in Delft, and found a reflection in, perhaps the first, methodical elucidation of the mathematical theory of plasticity given in 1927 by H. Mierzejewski [4].

After another decade it become clear that the yield condition (1.1) was clearly formulated by J.C. Maxwell in a private letter to prospect lord Kelvin [5]. After the explanation of the matter of the problem, the author of the letter additionally asserts: "*I think this notion will bear working out into a mathematical*

[1)] Written in 1984.

theory of plasticity when I have time ... ". It is a pity, that such a chance did not come about, we should remember however, that Maxwell's attention has turned to more important problems.

The mentioned above co-authors of the condition (1.1), (except for R. von Mises) based it on the charming in their simplicity, considerations on the energy. Let us recall the matter of the problem.

In a linearly elastic body under small strain $\boldsymbol{\varepsilon}$, the stored elastic energy is equal to the work performed by the stress $\boldsymbol{\sigma}$ on the strain $\boldsymbol{\varepsilon}$ and can be expressed as a quadratic form of stresses,

$$\Phi(\boldsymbol{\sigma}) \equiv \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{\sigma}). \tag{1.2}$$

If a body is isotropic, then this form should be invariant. Any **quadratic invariant** of the symmetric tensor, however, can be expressed as follows:

$$\Phi(\boldsymbol{\sigma}) = A\sigma^2 + B\mathbf{s} \cdot \mathbf{s}, \tag{1.3}$$

where

$$\boldsymbol{\sigma} = \sigma\mathbf{1} + \mathbf{s}, \qquad \sigma \equiv \frac{1}{3}\mathbf{1} \cdot \boldsymbol{\sigma}. \tag{1.4}$$

Indeed, σ, $\mathbf{s} \cdot \mathbf{s}$ and $\det \mathbf{s}$ comprise a complete (both functional and polynomial) system of invariants on the space of symmetric tensors $\mathscr{S}$, and (1.3) is the only possible quadratic expression which can be created using them. Giving a meaning to the constants, one obtains:

$$\Phi(\boldsymbol{\sigma}) = \frac{1}{2K}\sigma^2 + \frac{1}{4G}\mathbf{s} \cdot \mathbf{s}, \tag{1.5}$$

where K – compressibility modulus, G – shear modulus. It means that

$$\Phi(\sigma\mathbf{1} + \mathbf{s}) = \Phi(\sigma\mathbf{1}) + \Phi(\mathbf{s}), \tag{1.6}$$

i.e. the elastic energy is the sum of the **energy of the volume change** $\Phi(\sigma\mathbf{1})$ and the **energy of the shape change** $\Phi(\mathbf{s})$. Having performed this decomposition, J.C. Maxwell wrote: "*I have strong reasons for believing, that when* $\Phi(\mathbf{s})$ *reaches a certain limit..., then the element will begin to give way. ... Condition of not yielding*

$$\frac{1}{h}\Phi(\mathbf{s}) \leq 1, \tag{1.7}$$

where $h \equiv k^2/2G$. We took the liberty to change only the author's notation for the sake of similarity of the expressions (1.7) and (1.1).

Arising in that place and at that time (Cambridge, middle of the XIX century) of the ideas about the limit capacity of the elastic body for cumulating the energy of shape change seems to us by no means accidental. Some years earlier, one of the professors of that university, J. Green, laid the notion of the elastic energy in the foundations of the very definition of the elasticity [6]; J.G. Stokes pointed out quite clearly on the two kinds of the elasticity: the one trying to restore the volume, and another one tending to restore the shape [7].

M.T. Huber quotes the decomposition (1.6) referring to H. HELMHOLTZ [8] and writes: "*... można z wielkim prawdopodobieństwem uważać $\Phi(s)$ za miarę wytężenia materiału*" [1] ["*... one can in all probability consider $\Phi(s)$ as a measure of material effort*" – p. 185, [1] (English translation)]. He communicated his supposition to A. Föppl, who wrote in his well-known at that time textbook [9]: ["*Endlich ist noch darauf hinzugeweisen, daß mit den bisher genannten noch keineswegs alle Möglichkeiten erschöpft sind, die für die Bemessung der Bruchgefahr von vornherein offen stehen. Es ist auch sehr wohl möglich, daß wenigstens für gewisse Stoffe eine dieser anderen Möglichkeiten dem wirklichen Verhalten near kommt als die früheren. Namentlich liegt es nahe, in irgendeiner Weise die bezogene Formänderungsarbeit mit der Anstrengung des Stoffes in Verbindung zu bringen, da in ihr sowohl die auftretenden Spannungen als die von ihnen hervorgerufene Formänderung zur Geltung kommen*"].

"*In der Tat hat man dies wiederholt versucht, und eine besondere Form dieser Annahme, die von Herrn Professor Huber an der Technischen Hochschule in Lemberg aufgestellt wurde, erscheint durchaus beachtenswert, weshalb hier noch etwas näher darauf eingegangen werden soll. Die ursprüngliche Veröffentlichung von Huber ist uns nicht zugänglich, da sie in der polnischen Muttersprache ihres Verfassers geschrieben ist; wir können uns aber nach einer brieflichen Mitteilung mit einem ausführlichen Auszuge aus der Abhandlung richten, die wir Herrn Huber verdanken*" [p. 50 [9] (*Finally, one should mention that the before discussed measures of the risk of fracture by no means exhaust all possibilities that are at our disposal. It is also very possible that at least for certain materials one of the other possibilities approximates better the real behaviour than the earlier ones. It is namely conceivable to relate in some way the derived [specific] work of strain with material effort as well as to arrive at the assessment of induced stresses and the resulting deformation.*

In fact one has it repeatedly attempted and certain particular form of such an approach, which was exhibited by Professor Huber of the Technical University in Lemberg [Lwów Polytechnic] appears entirely worthy of our attention. Therefore, it should be brought closer here. The original publication of Huber is not accessible for us, for it was written in Polish mother tongue of the author. However, we can be guided by the comprehensive excerpt by letter, which we owe to Mr Huber.) – translation by sc. ed.].

R. von Mises proposed condition (1.1) starting from purely formal scheme of the quadratic approximation of the Tresca-Saint Venant yield condition. But as soon as the quadratic form is proposed, and a body is isotropic, then the limit function assumes form (1.3) and, discarding the influence of the hydrostatic pressure, one obtains (1.1), i.e. (1.7).

For completeness of the image we should mention, that E. Beltrami in 1885 [10] proposed boundedness of the complete elastic energy $\Phi(\sigma)$ as a limit criterion. This proposition was repeated later by B.P. Haigh [11]. Another old presumptions concerning the limit conditions one can find in the surveys by W. Burzyński [12] and M.M. Filonenko–Borodich [13].

Comment: as it is known, the condition (1.1) can be for isotropic body also differently interpreted. Particularly, V.V. Novozhilov [14] found, that the term $\mathbf{s} \cdot \mathbf{s}$ is proportional to the, averaged over all planes, square of the shear stress value.

2. Statement of the problem

According to the traditions of the old papers, we believe that it is justified, at a certain stage of the knowledge, to consider the form of **limit criterion**, understood as the bounding imposed on some **measure of the stress intensity** (called by M.T. Huber *miara wytężenia* [*material effort*]), without specifying the origins of the "element failure". The last can mean the transition to nonlinear elasticity, arising of permanent deformations (plastic, viscous, viscoplastic), disintegration on the micro- or macro-level, destruction of the composite structure configuration, attaining intolerable extent of deformation and so on.

Nowadays we know, more than the old time masters did, about the mechanisms of numerous effects. However in the same time, the following facts of the matter remain essential. Firstly: all the time increases the manifold of the engineering materials: of alloys, polymers, ceramics, concrete, composites, working mechanisms of their structures remaining as a rule inadequately recognized. Frequently thorough studies on them would be costly and time-consuming. Secondly: essentially, different structural effects can on the macro-level materialize quite similarly. For example, small strain crystal elasticity and elasticity of the solid polymers are based on quite different structural mechanisms, but their "macro-scale output" is identical. From there, the actuality and necessity of the phenomenological approach in the framework of the rational mechanics of materials comes out. Particularly, this remains true with respect to the limit criteria under consideration.

In this context, a phenomenological condition (1.7), for which the choice of the stress intensity measure is based on the fundamental notion of physics – a concept of energy, preserves in our opinion its heuristic attractiveness. *Basing* *on this start point, we set as our task, in the present paper, a comprehensive, from the formal viewpoint, clarification of the possibilities of direct generalization of the energy-based Maxwell–Huber condition (1.7) on linearly elastic anisotropic bodies of arbitrary symmetry.*

3. On the answer given by W. Olszak and W. Urbanowski

It seems to be evident, that the first attempt to enlighten the posed problem was taken by successors of M.T. Huber. My tutors, W. Olszak and W. Urbanowski proceeded on the way of extracting from the complete elastic energy of some its part, being an analogue of the energy of the shape change in isotropic body [15]. This study was continued lastly by J. Ostrowska in the lecture devoted to the memory of W. Olszak [16].

Let us consider an arbitrary elastic body described by the quadratic elastic potential

$$\Phi \equiv \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\sigma} \cdot \mathbf{S} \cdot \boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbf{C} \cdot \boldsymbol{\varepsilon}. \tag{3.1}$$

Here $\mathbf{S}$ is a **compliance tensor**, $\mathbf{C}$ – **stiffness tensor**,

$$\mathbf{C}^T = \mathbf{C}, \qquad \mathbf{S}^T = \mathbf{S}, \qquad \mathbf{C} \circ \mathbf{S} = \mathbf{S} \circ \mathbf{C} = \mathbf{I} \tag{3.2}$$

(see Appendix 1). From there, in virtue of Hooke's law, it follows:

$$\begin{aligned} \boldsymbol{\sigma} &= \partial_{\varepsilon}\Phi = \mathbf{C} \cdot \boldsymbol{\varepsilon}, \\ \boldsymbol{\varepsilon} &= \partial_{\sigma}\Phi = \mathbf{S} \cdot \boldsymbol{\sigma}. \end{aligned} \tag{3.3}$$

The idea suggested in [15] is attractive mainly because of its stimulating difficulties. Let us try to accomplish it with the aid of the standard decomposition (1.4). Unfortunately, this *does not lead* to the decomposition of the elastic energy. One has:

$$\Phi(\boldsymbol{\sigma}) = \frac{1}{2}(\sigma\mathbf{1} + \mathbf{s}) \cdot \mathbf{S} \cdot (\sigma\mathbf{1} + \mathbf{s}) = \frac{1}{2}\sigma^2\mathbf{1} \cdot \mathbf{S} \cdot \mathbf{1} + \sigma\mathbf{1} \cdot \mathbf{S} \cdot \mathbf{s} + \frac{1}{2}\mathbf{s} \cdot \mathbf{S} \cdot \mathbf{s}. \tag{3.4}$$

The first term

$$\frac{1}{2}\sigma^2\mathbf{1} \cdot \mathbf{S} \cdot \mathbf{1} = \Phi(\sigma\mathbf{1}) \geq 0 \tag{3.5}$$

describes the work of hydrostatic pressure $\sigma\mathbf{1}$ on the evoked by this stress state deformation $\sigma\mathbf{S} \cdot \mathbf{1}$, while the last one

$$\frac{1}{2}\mathbf{s} \cdot \mathbf{S} \cdot \mathbf{s} = \Phi(\mathbf{s}) \geq 0 \tag{3.6}$$

represents the work of the deviatoric part $\mathbf{s}$ on the deformation caused by it $\mathbf{S}\cdot\mathbf{s}$. However,

$$\Phi(\boldsymbol{\sigma}) \neq \Phi(\sigma\mathbf{1}) + \Phi(\mathbf{s}), \tag{3.7}$$

since there exists an additional term, which depends both on σ and $\mathbf{s}$

$$\sigma\mathbf{1}\cdot\mathbf{S}\cdot\mathbf{s}, \tag{3.8}$$

representing the sum of the work of $\sigma\mathbf{1}$ on the deformation $\mathbf{S}\cdot\mathbf{s}$ and, equal to it, work of $\mathbf{s}$ on the deformation $\sigma\mathbf{S}\cdot\mathbf{1}$. Let us notice that $\mathbf{1}\cdot\mathbf{S}\cdot\mathbf{s}$ describes a change of volume generated by the deviatoric part of load $\mathbf{s}$; it can be positive or negative depending on the sign of $\mathbf{s}$. An example of the case of pure shear is shown in Fig. 1. Difference $\Phi(\boldsymbol{\sigma}) - \Phi(\mathbf{s})$ *does not represent* energy of any stress state and it can assume negative values. Thus the use of $\Phi(\mathbf{s})$ as a measure of stress intensity appears out in general to be unsatisfactory.

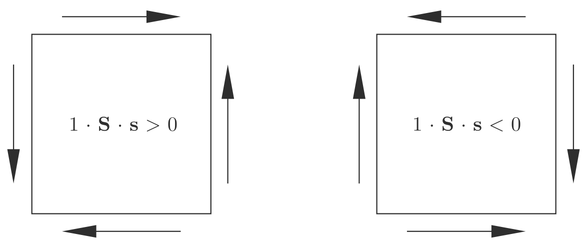

FIG. 1. Volume change $\mathbf{1}\cdot\mathbf{S}\cdot\mathbf{s}$ under the deviatoric load $\mathbf{s}$ depends on the orientation of $\mathbf{s}$ with respect to the elastic body.

Let us try to proceed on another way. We shall decompose the strain tensor

$$\boldsymbol{\varepsilon} = \varepsilon\mathbf{1} + \mathbf{e}, \qquad \varepsilon \equiv \frac{1}{3}\mathbf{1}\cdot\boldsymbol{\varepsilon} \tag{3.9}$$

and represent the elastic energy as follows:

$$\Phi(\boldsymbol{\sigma}) = \Phi_v(\boldsymbol{\sigma}) + \Phi_f(\boldsymbol{\sigma}), \tag{3.10}$$

where

$$\Phi_v(\boldsymbol{\sigma}) \equiv \frac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{e} = \frac{1}{2}\mathbf{s}\cdot\mathbf{e} = \frac{1}{2}\mathbf{s}\cdot\mathbf{S}\cdot\mathbf{s} + \frac{1}{2}\sigma\mathbf{1}\cdot\mathbf{S}\cdot\mathbf{s}, \tag{3.11}$$

$$\Phi_f(\boldsymbol{\sigma}) \equiv \frac{1}{2}\boldsymbol{\sigma}\cdot(\varepsilon\mathbf{1}) = \frac{3}{2}\sigma\varepsilon = \frac{1}{2}\sigma^2\mathbf{1}\cdot\mathbf{S}\cdot\mathbf{1} + \frac{1}{2}\sigma\mathbf{1}\cdot\mathbf{S}\cdot\mathbf{s}. \tag{3.12}$$

Term $\Phi_f(\boldsymbol{\sigma})$ is equal to *the work of the stress* $\boldsymbol{\sigma}$ *on the shape change*, while $\Phi_v(\boldsymbol{\sigma})$ represents *the work of the stress* $\boldsymbol{\sigma}$ *on the volume change*. In general, one can choose load $\boldsymbol{\sigma}$ in such a way, that one of the pair of terms $\Phi_f(\boldsymbol{\sigma})$ and $\Phi_v(\boldsymbol{\sigma})$ becomes negative, while total energy $\Phi(\boldsymbol{\sigma})$ remains positive. Neither $\Phi_f(\boldsymbol{\sigma})$ nor $\Phi_v(\boldsymbol{\sigma})$ represents an elastic energy of any state of stress. Thus $\Phi_f(\boldsymbol{\sigma})$ can not be used as a measure of stress intensity [material effort].

Let us notice that these difficulties disappear with vanishing of the term (3.8), i.e. for the elastic bodies fulfilling the condition

$$\mathbf{1} \cdot \mathbf{S} \cdot \mathbf{s} = 0, \tag{3.13}$$

for all deviatoric tensors $\mathbf{s}$.

This condition was, perhaps for the first time, proposed by the student of M.T. Huber – W. Burzyński [12]. In his opinion, it could turn up to obey in general for all elastic bodies and to replace the famous A. Cauchy condition [17], which evoked vigorous disputes in XIX century [18]. This is not true, of course. In [19] we have called the bodies obeying (3.13) *volumetrically-isotropic*. Condition (3.13) means that the hydrostatic load produces only volumetric deformation.

$$\begin{aligned} \mathbf{S} \cdot \mathbf{1} &= \frac{1}{3K}\mathbf{1}, \\ \frac{1}{K} &\equiv \mathbf{1} \cdot \mathbf{S} \cdot \mathbf{1}, \end{aligned} \tag{3.14}$$

i.e. the unit tensor $\mathbf{1}$ is a **proper elastic state**. Isotropic bodies are volumetrically-isotropic, because in this case

$$\mathbf{S} = \frac{1}{3K}\mathbf{I}_{\mathscr{P}} + \frac{1}{2G}\mathbf{I}_{\mathscr{D}}, \tag{3.15}$$

where

$$\mathbf{I}_{\mathscr{P}} \equiv \frac{1}{3}\mathbf{1} \otimes \mathbf{1}, \qquad \mathbf{I}_{\mathscr{D}} \equiv \mathbf{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1} \tag{3.16}$$

and (3.13) holds since $\mathbf{I}_{\mathscr{P}} \cdot \mathbf{1} = \mathbf{1}$, $\mathbf{I}_{\mathscr{D}} \cdot \mathbf{1} = \mathbf{0}$ [19][2)].

Let us come back however to the general case, when (3.13) does not hold. The authors of [15] passed over one idea which flashed across the thesis [12] (see p. 30), written under direct supervision of M.T. Huber. W shall attribute, for the beginning, a necessary clearness to this idea. Let us consider, beginning from this point, the stress $\boldsymbol{\sigma}$ as being related to some standard one and hence, being

[2)] Obviously, (3.13) is fulfilled for any incompressible material, since in this case $\mathbf{1} \cdot \mathbf{S} \cdot \boldsymbol{\sigma} = 0$ for any stress tensor $\boldsymbol{\sigma}$. Such a case is (from the formal viewpoint) not covered by equalities (3.14) (Translator's remark).

dimensionless. This enables us to consider $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ as elements of the space of symmetric tensors $\mathscr{S}$.

Definition. Two stress states $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ we shall call **energy-separated** for a given elastic body, if they decompose its elastic energy, i.e. if

$$\Phi(\boldsymbol{\alpha}+\boldsymbol{\beta}) = \Phi(\boldsymbol{\alpha}) + \Phi(\boldsymbol{\beta}). \tag{3.17}$$

Two subspaces $\mathscr{A}$, $\mathscr{B}$ in $\mathscr{S}$ we shall call energy-separated if all pairs $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}$ are energy-separated. The first example of the energy-separation is already known to us: for every isotropic linearly-elastic body, a one-dimensional space of spherical tensors $\mathscr{P}$ and five-dimensional space of deviators $\mathscr{D}$

$$\mathscr{S} = \mathscr{P} \oplus \mathscr{D}, \tag{3.18}$$

are energy-separated according to (1.6).

We have not assumed in our definition that a body is linearly-elastic, we should mention however that utility of the introduced notion in general case is rather doubtful. In the case of linearity, though it works excellently, as we shall make evident. Here

$$(\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\mathbf{S}\cdot(\boldsymbol{\alpha}+\boldsymbol{\beta}) = \boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\alpha} + \boldsymbol{\beta}\cdot\mathbf{S}\cdot\boldsymbol{\beta} + 2\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\beta} \tag{3.19}$$

and the energy-separation condition takes the form

$$\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\beta} = \boldsymbol{\beta}\cdot\mathbf{S}\cdot\boldsymbol{\alpha} = 0. \tag{3.20}$$

Thus, for the case of linear elasticity, *energy-separation of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *means that the stress* $\boldsymbol{\alpha}$ *does not perform work on the strain caused by the stress* $\boldsymbol{\beta}$, and equally: $\boldsymbol{\beta}$ does not work on $\mathbf{S}\cdot\boldsymbol{\alpha}$.

Now, everything is ready for a description of the following simple case. Assume that, for the class of elastic bodies under consideration, there exists such a tensor $\boldsymbol{\alpha}$, that the stress $c\boldsymbol{\alpha}$ of any intensity c does not cause a failure of the element (in the particular sense under consideration). We shall call the states $c\boldsymbol{\alpha}$ **the safe** ones. They constitute a one-dimensional space

$$\mathscr{E} \equiv \{c\boldsymbol{\alpha} \mid c:\ \text{arbitrary number}\}. \tag{3.21}$$

Let us introduce its orthogonal complement

$$\mathscr{E}^{\perp} \equiv \{\boldsymbol{\beta} \mid \boldsymbol{\beta}\cdot\boldsymbol{\alpha} = 0\}. \tag{3.22}$$

We choose now *all states energy-separated from* $\boldsymbol{\alpha}$. They constitute a five-dimensional space

$$\mathscr{E}^{\dot{\perp}} \equiv \{\boldsymbol{\omega} \mid \boldsymbol{\omega}\cdot\mathbf{S}\cdot\boldsymbol{\alpha} = 0\}. \tag{3.23}$$

Evidently

$$\mathscr{E}^{\dot\perp} = \mathbf{C}\cdot\mathscr{E}^{\perp} = \{\mathbf{C}\cdot\boldsymbol{\beta} \mid \boldsymbol{\beta}\cdot\boldsymbol{\alpha} = 0\}. \tag{3.24}$$

The state $\boldsymbol{\alpha}$ is a normal of $\mathscr{E}^{\perp}$ while the state $\mathbf{S}\cdot\boldsymbol{\alpha}$ is a normal of $\mathscr{E}^{\dot\perp}$, Fig. 2.

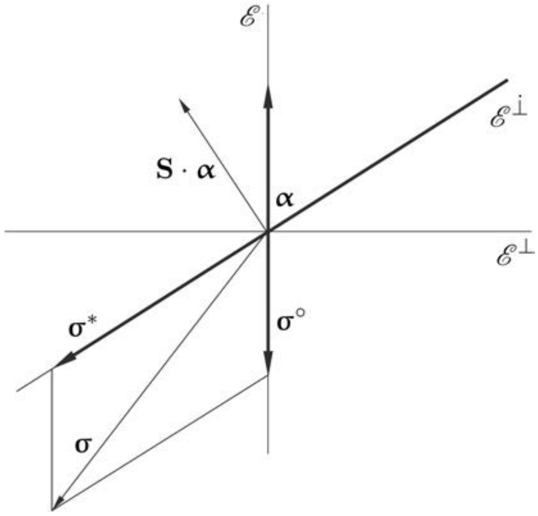

FIG. 2. $\mathscr{E}$ – space of safe states, $\mathscr{E}^{\dot\perp}$– space of the states energy-separated from the safe ones, $\mathscr{E}^{\perp}$ – space of the states orthogonal to the safe states.

Let us introduce a decomposition into a direct sum

$$\mathscr{S} = \mathscr{E} \oplus \mathscr{E}^{\dot\perp} \tag{3.25}$$

i.e. we shall represent every stress $\boldsymbol{\sigma}$ as a sum of energy-separated parts, the first of them being a safe state

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^\circ + \boldsymbol{\sigma}^*, \qquad \boldsymbol{\sigma}^\circ \in \mathscr{E}, \qquad \boldsymbol{\sigma}^* \in \mathscr{E}^{\dot\perp}. \tag{3.26}$$

The component $\boldsymbol{\sigma}^\circ$ will be called the **safe part** of the stress $\boldsymbol{\sigma}$, and the component $\boldsymbol{\sigma}^*$ – **the hazardous** one. Making use of the condition of energy-separation $\boldsymbol{\sigma}^\circ\cdot\mathbf{S}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}^\circ) = 0$, one obtains

$$\boldsymbol{\sigma}^\circ = \sigma^\circ\boldsymbol{\alpha}, \qquad \sigma^\circ \equiv \frac{\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\sigma}}{\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\alpha}}. \tag{3.27}$$

Operation $\boldsymbol{\sigma} \to \boldsymbol{\sigma}^\circ$ is a projection, parallel with respect to the space $\mathscr{E}^{\dot\perp}$, on the straight line $\mathscr{E}$. It is performed with the aid of projector $\mathbf{E}^\circ \in \mathscr{T}$, which is uniquely defined as follows:

$$\mathbf{E}^\circ\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}^\circ \qquad \text{for all} \qquad \boldsymbol{\sigma} \in \mathscr{S}. \tag{3.28}$$

It is not difficult to show that the projector $\mathbf{E}^\circ$ is equal to

$$\mathbf{E}^\circ = \frac{1}{\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\alpha}}\boldsymbol{\alpha}\otimes\mathbf{S}\cdot\boldsymbol{\alpha}. \tag{3.29}$$

The elastic energy can be decomposed in the following way:

$$\varPhi(\boldsymbol{\sigma}) = \varPhi(\boldsymbol{\sigma}^\circ) + \varPhi(\boldsymbol{\sigma}^*), \tag{3.30}$$

the following equality being true:

$$\varPhi(\boldsymbol{\sigma}^\circ) = \frac{1}{2}(\sigma^\circ)^2\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\alpha} = \frac{(\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\alpha})^2}{2\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\alpha}}. \tag{3.31}$$

Elastic energy of the hazardous part of stress is equal to

$$\varPhi(\boldsymbol{\sigma}^*) = \frac{1}{2}\boldsymbol{\sigma}^*\cdot\mathbf{S}\cdot\boldsymbol{\sigma}^* = \frac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{S}^*\cdot\boldsymbol{\sigma}, \tag{3.32}$$

where

$$\mathbf{S}^* \equiv (\mathbf{E}^*)^T\circ\mathbf{S}\circ\mathbf{E}^* = \mathbf{S} - (\mathbf{E}^\circ)^T\circ\mathbf{S}\circ\mathbf{E}^\circ = \mathbf{S} - \frac{1}{\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\alpha}}\mathbf{S}\cdot\boldsymbol{\alpha}\otimes\mathbf{S}\cdot\boldsymbol{\alpha}. \tag{3.33}$$

Here $\mathbf{E}^*$ is a projector onto $\mathscr{E}^\perp$ parallel to $\mathscr{E}$, i.e.

$$\mathbf{E}^*\cdot\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}^* \qquad \text{for every} \qquad \boldsymbol{\sigma}\in\mathscr{S}. \tag{3.34}$$

Decomposition of the space (3.25) is associated with the corresponding decomposition of unit operator

$$\mathbf{I} = \mathbf{E}^\circ + \mathbf{E}^*, \qquad \mathbf{E}^\circ\circ\mathbf{E}^* = \mathbf{E}^*\circ\mathbf{E}^\circ = \mathbf{0}. \tag{3.35}$$

For the bodies under consideration, the following **energy limit criterion** can be proposed

$$\frac{1}{h}\varPhi(\boldsymbol{\sigma}^*) \le 1, \tag{3.36}$$

where h is the limit value of elastic energy under loading with the stress $\boldsymbol{\sigma}^*\in\mathscr{E}^\perp$. For the isotropic body with the spherical safe state, one has

$$\begin{gathered}\boldsymbol{\alpha} = \mathbf{1}, \qquad \mathbf{S} = \frac{1}{3K}\mathbf{I}_{\mathscr{P}} + \frac{1}{2G}\mathbf{I}_{\mathscr{D}},\\ \mathbf{E}^\circ = \mathbf{I}_{\mathscr{P}}, \qquad \mathbf{E}^* = \mathbf{I}_{\mathscr{D}}.\end{gathered} \tag{3.37}$$

Hence $\boldsymbol{\sigma}^* = \mathbf{s}$ and

$$\varPhi(\boldsymbol{\sigma}^*) = \varPhi(\mathbf{s}) \tag{3.38}$$

becomes the energy of shape change, and limit condition (3.36) turns out to be the Maxwell-Huber condition (1.7). Proposals [15] rely on the following two examples which excellently exhibit a difference between the isotropic and anisotropic bodies.

Example. *Let the hydrostatic stress states be safe*, i.e.

$$\boldsymbol{\alpha} = \mathbf{1}, \tag{3.39}$$

(see Fig. 1). Here

$$\boldsymbol{\sigma}^{\circ} = \frac{\mathbf{1}\cdot\mathbf{S}\cdot\boldsymbol{\sigma}}{\mathbf{1}\cdot\mathbf{S}\cdot\mathbf{1}}\mathbf{1} = (9K\varepsilon)\mathbf{1}, \tag{3.40}$$

$$\boldsymbol{\sigma}^{*} = \boldsymbol{\sigma} - (9K\varepsilon)\mathbf{1}. \tag{3.41}$$

The space $\mathscr{E}^{\dot\perp}$ is composed of the preserving volume stress states $\mathbf{1}\cdot\mathbf{S}\cdot\boldsymbol{\sigma}^{*} = 0$.

Limit condition (3.36) takes the form

$$\boldsymbol{\sigma}\cdot\mathbf{S}\cdot\boldsymbol{\sigma} - \frac{(\mathbf{1}\cdot\mathbf{S}\cdot\boldsymbol{\sigma})^2}{\mathbf{1}\cdot\mathbf{S}\cdot\mathbf{1}} \le 2h. \tag{3.42}$$

This corresponds exactly to the first of the two possibilities proposed in [15].

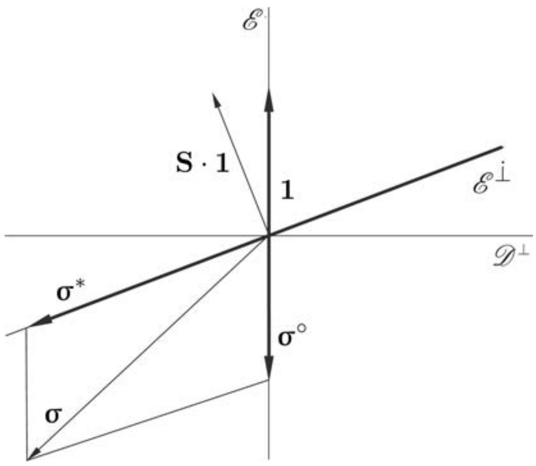

FIG. 3. $\mathscr{E}$ – space of spherical tensors, $\mathscr{E}^{\dot\perp}$ – space of the preserving volume stresses, $\mathscr{D}^{\perp}$ – space of deviators.

Example. *Let every stress, which causes volume changes only, be safe.* Then

$$\boldsymbol{\alpha} = \mathbf{C}\cdot\mathbf{1} \tag{3.43}$$

(see Fig. 4). Here

$$\boldsymbol{\sigma}^{\circ} = \frac{3\sigma}{\mathbf{1}\cdot\mathbf{C}\cdot\mathbf{1}}\mathbf{C}\cdot\mathbf{1}, \qquad \sigma \equiv \frac{1}{3}\mathbf{1}\cdot\boldsymbol{\sigma}, \tag{3.44}$$

$$\boldsymbol{\sigma}^{*} = \boldsymbol{\sigma} - \frac{3\sigma}{\mathbf{1}\cdot\mathbf{C}\cdot\mathbf{1}}\mathbf{C}\cdot\mathbf{1}. \tag{3.45}$$

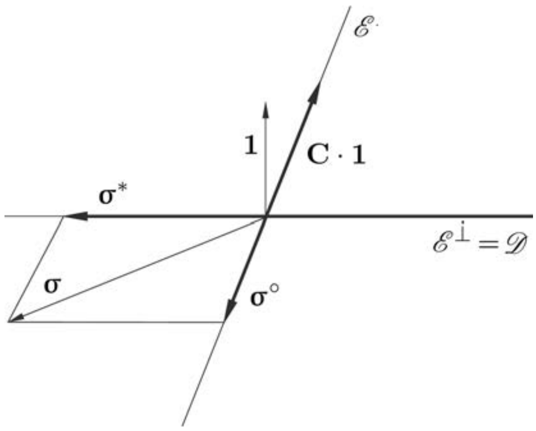

FIG. 4. $\mathscr{E}$ – space of stresses causing volume changes only, $\mathscr{D}$ – space of deviators.

Let us notice that $\boldsymbol{\sigma}^{*}$ is a deviator, $\mathbf{1}\cdot\boldsymbol{\sigma}^{*} = 0$ i.e. $\mathscr{E}^{\perp} = \mathscr{D}$. Deviator $\boldsymbol{\sigma}^{*}$ was introduced and interestingly implemented by V.A. LOMAKIN [20]. Limit condition in this case takes the form

$$\boldsymbol{\sigma}\cdot\mathbf{S}\cdot\boldsymbol{\sigma} - \frac{9\sigma^2}{\mathbf{1}\cdot\mathbf{C}\cdot\mathbf{1}} \le 2h. \tag{3.46}$$

This is the second possibility pointed out in [15].

For every *volumetrically-isotropic* body, particularly for an isotropic one, we have

$$\mathbf{S}\cdot\mathbf{1} = \frac{1}{3K}\mathbf{1}, \qquad \mathbf{C}\cdot\mathbf{1} = (3K)\mathbf{1} \tag{3.47}$$

and both conditions (3.42) and (3.46) coincide.

Unfortunately, criterion (3.36) is of a very particular nature. Limit properties are described here, with the exception of one constant h fixing the scale of stress, with the elastic tensor $\mathbf{S}^{*}$ alone. Such close bonds between the limit and the elastic properties seem to be very particular and can not take place in a general case of anisotropic body.

We should notice also that it is not difficult to generalize the obtained relations on the case when the space of the safe states is not one-dimensional.

4. Principal decomposition of the elastic energy

We need now a short, but crucial for the further considerations, excursion to some section of the algebra of Euclidean tensors, which is not known well enough and not sufficiently exploited in mechanics.

The set $\mathscr{S}$ can be considered as a linear 6-dimensional space with the scalar product

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) \to \boldsymbol{\alpha} \cdot \boldsymbol{\beta}. \tag{4.1}$$

Let us take two arbitrary bases in $\mathscr{S}$, i.e. two linearly independent sets of symmetric second-rank tensors

$$\begin{aligned} &\boldsymbol{\nu}_G, \qquad G = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI}, \\ &\boldsymbol{\mu}_l, \qquad l = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI}. \end{aligned} \tag{4.2}$$

According to the very definition of the tensor product of the linear spaces, a system of 36 *fourth* rank tensors

$$\boldsymbol{\nu}_G \otimes \boldsymbol{\mu}_l \qquad G, l = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI} \tag{4.3}$$

constitute a basis in $\mathscr{T} \equiv \mathscr{S} \overline{\otimes} \mathscr{S}$. Hence any tensor $\mathbf{L} \in \mathscr{T}$ can be uniquely denoted as

$$\mathbf{L} = \sum_{G,l=\mathrm{I}}^{\mathrm{VI}} L_{Gl} \boldsymbol{\nu}_G \otimes \boldsymbol{\mu}_l. \tag{4.4}$$

Moreover, it is convenient to regard any tensor $\mathbf{L} \in \mathscr{T}$ as *a linear operator* from $\mathscr{S}$ into $\mathscr{S}$, acting according to the rule

$$\boldsymbol{\alpha} \to \mathbf{L} \cdot \boldsymbol{\alpha}, \tag{4.5}$$

where $(\boldsymbol{\omega} \otimes \boldsymbol{\tau}) \cdot \boldsymbol{\alpha} \equiv (\boldsymbol{\tau} \cdot \boldsymbol{\alpha}) \boldsymbol{\omega}$.

Let us introduce, for the basis $\boldsymbol{\mu}_l$, its standard reciprocal basis $\boldsymbol{\mu}^l$. It is defined as the unique solution of the system of equations

$$\boldsymbol{\mu}_l \cdot \boldsymbol{\mu}^k = \delta_l^k \equiv \begin{cases} 1 & l = k, \\ 0 & l \neq k. \end{cases} \tag{4.6}$$

Now $\mathbf{L} \cdot \boldsymbol{\mu}^l = L_{\mathrm{I}\,l} \boldsymbol{\nu}_{\mathrm{I}} + L_{\mathrm{II}\,l} \boldsymbol{\nu}_{\mathrm{II}} + \ldots$ and, hence, Eq. (4.4) can be expressed as the *fundamental identity*: for any $\mathbf{L} \in \mathscr{T}$ and any basis $\boldsymbol{\mu}_{\mathrm{K}}$ in $\mathscr{S}$,

$$\mathbf{L} = \mathbf{L} \cdot \boldsymbol{\mu}^{\mathrm{I}} \otimes \boldsymbol{\mu}_{\mathrm{I}} + \ldots + \mathbf{L} \cdot \boldsymbol{\mu}^{\mathrm{VI}} \otimes \boldsymbol{\mu}_{\mathrm{VI}} = \mathbf{L} \cdot \boldsymbol{\mu}_{\mathrm{I}} \otimes \boldsymbol{\mu}^{\mathrm{I}} + \ldots + \mathbf{L} \cdot \boldsymbol{\mu}_{\mathrm{VI}} \otimes \boldsymbol{\mu}^{\mathrm{VI}}. \tag{4.7}$$

Let us come back to the idea of energy decomposition, which by no means can be reduced to the examples quoted in Sec. 3.

Let us analyze a symmetric bilinear form $\boldsymbol{\alpha} \cdot \mathbf{S} \cdot \boldsymbol{\beta}$. It is a polar form of the positive defined quadratic form $\boldsymbol{\alpha} \cdot \mathbf{S} \cdot \boldsymbol{\alpha}$ on $\mathscr{S}$. Therefore an operation

$$(\boldsymbol{\alpha} \cdot \boldsymbol{\beta}) \rightarrow \boldsymbol{\alpha} \times \boldsymbol{\beta} \equiv \boldsymbol{\alpha} \cdot \mathbf{S} \cdot \boldsymbol{\beta} \tag{4.8}$$

defines another correct scalar product of symmetric tensors of rank two. We shall call it **energy-scalar product**, contrary to the **standard scalar product** $\boldsymbol{\alpha} \cdot \boldsymbol{\beta}$. Energy product $\boldsymbol{\alpha} \times \boldsymbol{\beta}$ is tailored to the particular elastic body under consideration, it is defined by its compliance tensor $\mathbf{S}$. The condition of the energy-separation $\boldsymbol{\alpha} \cdot \mathbf{S} \cdot \boldsymbol{\beta} = 0$ achieves the geometric meaning of orthogonality in energy sense

$$\boldsymbol{\alpha} \dot{\perp} \boldsymbol{\beta}, \quad \text{i.e.} \quad \boldsymbol{\alpha} \times \boldsymbol{\beta} = 0.$$

Elastic energy of the stress is equal to one half of the square of the energy-norm of $\boldsymbol{\sigma}$:

$$\Phi(\boldsymbol{\sigma}) = \frac{1}{2} \boldsymbol{\sigma} \times \boldsymbol{\sigma}. \tag{4.9}$$

Definition. Every decomposition

$$\mathscr{S} = \mathscr{E}_1 \oplus, \ldots, \oplus \mathscr{E}_\varkappa, \qquad \varkappa \leq 6, \tag{4.10}$$

for which any two components of the direct sum are energy-orthogonal (separated)

$$\mathscr{E}_\alpha \dot{\perp} \mathscr{E}_\beta \quad \text{for} \quad \alpha \neq \beta, \tag{4.11}$$

we shall call an **energy-orthogonal decomposition** of the stress space for a given elastic body.

Distributing any stress tensor over subspaces (4.10)

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \ldots + \boldsymbol{\sigma}_\varkappa, \qquad \boldsymbol{\sigma}_\alpha \in \mathscr{E}_\alpha, \tag{4.12}$$

we have

$$\boldsymbol{\sigma}_\alpha \times \boldsymbol{\sigma}_\beta = 0 \quad \text{for} \quad \alpha \neq \beta \tag{4.13}$$

and, hence, as it should be,

$$\Phi(\boldsymbol{\sigma}_1 + \ldots + \boldsymbol{\sigma}_\varkappa) = \Phi(\boldsymbol{\sigma}_1) + \ldots + \Phi(\boldsymbol{\sigma}_\varkappa). \tag{4.14}$$

Of course, there exist many energy-orthogonal decompositions at will.

Let us introduce a projector $\mathbf{E}_\alpha$ on the subspace $\mathscr{E}_\alpha$ parallel to $\mathscr{E}^{\dot\perp} = \mathscr{E}_1 \oplus \ldots \oplus \mathscr{E}_{\alpha-1} \oplus \mathscr{E}_{\alpha+1} \oplus \ldots \oplus \mathscr{E}_\varkappa$

$$\mathbf{E}_\alpha \cdot \boldsymbol{\omega} = \begin{cases} \boldsymbol{\omega} & \text{if} \quad \boldsymbol{\omega} = \mathscr{E}_\alpha, \\ \mathbf{0} & \text{if} \quad \boldsymbol{\omega} \dot\perp \mathscr{E}_\alpha. \end{cases} \tag{4.15}$$

Decomposition of the unity operator

$$\mathbf{I} = \mathbf{E}_1 + \ldots + \mathbf{E}_\varkappa, \tag{4.16}$$

corresponds to the decomposition (4.10), here

$$\mathbf{E}_\alpha \circ \mathbf{E}_\alpha = \mathbf{E}_\alpha, \qquad \mathbf{E}_\alpha \circ \mathbf{E}_\beta = \mathbf{0} \quad \text{for} \quad \alpha \neq \beta. \tag{4.17}$$

It is not difficult to express projectors $\mathbf{E}_\alpha$ explicitly. Let us take any energy-orthonormal basis

$$\begin{gathered} \mathbf{\partial}_{\mathrm{I}}, \ldots, \mathbf{\partial}_{\mathrm{VI}}, \\ \mathbf{\partial}_K \times \mathbf{\partial}_L = \delta_{KL} \equiv \begin{cases} 1 & K = L, \\ 0 & K \neq L, \end{cases} \end{gathered} \tag{4.18}$$

chosen in such a way, that the first s_1 elements. $\mathbf{\partial}_1, \ldots, \mathbf{\partial}_{s1}$ belong to $\mathscr{E}_1, s_1 = \dim \mathscr{E}_1$, the next s_2 belong to $\mathscr{E}_2$, etc. The reciprocal basis will assume the following form:

$$\mathbf{\partial}^{\mathrm{I}} \equiv \mathbf{S} \cdot \mathbf{\partial}_{\mathrm{I}}, \ldots, \mathbf{\partial}^{\mathrm{VI}} \equiv \mathbf{S} \cdot \mathbf{\partial}_{\mathrm{VI}}. \tag{4.19}$$

Indeed

$$\mathbf{\partial}_K \cdot \mathbf{\partial}^L = \mathbf{\partial}_K \times \mathbf{\partial}_L = \delta_{KL}. \tag{4.20}$$

If one considers $\mathbf{\partial}_K$ as a stress, then $\mathbf{\partial}^K$ will be the strain caused by this stress. Using identity (4.7), one obtains promptly

$$\mathbf{E}_1 = \mathbf{\partial}_{\mathrm{I}} \otimes \mathbf{\partial}^{\mathrm{I}} + \ldots + \mathbf{\partial}_{s_1} \otimes \mathbf{\partial}^{s_1}. \tag{4.21}$$

Let us notice the following relations yielding from (4.7):

$$\mathbf{S} = \mathbf{\partial}^{\mathrm{I}} \otimes \mathbf{\partial}^{\mathrm{I}} + \ldots + \mathbf{\partial}^{\mathrm{VI}} \otimes \mathbf{\partial}^{\mathrm{VI}}, \tag{4.22}$$

$$\mathbf{C} = \mathbf{\partial}_{\mathrm{I}} \otimes \mathbf{\partial}_{\mathrm{I}} + \ldots + \mathbf{\partial}_{\mathrm{VI}} \otimes \mathbf{\partial}_{\mathrm{VI}}. \tag{4.23}$$

The most remarkable among the energy-decompositions is the decomposition pointed out and applied in papers [19, 21–24]. It is given by the following theorem being some implementation of the general spectral theorem (see c.f. [25, 26]).

Theorem: *For every elastic body, defined with its compliance tensor* $\mathbf{S}$, *there exists exactly one energy-orthogonal and orthogonal decomposition*

$$\mathscr{S} = \mathscr{P}_1 \oplus \ldots \oplus \mathscr{P}_\varrho, \qquad \varrho \leq 6, \tag{4.24}$$

$$\mathscr{P}_\alpha \mathbin{\dot{\perp}} \mathscr{P}_\beta, \qquad \mathscr{P}_\alpha \perp \mathscr{P}_\beta, \qquad \text{for} \quad \alpha \neq \beta \tag{4.25}$$

and exactly one set of pair-wise unequal constants

$$\lambda_1, \ldots, \lambda_\varrho, \qquad \lambda_\alpha \neq \lambda_\beta \qquad \text{for} \quad \alpha \neq \beta \tag{4.26}$$

such, that

$$\mathbf{S} = \frac{1}{\lambda_1}\mathbf{P}_1 + \ldots + \frac{1}{\lambda_\varrho}\mathbf{P}_\varrho, \tag{4.27}$$

where $\mathbf{P}_\alpha$ is an orthogonal projector on $\mathscr{P}_\alpha$, $\alpha = 1, \ldots, \varrho$.

Proof. Since $\mathbf{S}$ is a symmetric operator acting in the space $\mathscr{S}$ with the scalar product (4.1), the equation

$$\mathbf{S} \cdot \boldsymbol{\omega} = \frac{1}{\lambda}\boldsymbol{\omega} \tag{4.28}$$

has an orthonormal set of solutions

$$\begin{gathered} \boldsymbol{\omega}_{\mathrm{I}}, \ldots, \boldsymbol{\omega}_{\mathrm{VI}}, \\ \boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL}, \end{gathered} \tag{4.29}$$

each $\boldsymbol{\omega}_K$ being related to the proper value λ_K^{-1}. This orthonormal basis is accompanied by the energy-orthonormal one

$$\mathbf{э}_1 = \lambda_{\mathrm{I}}^{1/2}\boldsymbol{\omega}_{\mathrm{I}}, \ \ldots, \quad \mathbf{э}_{\mathrm{VI}} = \lambda_{\mathrm{VI}}^{1/2}\boldsymbol{\omega}_{\mathrm{VI}} \tag{4.30}$$

and

$$\mathbf{э}^{\mathrm{I}} = \frac{1}{\lambda_{\mathrm{I}}^{1/2}}\boldsymbol{\omega}_{\mathrm{I}}, \ \ldots, \quad \mathbf{э}^{\mathrm{VI}} = \frac{1}{\lambda_{\mathrm{VI}}^{1/2}}\boldsymbol{\omega}_{\mathrm{VI}}. \tag{4.31}$$

From (4.22) it follows that

$$\mathbf{S} = \frac{1}{\lambda_{\mathrm{I}}}\boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \frac{1}{\lambda_{\mathrm{VI}}}\boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}}. \tag{4.32}$$

Let $\boldsymbol{\omega}_K$ are labeled in such a way, that

$$\lambda_{\mathrm{I}} = \lambda_{\mathrm{II}} = \ldots = \lambda_{s_1} = \lambda_1, \quad \lambda_{s_{1+1}} = \ldots = \lambda_{s_1+s_2} = \lambda_2$$

etc., then

$$\boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \boldsymbol{\omega}_{s_1} \otimes \boldsymbol{\omega}_{s_1} = \mathbf{P}_1, \tag{4.33}$$

are orthogonal projectors on some subspaces

$$\mathscr{P}_1, \ldots; \dim \mathscr{P}_1 = s_1, \ldots$$

Expression (4.32) takes the form of (4.27). It is evident that the decomposition (4.24) is unique, orthogonal and energy-orthogonal. From (4.23) it follows, that

$$\mathbf{C} = \lambda_1 \mathbf{P}_1 + \ldots + \lambda_\varrho \mathbf{P}_\varrho. \tag{4.34}$$

Solutions $\boldsymbol{\omega}$ of the Eq. (4.28) we designed in [19] as **proper elastic states** of the elastic body under consideration, while parameters λ were called **true** (**proper**) **stiffness moduli**[3)]. The proper elastic states have been found for all symmetries of crystals and the anisotropic engineering materials.

Spaces $\mathscr{P}_\alpha$ consist of the proper elastic states and to each of these spaces is prescribed its own true stiffness modulus λ_α. We shall call (4.24) **the proper energy-decomposition**, for the body under consideration.

Let us represent arbitrary stress $\boldsymbol{\sigma}$ according to the proper decomposition:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \ldots + \boldsymbol{\sigma}_\varrho, \qquad \boldsymbol{\sigma}_\alpha \equiv \mathbf{P}_\alpha \cdot \boldsymbol{\sigma} \in \mathscr{P}_\alpha; \tag{4.35}$$

$$\boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\sigma}_\beta = 0, \qquad \text{for} \quad \alpha \neq \beta. \tag{4.36}$$

We introduce also values of projections

$$\sigma_\alpha \equiv (\boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\sigma}_\alpha)^{1/2} = (\boldsymbol{\sigma} \cdot \mathbf{P}_\alpha \cdot \boldsymbol{\sigma})^{1/2}. \tag{4.37}$$

The elastic energy corresponding to the α-th part of the stress is equal to

$$\Phi(\boldsymbol{\sigma}_\alpha) \equiv \frac{1}{2} \boldsymbol{\sigma}_\alpha \cdot \mathbf{S} \cdot \boldsymbol{\sigma} = \frac{\sigma_\alpha^2}{2\lambda_\alpha}, \qquad \alpha = 1, \ldots, \varrho \tag{4.38}$$

and, therefore, **the proper decomposition of energy** (corresponding to the proper space decomposition) takes a very simple form

$$\Phi(\boldsymbol{\sigma}) = \frac{\sigma_1^2}{2\lambda_1} + \ldots + \frac{\sigma_\varrho^2}{2\lambda_\varrho}. \tag{4.39}$$

[3)]The Author referred to them also as to **Kelvin moduli**, cf. J. Rychlewski, On Hooke's Law, *Journal of Applied Mathematics and Mechanics*, 48(3), 303–314, 1984 (translator's remark).

Basing on this foundation, one can propose the following *particular quadratic energy-criterion of the limit state*:

$$\frac{\sigma_1^2}{k_1^2} + \ldots + \frac{\sigma_\varrho^2}{k_\varrho^2} \leq 1, \tag{4.40}$$

where $h_\alpha \equiv k_\alpha^2/2\lambda_\alpha$ is the limit value of energy of the load $\boldsymbol{\sigma}_\alpha \in \mathscr{P}_\alpha$. If $k_\alpha = \infty$, then the space $\mathscr{P}_\alpha$ is composed of the safe states.

For the materials being isotropic with respect to elastic properties, the proper elastic states are following: any hydrostatic stress $\boldsymbol{\sigma}_1 = \sigma\mathbf{1}$ with the stiffness modulus $\lambda_1 = 3K$ and any deviator $\boldsymbol{\sigma}_2 = \mathbf{s}$, with the stiffness modulus $\lambda_2 = 2G$. Principal decomposition is given by the expression (3.18) and structural one (4.27) – by the relation (3.15). Hence, the limit criterion (4.40) can be expressed as

$$\frac{\sigma^2}{\sigma_0^2} + \frac{\mathbf{s}\cdot\mathbf{s}}{2k^2} \leq 1. \tag{4.41}$$

If any hydrostatic state is safe, then $\sigma_0 = \infty$ and we obtain the Maxwell–Huber criterion (1.1).

Criterion (4.40) assumes some *weak coupling between elastic and limit properties.* In many cases such a coupling probably takes place, e.g. due to the symmetry of the structure of the body under consideration. However it can not be truthfully in a general case. The simplest counterexample supplies a body which is isotropic as regards its elastic properties, being anisotropic regarding limit properties. We shall fully clarify the nature of the specific connection between the elastic and the limit properties in the Sec. 6.

Comment. On the ground of (4.39) one can propose, of course, a more general energy-criterion

$$F(\sigma_1, \ldots, \sigma_\varrho) \leq 1; \tag{4.42}$$

it would, however still, enclose an assumption about the mentioned interconnection between the elastic and the limit properties.

5. Energy related meaning of the quadratic limit criteria

Noticeable, ahead evident, generalization of the condition (1.1) was proposed in the classic work of R. von Mises [27]. He has chosen the yield condition in the form

$$\mathbf{s}\cdot\mathbf{H}\cdot\mathbf{s} \leq 1, \tag{5.1}$$

where $\mathbf{H} \in \mathscr{T}$. For $\mathbf{H} = (1/2k^2)\mathbf{I}_{\mathscr{D}}$ one obtains (1.1). It was repeated by R. HILL for orthotropic bodies in [28].

We shall discuss the most general **quadratic condition of limit state**

$$\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} \leq 1. \tag{5.2}$$

We shall call tensor $\mathbf{H}$ the **limit state tensor** and the quadratic form $\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma}$ will be called a **quadratic measure of stress intensity**. Without loss of generality one can assume, that $\mathbf{H}$ is a symmetric tensor of $\mathscr{T}$. Besides this, the measure of intensity of an arbitrary load $\boldsymbol{\sigma}$ should be non-negative. Thus, the following conditions are imposed on $\mathbf{H}$:

$$\mathbf{H} = \mathbf{H}^T, \qquad \boldsymbol{\alpha} \cdot \mathbf{H} \cdot \boldsymbol{\alpha} \geq 0 \qquad \text{for any} \quad \alpha \in \mathscr{S}. \tag{5.3}$$

As always, with the quadratic form $\boldsymbol{\alpha} \cdot \mathbf{H} \cdot \boldsymbol{\alpha}$ are associated: its polar – bilinear symmetric form $\boldsymbol{\alpha} \cdot \mathbf{H} \cdot \boldsymbol{\beta}$ and the symmetric linear operator $\mathbf{H}\cdot$, $\boldsymbol{\alpha} \to \mathbf{H} \cdot \boldsymbol{\alpha}$. We shall make use of this operator without delay.

A stress state $\boldsymbol{\sigma}$ we shall call the **safe state** for the elastic body, whose limit properties are given by the limit state tensor $\mathbf{H}$ if $\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = 0$. Let us recall the simple theorem of linear algebra.

Theorem. *The set of the safe stress states constitutes a kernel of the operator* $\mathbf{H}\cdot$, *i.e. it is composed of the stresses* $\boldsymbol{\sigma}$, *fulfilling the condition*

$$\mathbf{H} \cdot \boldsymbol{\sigma} = \mathbf{0}. \tag{5.4}$$

Proof. Let us examine measure of stress intensity on a unit sphere $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1$. Due to non-negativeness, its null value is minimal. Therefore: if $\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = 0$, then the derivative of the Lagrange function $\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} - \mu(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - 1)$ must vanish, this yields $\mathbf{H} \cdot \boldsymbol{\sigma} = \mu\boldsymbol{\sigma}$, at the same time $\mu = \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma}/\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 0$.

Von Mises criterion (5.1) is in fact a criterion (5.2) for which it was assumed that an arbitrary hydrostatic stress is safe. Then $\mathbf{1} \cdot \mathbf{H} \cdot \mathbf{1} = 0$, i.e.

$$\mathbf{H} \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{H} = \mathbf{0} \tag{5.5}$$

and, therefore

$$\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \mathbf{s} \cdot \mathbf{H} \cdot \mathbf{s}.$$

This assumption does not seem to us to be naturally innate in general case. There are no a priori reasons for considering a spherical tensor as something exceptional for anisotropic media. *The habit to separate the spherical part of stress* $\sigma\mathbf{1}$ *and to consider the pressure* $-\sigma$ *as some universal thermodynamic parameter, has come into mechanics of the solid deformable body from the mechanics and thermodynamics of liquids and gases and has rooted on fertile soil* *of isotropic bodies. This habit should be revised, in our opinion.* In particular, for the anisotropic body, say a composite, quite different stress states related to its structure (reinforcement, etc.) can emerge as the safe ones.

Concerning the limit state tensor $\mathbf{H}$ we shall not assume anything more than (5.3).

R. von Mises have not ascribed any specific interpretation to his condition. In particular, any connection of his measure of stress intensity $\mathbf{s} \cdot \mathbf{H} \cdot \mathbf{s}$ with elastic energy was not discernible. We shall prove that an *arbitrary quadratic measure of stress intensity* $\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma}$ *possesses uniquely defined, in terms of energy, interpretation.* This is contained in the following theorem, which is an implementation of the idea o simultaneous reduction of two quadratic forms (here $\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma}$ and $\boldsymbol{\sigma} \cdot \mathbf{S} \cdot \boldsymbol{\sigma}$) into a sum of squares (cf. [29]).

Theorem. *For every elastic body defined by its compliance tensor* $\mathbf{S}$ *and limit state tensor* $\mathbf{H}$, *there exist: exactly one energy-orthogonal decomposition*:

$$\mathscr{S} = \mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_\chi, \qquad \chi \leq 6, \tag{5.6}$$

$$\mathscr{H}_\alpha \ \dot{\perp}\ \mathscr{H}_\beta \qquad \text{for} \qquad \alpha \neq \beta \tag{5.7}$$

and exactly one set of pair-wise unequal constants

$$h_1, \ldots, h_\chi, \qquad h_\alpha \neq h_\beta, \qquad \text{for} \qquad \alpha \neq \beta, \tag{5.8}$$

such that, for an arbitrary stress

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \ldots + \boldsymbol{\sigma}_\chi, \qquad \boldsymbol{\sigma}_\alpha \in \mathscr{H}_\alpha, \tag{5.9}$$

the measure of stress intensity is equal to

$$\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \frac{1}{h_1} \varPhi(\boldsymbol{\sigma}_1) + \ldots + \frac{1}{h_\chi} \varPhi(\boldsymbol{\sigma}_\chi), \tag{5.10}$$

where

$$\varPhi(\boldsymbol{\sigma}_1) + \ldots + \varPhi(\boldsymbol{\sigma}_\chi) = \varPhi(\boldsymbol{\sigma}). \tag{5.11}$$

Proof. We shall present a constructive one. We shall entirely use the energy-scalar product (4.8) instead of the standard one (4.1). Let us introduce linear operations

$$\mathbf{L} \times \boldsymbol{\alpha} \equiv \mathbf{L} \cdot (\mathbf{S} \cdot \boldsymbol{\alpha}) = (\mathbf{L} \circ \mathbf{S}) \cdot \boldsymbol{\alpha}, \tag{5.12}$$

$$\boldsymbol{\alpha} \times \mathbf{L} \equiv (\mathbf{S} \cdot \boldsymbol{\alpha}) \cdot \mathbf{L} = (\boldsymbol{\alpha} \cdot \mathbf{S}) \cdot \mathbf{L} = \boldsymbol{\alpha} \cdot (\mathbf{S} \circ \mathbf{L}), \tag{5.13}$$

$$\mathbf{L} \,\square\, \mathbf{N} \equiv \mathbf{L} \circ \mathbf{S} \circ \mathbf{N}. \tag{5.14}$$

We shall identify tensors $\mathbf{L} \in \mathscr{T}$ with linear operators acting from $\mathscr{S}$ into $\mathscr{S}$, according to the rule

$$\boldsymbol{\alpha} \to \mathbf{L} \times \boldsymbol{\alpha} \tag{5.15}$$

instead of (4.5). Unit operator $\boldsymbol{\alpha} \to \boldsymbol{\alpha}$ under this convention is realized not by the tensor $\mathbf{I}$, but by the stiffness tensor $\mathbf{C}$. Indeed,

$$\mathbf{C} \times \boldsymbol{\alpha} = (\mathbf{C} \circ \mathbf{S}) \cdot \boldsymbol{\alpha} = \mathbf{I} \cdot \boldsymbol{\alpha} = \boldsymbol{\alpha}. \tag{5.16}$$

Let us write the stress intensity measure in the following form:

$$\boldsymbol{\alpha} \times (\mathbf{C} \circ \mathbf{H} \circ \mathbf{C}) \times \boldsymbol{\alpha} = \boldsymbol{\alpha} \cdot \mathbf{H} \cdot \boldsymbol{\alpha}. \tag{5.17}$$

The operation completed by the tensor $\mathbf{C} \circ \mathbf{H} \circ \mathbf{C}$ is symmetric with respect to the energy-scalar product, i.e.

$$\boldsymbol{\alpha} \times (\mathbf{C} \circ \mathbf{H} \circ \mathbf{C}) \times \boldsymbol{\beta} = \boldsymbol{\beta} \times (\mathbf{C} \circ \mathbf{H} \circ \mathbf{C}) \times \boldsymbol{\alpha} \tag{5.18}$$

for every $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \mathscr{S}$. Let us analyze proper elements $\boldsymbol{\chi}$ and proper values $(2h)^{-1}$ of this operator

$$(\mathbf{C} \circ \mathbf{H} \circ \mathbf{C}) \times \boldsymbol{\chi} = \frac{1}{2h} \boldsymbol{\chi}. \tag{5.19}$$

In view of the symmetry (5.18) one can evidently find an energy-orthonormal set of solutions

$$\boldsymbol{\chi}_{\mathrm{I}}, \ldots, \boldsymbol{\chi}_{\mathrm{VI}}, \qquad \boldsymbol{\chi}_K \times \boldsymbol{\chi}_L = \delta_{KL}, \tag{5.20}$$

where $\boldsymbol{\chi}_L$ corresponds to the proper value $(2h_L)^{-1}$. The set of $\boldsymbol{\chi}_K \otimes \boldsymbol{\chi}_L$ constitutes a basis in $\mathscr{T}$ (see (4.4)) i.e. for any $\mathbf{L} \in \mathscr{T}$

$$\mathbf{L} = \sum_{K,L=\mathrm{I}}^{\mathrm{VI}} L_{KL} \boldsymbol{\chi}_K \otimes \boldsymbol{\chi}_L; \tag{5.21}$$

here

$$L_{KL} \equiv \boldsymbol{\chi}_K \times \mathbf{L} \times \boldsymbol{\chi}_L. \tag{5.22}$$

In particular, taking into account (5.19) and (5.20), for $\mathbf{L} = \mathbf{C} \circ \mathbf{H} \circ \mathbf{C}$ one has:

$$\mathbf{C} \circ \mathbf{H} \circ \mathbf{C} = \frac{1}{2h_{\mathrm{I}}} \boldsymbol{\chi}_{\mathrm{I}} \otimes \boldsymbol{\chi}_{\mathrm{I}} + \ldots + \frac{1}{2h_{\mathrm{VI}}} \boldsymbol{\chi}_{\mathrm{VI}} \otimes \boldsymbol{\chi}_{\mathrm{VI}}. \tag{5.23}$$

Let $\boldsymbol{\chi}_K$ be labeled in such a way that $h_{\mathrm{I}} = \ldots = h_{G_1} = h_1$, $h_{G_1+1} = \ldots = h_{G_2+2} = h_2$ etc. Let us consider

$$\underline{\mathbf{H}_1 \equiv \boldsymbol{\chi}_1 \otimes \boldsymbol{\chi}_1 + \ldots + \boldsymbol{\chi}_{G_1} \otimes \boldsymbol{\chi}_{G_1}} \tag{5.24}$$

and denote

$$\mathscr{H}_1 \equiv \operatorname{Im} H_1 \ldots, \qquad \dim \mathscr{H}_1 = G_1, \ldots \tag{5.25}$$

It is evident that

$$\mathbf{H}_\alpha \times \boldsymbol{\chi} = \begin{cases} \boldsymbol{\chi} & \text{for} \quad \boldsymbol{\chi} \in H_\alpha, \\ \mathbf{0} & \text{for} \quad \boldsymbol{\chi} \in H_\beta, \qquad \beta \neq \alpha \end{cases} \tag{5.26}$$

and $\mathscr{H}_\alpha \dot{\perp} \mathscr{H}_\beta$ for $\alpha \neq \beta$. In other words, $\mathbf{H}_\alpha$ are energy-orthoprojectors and constitute energy-orthogonal decomposition of the unity operator

$$\mathbf{C} = \mathbf{H}_1 + \ldots + \mathbf{H}_{\boldsymbol{\chi}}, \tag{5.27}$$

$$\mathbf{H}_\alpha \,\Box\, \mathbf{H}_\alpha = \mathbf{H}_\alpha, \qquad \mathbf{H}_\alpha \,\Box\, \mathbf{H}_\beta = \mathbf{0} \qquad \text{for} \quad \alpha \neq \beta. \tag{5.28}$$

It corresponds to space decomposition (5.6), (5.7).

Collecting together in (5.23) the terms with the same h_L, one obtains

$$\mathbf{C} \circ \mathbf{H} \circ \mathbf{C} = \frac{1}{2h_1}\mathbf{H}_1 + \ldots + \frac{1}{2h_{\boldsymbol{\chi}}}\mathbf{H}_{\boldsymbol{\chi}}. \tag{5.29}$$

Taking (5.9) we get (5.11). Now

$$\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \frac{1}{2h_1}\boldsymbol{\sigma} \times \mathbf{H}_1 \times \boldsymbol{\sigma} + \ldots + \frac{1}{2h_{\boldsymbol{\chi}}}\boldsymbol{\sigma} \times \mathbf{H}_{\boldsymbol{\chi}} \times \boldsymbol{\sigma} \tag{5.30}$$

and, since

$$\boldsymbol{\sigma} \times \mathbf{H}_\alpha \times \boldsymbol{\sigma} = \boldsymbol{\sigma}_\alpha \times \boldsymbol{\sigma}_\alpha = 2\Phi(\boldsymbol{\sigma}_\alpha) \tag{5.31}$$

we obtain (5.10).

Everything here is constructed effectively, together with the definitions of h_K and $\boldsymbol{\chi}_K$ using the Eq. (5.19), which can be rewritten as

$$(2h\mathbf{H} - \mathbf{S}) \cdot \boldsymbol{\chi} = \mathbf{0}. \tag{5.32}$$

The expression for the limit state tensor can be expressed as follows:

$$\mathbf{H} = \frac{1}{2h_{\mathrm{I}}}\mathbf{S} \cdot \boldsymbol{\chi}_{\mathrm{I}} \otimes \mathbf{S} \cdot \boldsymbol{\chi}_{\mathrm{I}} + \ldots + \frac{1}{2h_{\mathrm{VI}}}\mathbf{S} \cdot \boldsymbol{\chi}_{\mathrm{VI}} \otimes \mathbf{S} \cdot \boldsymbol{\chi}_{\mathrm{VI}}, \tag{5.33}$$

or

$$\mathbf{H} = \frac{1}{2h_{\mathrm{I}}}\mathbf{S}_1 + \ldots + \frac{1}{2h_{\chi}}\mathbf{S}_{\chi}, \tag{5.34}$$

where

$$\mathbf{S}_\alpha \equiv \mathbf{S} \circ \mathbf{H}_\alpha \circ \mathbf{S}, \qquad \alpha = 1, \ldots, \chi, \tag{5.35}$$

$$\mathbf{S} = \mathbf{S}_1 + \ldots + \mathbf{S}_{\chi}. \tag{5.36}$$

Thus, **any arbitrary quadratic criterion of limit state can be expressed in the form of energy-inequality**

$$\frac{1}{h_1}\Phi(\boldsymbol{\sigma}_1) + \ldots + \frac{1}{h_{\chi}}\Phi(\boldsymbol{\sigma}_{\chi}) \leq 1. \tag{5.37}$$

Parameters

$$h_\alpha \equiv \frac{\boldsymbol{\chi}\cdot\mathbf{S}\cdot\boldsymbol{\chi}}{2\boldsymbol{\chi}\cdot\mathbf{H}\cdot\boldsymbol{\chi}} = \mathrm{const} \qquad \text{for any} \quad \boldsymbol{\chi} \in \mathscr{H}_\alpha \tag{5.38}$$

are the limit values of elastic energy for the loads $\boldsymbol{\sigma}_\alpha \in \mathbf{H}_\alpha$.

If $h_\alpha = \infty$, then the loads $\boldsymbol{\sigma}_\alpha \in \mathscr{H}_\alpha$ are safe. Criteria (3.42) and (3.46) represent particular cases of (5.37) for

$$\chi = 2, \qquad h_1 = \infty, \qquad \dim \mathscr{H}_1 = 1. \tag{5.39}$$

Let us notice that expression (5.33) presents the most general form of limit state tensor in such a sense, that $\mathbf{H}$ is uniquely determined by some energy-orthogonal basis $\boldsymbol{\chi}_K$ and a set of h_K

$$(h_1, \ldots, h_{\mathrm{VI}};\ \boldsymbol{\chi}_{\mathrm{I}}, \ldots, \boldsymbol{\chi}_{\mathrm{VI}}) \to \mathbf{H}. \tag{5.40}$$

6. On possible forms of coupling between the elastic and the limit properties

Theorem (5.10) does not assume any connection between the directional distribution of the elastic properties described by the compliance tensor $\mathbf{S}$ and the limit properties described by the limit state tensor $\mathbf{H}$. Though, some coupling may appear due to the structure of the body. Our theorem points out a form through which this coupling may possibly reveal itself.

The principal decomposition of elastic energy (4.24) differs from the decomposition with regard to limit properties (5.6) by its orthogonality. Let us make clear when (5.6) is also orthogonal. We shall start from the following lemma:

Lemma. *Energy-orthogonal decomposition*

$$\mathscr{S} = \mathscr{E}_1 \oplus \ldots \oplus \mathscr{E}_{\varkappa}, \qquad \varkappa \leq 6 \tag{6.1}$$

is orthogonal if and only if it is stable (invariant) with respect to the compliance tensor $\mathbf{S}$.

Proof. Sufficiency. Let the energy-orthogonal decomposition be stable with respect to $\mathbf{S}$, i.e. $\mathbf{S}\cdot\boldsymbol{\omega}\in\mathscr{E}_\alpha$ for each $\boldsymbol{\omega}\in\mathscr{E}_\alpha$ and for all $\alpha=1,\ldots,\varkappa$. Then in any $\mathscr{E}_\alpha$ one can find a basis $\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_q$, $q=\dim\mathscr{E}_\alpha$ consisting of the proper elements of $\mathbf{S}$,

$$\mathbf{S}\cdot\boldsymbol{\omega}_K=\mu_K\boldsymbol{\omega}_K,\qquad K=1,\ldots,q, \tag{6.2}$$

where $\mu_K>0$, i.e. the form $\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\alpha}$, is positive definite. Let us take any $\boldsymbol{\alpha}\in\mathscr{E}_\alpha$ and any $\boldsymbol{\beta}\in\mathscr{E}_\beta$, $\boldsymbol{\beta}\neq\boldsymbol{\alpha}$. Since $\mathscr{E}_\alpha\dot{\perp}\mathscr{E}_\beta$, then

$$\begin{aligned}\boldsymbol{\alpha}\cdot\boldsymbol{\beta}=(\alpha_1\boldsymbol{\omega}_1+\ldots+\alpha_q\boldsymbol{\omega}_q)\cdot\boldsymbol{\beta}&=\left(\frac{\alpha_1}{\mu_1}\mathbf{S}\cdot\boldsymbol{\omega}_1+\ldots+\frac{\alpha_q}{\mu_q}\mathbf{S}\cdot\boldsymbol{\omega}_q\right)\cdot\boldsymbol{\beta}\\&=\frac{\alpha_1}{\mu_1}\boldsymbol{\omega}_q\times\boldsymbol{\beta}+\ldots+\frac{\alpha_q}{\mu_q}\boldsymbol{\omega}_q\times\boldsymbol{\beta}=0,\end{aligned} \tag{6.3}$$

i.e. $\mathscr{E}_\alpha\perp\mathscr{E}_\beta$.

Necessity. Let any two terms of energy-orthogonal decomposition be orthogonal, i.e.

$$\mathscr{E}_\alpha\dot{\perp}\mathscr{E}_\beta\qquad\text{and}\qquad\mathscr{E}_\alpha\perp\mathscr{E}_\beta\qquad\text{for all}\qquad\alpha\neq\beta. \tag{6.4}$$

Let us take arbitrary $\mathscr{E}_\alpha$. In virtue of (6.4), an orthogonal complement $\mathscr{E}_\alpha^\perp$ coincides with the energy-orthogonal one $\mathscr{E}_\alpha^{\dot{\perp}}$. Therefore for any $\boldsymbol{\alpha}\in\mathscr{E}_\alpha$, taking any $\boldsymbol{\tau}\in\mathscr{E}_\alpha^\perp$ we have $\boldsymbol{\alpha}\times\boldsymbol{\tau}=\boldsymbol{\alpha}\cdot\mathbf{S}\cdot\boldsymbol{\tau}=0$, i.e. $\mathbf{S}\cdot\boldsymbol{\alpha}\in\mathscr{E}_\alpha^{\perp\perp}=\mathscr{E}_\alpha$. Therefore any $\mathscr{E}_\alpha$ is invariant with respect to $\mathbf{S}$.

Definition. We shall tell that $\mathbf{A},\mathbf{B}\in\mathscr{T}$ are **coaxial** if there exists an orthonormal basis

$$\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_{\mathrm{VI}},\qquad\boldsymbol{\omega}_K\cdot\boldsymbol{\omega}_L=\delta_{KL}, \tag{6.5}$$

composed of the proper elements of both tensors, i.e. such, that for some sets $\alpha_{\mathrm{I}},\ldots,\alpha_{\mathrm{VI}}$ and $\beta_{\mathrm{I}},\ldots,\beta_{\mathrm{VI}}$:

$$\begin{aligned}\mathbf{A}&=\alpha_{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{I}}\otimes\boldsymbol{\omega}_{\mathrm{I}}+\ldots+\alpha_{\mathrm{VI}}\boldsymbol{\omega}_{\mathrm{VI}}\otimes\boldsymbol{\omega}_{\mathrm{VI}},\\\mathbf{B}&=\beta_{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{I}}\otimes\boldsymbol{\omega}_{\mathrm{I}}+\ldots+\beta_{\mathrm{VI}}\boldsymbol{\omega}_{\mathrm{VI}}\otimes\boldsymbol{\omega}_{\mathrm{VI}}.\end{aligned} \tag{6.6}$$

Let us notice that the numbers of equal values in the array $\alpha_{\mathrm{I}},\ldots,\alpha_{\mathrm{VI}}$ can be entirely different from those of $\beta_{\mathrm{I}},\ldots,\beta_{\mathrm{VI}}$. Hence, decompositions of $\mathscr{S}$ into direct sums of the proper subspaces of $\mathbf{A}$ and $\mathbf{B}$ can be quite different.

Theorem. *Energy-orthogonal decomposition of the stress space with regard to the limit properties of the body (5.6) is orthogonal if and only if the limit state tensor* $\mathbf{H}$ *and the compliance tensor* $\mathbf{S}$ *are coaxial.*

Proof. Necessity. Let a decomposition (5.6) be orthogonal. This means, according to the lemma, that it is stable with respect to the $\mathbf{S}$. Therefore, there exists such an orthonormal basis $\boldsymbol{\omega}_{\mathrm{I}}, \ldots, \boldsymbol{\omega}_{\mathrm{VI}}$, composed of the proper elements of $\mathbf{S}$, that any $\boldsymbol{\omega}_K$ belongs to one of $\mathscr{E}_\alpha$. For any $\boldsymbol{\chi} \in \mathscr{E}_\alpha$, however, Eq. (5.32) is satisfied with some value of h. Thus, for any $\boldsymbol{\omega}_K$, $\mathbf{H} \cdot \boldsymbol{\omega}_K = \nu_K \boldsymbol{\omega}_K$ for a certain ν_K, i.e. basis $\boldsymbol{\omega}_K$ is composed of proper elements of the tensor $\mathbf{H}$. Sufficiency. Assume that $\mathbf{H}$ and $\mathbf{S}$ are coaxial, $\boldsymbol{\omega}_K$ being an orthonormal basis composed of proper elements of $\mathbf{S}$ and $\mathbf{H}$. Then $\boldsymbol{\chi}_K = \boldsymbol{\omega}_K$ satisfy Eq. (5.32). Carried out according to relations (5.24), decomposition (5.6) is, therefore, orthogonal.

Coaxiality of the tensors $\mathbf{S}$, $\mathbf{H}$

$$\mathbf{S} = \frac{1}{\lambda_{\mathrm{I}}} \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \frac{1}{\lambda_{\mathrm{VI}}} \boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}}, \tag{6.7}$$

$$\mathbf{H} = \frac{1}{2h_{\mathrm{I}}} \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \frac{1}{2h_{\mathrm{VI}}} \boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}}, \tag{6.8}$$

reflects some coupling between the elastic and the limit properties. It does not seem to be very rigid.

We shall present the following interpretation of the coaxiality (6.7), (6.8). For any quadratic form $\boldsymbol{\sigma} \cdot \mathbf{A} \cdot \boldsymbol{\sigma}$, the states of local extremality, defined by the condition

$$\boldsymbol{\sigma} \cdot \mathbf{A} \cdot \boldsymbol{\sigma} = \text{ext} \quad \text{at} \quad \boldsymbol{\omega} \cdot \boldsymbol{\omega} = 1, \quad \text{under condition} \quad \boldsymbol{\sigma} \cdot \mathbf{A} \cdot \boldsymbol{\sigma} = 1 \tag{6.9}$$

are the proper elements, i.e.

$$\mathbf{A} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{\sigma}. \tag{6.10}$$

Indeed, (6.9) is equivalent to

$$\partial_\sigma F = 0 \quad \text{for} \quad F \equiv \boldsymbol{\sigma} \cdot \mathbf{A} \cdot \boldsymbol{\sigma} - \alpha(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - 1) \tag{6.11}$$

which yields Eq. (6.10).

Proper elastic states $\mathbf{S} \cdot \boldsymbol{\omega} = (1/\lambda)\boldsymbol{\omega}$ are, therefore, **the states of extremal energy** and the proper states of $\mathbf{H}$, $\mathbf{H} \cdot \boldsymbol{\omega} = \chi \boldsymbol{\omega}$ are **the states of extremal limit stress measure**.

Coaxiality of $\mathbf{S}$ and $\mathbf{H}$, therefore, means that there exists an orthonormal set of states $\boldsymbol{\omega}_K$ being simultaneously the states of extremal energy and the states of maximal limit stress intensity, see Fig. 5.

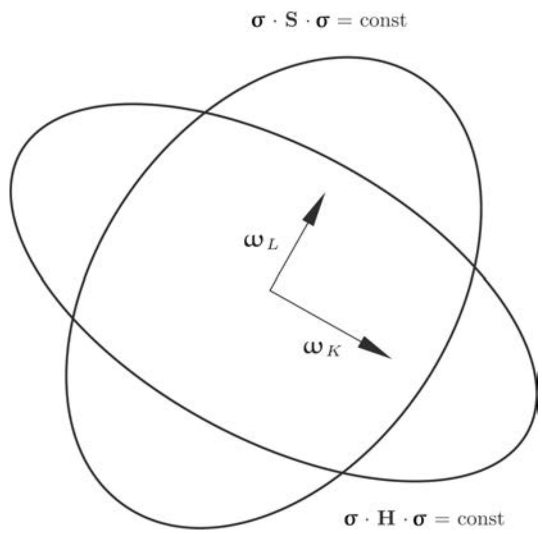

FIG. 5. Coaxiality of the compliance tensor $\mathbf{S}$ and the limit state tensor $\mathbf{H}$, the cross-section of the surface of equal energy and the surface of the equal limit stress intensity with the plane $(\boldsymbol{\omega}_L, \boldsymbol{\omega}_K)$ in the stress space.

Let us consider a more restrictive modification of a coupling between the elastic and the limit properties, where *any state of extremal energy is simultaneously the state of extremal limit stress measure*. This is a particular case of coaxiality when

$$\lambda_K = \lambda_L \Rightarrow h_K = h_L. \tag{6.12}$$

Taking the structural decomposition of the compliance tensor

$$\mathbf{S} = \frac{1}{\lambda_1}\mathbf{P}_1 + \ldots + \frac{1}{\lambda_\varrho}\mathbf{P}_\varrho, \tag{6.13}$$

where $\lambda_\alpha \neq \lambda_\beta$ for $\alpha \neq \beta$, one has

$$\mathbf{H} = \frac{1}{2h_1}\mathbf{P}_1 + \ldots + \frac{1}{2h_\varrho}\mathbf{P}_\varrho, \tag{6.14}$$

values of h_α being not necessarily different. Just in this case, a quadratic limit criterion takes the form (4.40).

7. Expression of any limit criterion in the E. Beltrami form

As we have already mentioned, E. BELTRAMI in his unjustly forgotten paper [10] posed a limit criterion

$$\Phi(\boldsymbol{\sigma}) \leq h = \text{const}. \tag{7.1}$$

This proposal meets at once the following objection: for overwhelming majority of circumstances, hydrostatic stress can be considered as safe. We shall, however, *not throw the baby out with the bathwater.* A rational root of E. Beltrami proposal consists in the fact that for the failure of element, one always needs to spend some work, which, in the elastic element, should be equal to the elastic energy stored until that instant. Shortcoming of the direct realization of this concept in the form of (7.1) lies only in the assumption that this work does not depend on the kind of stress state.

Let us introduce and denote as $h(\boldsymbol{\omega})$, $\boldsymbol{\omega} \cdot \boldsymbol{\omega} = 1$, *the limit value of elastic energy which the element is capable to accumulate under the load* $\boldsymbol{\sigma} = c\boldsymbol{\omega}$, $c > 0$. For the usual elastic body (hyperelastic), $h(\boldsymbol{\omega})$ is equal to the work, which is necessary for failure of the element under such stress. W shall modify the proposal of E. Beltrami as follows:

$$\Phi(\boldsymbol{\sigma}) \leq h\left(\frac{\boldsymbol{\sigma}}{(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})^{1/2}}\right). \tag{7.2}$$

But such a form can assume any limit criterion for which the region of the safe states in $\mathscr{S}$ has a stellar shape and includes unstressed natural state; an overdone example is shown in Fig. 6. Indeed, any such a region is described by inequality

$$\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \leq k^2\left(\frac{\boldsymbol{\sigma}}{(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})^{1/2}}\right), \tag{7.3}$$

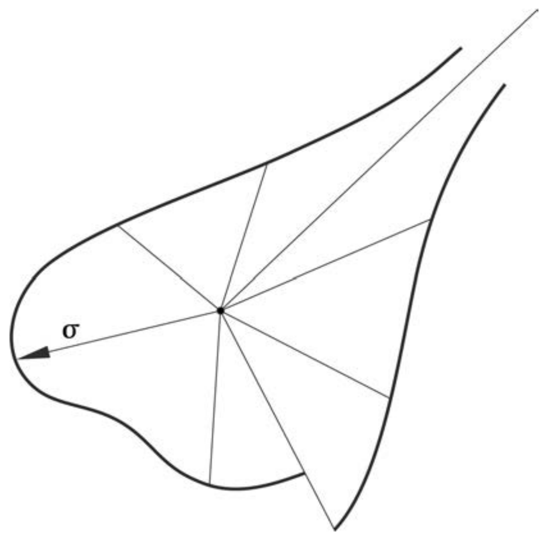

FIG. 6. An exotic example of a star-shape region of safe states, containing unstressed natural state.

where k is the limit function defined on the unit sphere in $\mathscr{S}$. Since for linearly elastic body one has

$$\Phi(\boldsymbol{\sigma}) = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})\Phi\left(\frac{\boldsymbol{\sigma}}{(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{1/2}}\right), \tag{7.4}$$

then (7.3) is equivalent to (7.2) with

$$h(\boldsymbol{\omega}) = \Phi(\boldsymbol{\omega})k^2(\boldsymbol{\omega}). \tag{7.5}$$

Since the expressed assumption regarding the shape of the region of safe states is fulfilled for all thinkable cases (materials, processes, kinds of limit properties), representation (7.2) is not confined to any kind of particular form of limit criterion.

For the quadratic criteria of limit states (5.2) considered above

$$k^2(\boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \mathbf{H} \cdot \boldsymbol{\omega})^{-1} = [h_1\Phi(\boldsymbol{\omega}_1) + \ldots + h_\chi\Phi(\boldsymbol{\omega}_\chi)]^{-1}. \tag{7.6}$$

where

$$\begin{gathered} \boldsymbol{\omega} = \boldsymbol{\omega}_1 + \ldots + \boldsymbol{\omega}_\chi, \qquad \boldsymbol{\omega}_\alpha \in \mathscr{H}_\alpha, \\ \boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_1 = \ldots = \boldsymbol{\omega}_\chi \cdot \boldsymbol{\omega}_\chi = 1. \end{gathered} \tag{7.7}$$

The Maxwell–Huber criterion represented in E. Beltrami form looks as follows:

$$\Phi(\boldsymbol{\sigma}) \le h + l\frac{\sigma^2}{\mathbf{s} \cdot \mathbf{s}}, \tag{7.8}$$

where

$$h \equiv \frac{k^2}{2G}, \qquad l \equiv \frac{k^2}{K}. \tag{7.9}$$

Quadratic criteria are discussed from somewhat different viewpoint in [30].

8. Conclusions

Simple, but unexpected theorem (5.10) exhausts in a formal sense the problem of energy-criteria of the limit state which, for isotropic bodies, was posed by Maxwell, Beltrami and Huber.

This problem was transferred in the Polish school of mechanics along the relay of generations, from M.T. Huber to W. Burzyński and W. Olszak, from W. Olszak to W. Urbanowski, an from them to J. Ostrowska. I am obliged to her for discussions which helped me to clarify this history of the subject. I am also grateful to N.N. Malinin for his kind discussions, during which he proved to be an expert on Polish works.

Appendix 1

List of equivalences, making possible to transform readily any expression of this paper to the standard Cartesian index notation with the usual summation rules:

$$\begin{aligned}
\boldsymbol{\sigma},\ \boldsymbol{\varepsilon},\ \boldsymbol{\alpha} &\leftrightarrow \sigma_{ij},\ \varepsilon_{ij},\ \alpha_{ij},\\
\mathbf{1},\ \mathbf{s} &\leftrightarrow \delta_{ij},\ s_{ij},\\
\mathbf{C},\ \mathbf{S},\ \mathbf{H} &\leftrightarrow C_{ijkl},\ S_{ijkl},\ H_{ijkl},\\
\boldsymbol{\alpha}\cdot\boldsymbol{\beta} &\leftrightarrow \alpha_{ij}\beta_{ij},\\
\boldsymbol{\alpha}\times\boldsymbol{\beta} &\leftrightarrow S_{ijkl}\alpha_{ij}\beta_{kl},\\
\boldsymbol{\alpha}\otimes\boldsymbol{\beta} &\leftrightarrow \alpha_{ij}\beta_{kl},\\
\mathbf{L}\cdot\boldsymbol{\alpha} &\leftrightarrow L_{ijkl}\alpha_{kl},\\
\boldsymbol{\alpha}\cdot\mathbf{L}\cdot\boldsymbol{\beta} &\leftrightarrow L_{ijkl}\alpha_{ij}\beta_{kl},\\
\mathbf{A}^T=\mathbf{A} &\leftrightarrow A_{ijkl}=A_{klij},\\
\mathbf{A}\circ\mathbf{B} &\leftrightarrow A_{ijpq}B_{pqkl},\\
\partial_\varepsilon\Phi &\leftrightarrow \frac{\partial\Phi}{\partial\varepsilon_{kl}}
\end{aligned}$$

$$\mathbf{I}_{ijkl}=\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{kj}).$$

In the whole paper, $\mathscr{S}$ is the space of the symmetric Euclidean tensors of second rank and $\mathscr{T}$ denotes its tensorial product

$$\mathscr{T}\equiv\mathscr{S}\otimes\mathscr{S}.$$

Tensors of stiffness $\mathbf{C}$, compliance $\mathbf{S}$ and limit state $\mathbf{H}$ obey conditions of internal symmetry

$$A_{ijkl}=A_{jikl}=A_{klij}.$$

In the whole paper, besides the present Appendix, the summation over repeated indices is not applied.

Appendix 2

We quote this fragment of the letter of J.C. Maxwell, [from Origins of Clerk Maxwell's Electric Ideas. . . [5], pp. 31–33], which had a chance to become a foundation of contemporary mathematical theory of plasticity.

129 Union Street
Aberdeen
18 Dec. 1856

Dear Thomson
. . . .

Here is my present notion about plasticity of homogeneous amorphous solids.

Let $\alpha\beta\gamma$ be the 3 principal strains at any point PQR the principal stresses connected with $\alpha\beta\gamma$ by symmetrical linear equations the same for all axes. Then the whole work done by PQR in developing may be written

$$U = A(\alpha^2 + \beta^2 + \gamma^2) + B(\beta\gamma + \gamma\alpha + \alpha\beta)$$

where A and B are coeffts, the nature of which is foreign to our inquiry. Now we may put

$$U = U_1 + U_2,$$

where U_1 is due to a symmetrical compression ($\alpha_1 = \beta_1 = \gamma_1$) and U_2 to distortion without compression ($\alpha_2 + \beta_2 + \gamma_2 = 0$)

$$\alpha = \alpha_1 + \alpha_2, \qquad \beta = \beta_1 + \beta_2, \qquad \gamma = \gamma_1 + \gamma_2.$$

It follows that $\qquad U_1 = 1/3(A + B)(\alpha + \beta + \gamma)^2$

$$U_2 = \frac{2A - B}{3}\left(\alpha^2 + \beta^2 + \gamma^2 - (\beta\gamma + \gamma\alpha + \alpha\beta)\right).$$

Now my *opinion* is that these two parts may be considered as independent U_1 being the work done in condensation and U_2 that done in distortion. Now I would use the old word "Resilience" to denote the work necessary to be done on a body to overcome its elastic forces.

The cubical resilience R is a measure of the work necessary to be expended in compression in order to increase the density permanently. This *must* increase rapidly as the body is condensed, whether it be wood or lead or iron.

The resilience of rigidity R_2 (which is the converse of plasticity) is the work required to be expended in pure distortion in order to produce a permanent change of form in the element. I have strong reasons for believing that when

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta$$

reaches a certain limit $= R_2$ then the element will begin to give way. If the body be tough the disfigurement will go on till this function U_2 (which truly represents the work which the element *would* do in recovering its form) has diminished to R by an alteration of the *permanent dimensions*.

Now let $a\,b\,c$ be the *very small* permanent alterations due to the fact that $U_2 > R_2$ for an instant. Whenever $U_2 = R_2$ the element has as much work done to it as it can bear. Any more work done to the element will be consumed in parmanent alterations.

Therefore if $U_2 = R_2$, and in the next instant, U be increased, dU must be lost in some way.

My rough notion on this subject is that

$$a = \frac{dU}{U}\alpha, \qquad b = \frac{dU}{U}\beta, \qquad c = \frac{dU}{U}\gamma$$

the new values of $\alpha\ \beta\ \gamma$ will be

$$\alpha' = \alpha - a, \qquad \beta' = \beta - b, \qquad \gamma' = \gamma - c.$$

This is the first time that I have put pen to paper on this subject. I have never seen any investigation of the question, „Given the mechanical strain in 3 directions on an element, when will it give way?" I think this notion will bear working out into a mathemat. theory of plasticity when I have time; to be compared with experiment when I know the right experiments to make.

Condition of not yielding

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta < R_2.$$

Yours
J.C. Maxwell

References

1. M. T. Huber, *Specific work of strain as a measure of material effort* [in Polish], Czasopismo Techniczne, XXII, Lwow, 1904, also English translation: Arch. Mech., **56**, 173–190, 2004.
2. R. Mises, *Mechanik der festen Körper im plastisch-deformablen Zustand*, Nachr. von der Königlichen Geselschaft der Wissenschaften zu Götingen, Math. Phys., 1, Kl. 4, 582–592, 1913.
3. H. Hencky, *Zur Theorie plastischer Deformationen und der hierdurch im Material hervorgerufenen Nachspannungen*, ZAMM, **4**, 323–334, 1924.
4. H. Mierzejewski, *Foundations of Mechanics of Plastic Solids* [in Polish], Warszawa, 1927.
5. J. C. Maxwell, Proc. Cambridge Phil. Soc., 32, 1936 (cf. also: Origins of Clerk Maxwell's electric ideas as described in familiar letters to William Thompson, ed. by Sir J. Larmor, Cambridge at Univ. Press, 1937).
6. G. Green, *On the laws of the reflection and refraction of light at the common surface of two non-crystallized media*, Trans. Of the Cambridge Phil. Soc., VII, 1–24, 1839 (cf. also: Math. Papers of the late George Green, 1971).
7. G. G. Stokes, *On the theories of the internal friction of fluids in motion of elastic solids*, Trans. of the Cambridge Phil. Soc., VIII, 287–319, 1849 (cf. also Math. Papers, I, 75–129).
8. H. Helmhotz, *Dynamic continuerlich verbreiteten Massen*, Leipzig, 1902.
9. A. Föppl, L. Föppl, *Drang und Zwang*, I, München u. Berlin, 1920.

10. E. Beltrami, *Sulla conditioni di resistenza dei corpi elastici*, Rend. Ist. Lomb., II, 18, 1885.

11. B. P. High, *Strain-energy function and elastic limit*, Rpt. Brit.Association for the Advancement of Sciences, 486, 1919, Engineering, CIX, 158–160, 1920.

12. W. Burzyński, *Studium nad hipotezami wytężenia*, Lwów, 1928 (cf. also: Dzieła Wybrane, t. I, PWN, Warszawa, 1982, 67–258; and English translation: *Selected passages from Włodzimierz Burzyński's doctoral dissertation Study on Material Effort Hypotheses*, Engineering Transactions, **57**, 3–4, 185–215, 2009.

13. M. M. Filonenko–Borodich, *Mechanical theories of strength* [in Russian], Izd. MGU, Moskva 1961.

14. V. V. Novozhilov, *About physical meaning of stress invariants* [in Russian], Prikl. Mat. Mekh, **15**, 2, 1951.

15. W. Olszak, W. Urbanowski, *The plastic potential and the generalized distortion energy*, Arch. Mech. Stos., **8**, 4, 1956.

16. W. Olszak, J. Ostrowska–Maciejewska, *The plastic potential in the theory of anisotropic elastic-plastic bodies*, Engineering Fracture Mechanics, **21**, 625–632, 1985.

17. A. L. Cauchy, *Sur les équations différentielles d'equilibre ou de mouvement pour un systéme de points matériels sollicités par des forces d'attraction ou de répulsion mutuelle*, Exercises de mathématiques, **4**, 129–139, 1829.

18. I. Todhunter, K. Pearson, *A history of the theory of elasticity and of the strength of materials*, I, Univ. Press, Cambridge, 1886.

19. J. Rychlewski, ≪*CEIIINOSSSTTUV*≫ *Mathematical structure of elastic bodies* [in Russian], Institut Problem Mekhaniki AN SSSR, preprint 217, 1983.

20. V. A. Lomakhin, *About the non-linear theory of elasticity and plasticity of anisotropic continua* [in Russian], Izv. AN SSSR, OTN Mekh. i Mashinnostr., **4**, 60–64, 1960.

21. J. Rychlewski, *Dinh laut Hooke hong cach mo ta moi*, Bao cao tai HOI THAO QUOC TE CO HOC, Hanoi, **4**, 252–256, 1983.

22. J. Rychlewski, *On thermoelastic constants*, Arch. Mech., **36**, 1, 77–95, 1984.

23. J. Rychlewski, *About Hooke's law* [in Russian], Prikl. Mat. Mekh., 48, 420–435, 1984; also in English: J. Applied Mathematics and Mechanics, **48**, 303–314, 1984.

24. J. Rychlewski, *About the non-coaxiality of elastic deformations and stresses* [in Russian], Izv. RAN MTT, 101–104, 1984.

25. A. I. Malcev, *Foundations of Linear Algebra* [in Russian], Gostexizdat, Moskva, 1956.

26. I. M. Glazman, Yu. I. Lyubich, *Finite dimensional linear analysis* [in Russian], Nayka, Moskva, 1969.

27. R. Mises, *Mechanik der plastischen Formänderung von Kristallen*, ZAMM, **8**, 161–185, 1928.

28. R. Hill, *The Mathematical Theory of Plasticity*, Clarendon Press, Oxford, 1950.

29. I. M. Gelfand, *Lectures on Linear Algebra* [in Russian], Nayka, Moskva, 1971.

30. J. Rychlewski, *About quadratic limit criteria of stressed state* [in Russian] – manuscript.

Summary

The well-known yield condition for isotropic materials, known as the M.T. Huber (and R. von Mises, H. Hencky) yield condition, has oryginally been proposed by J.C. Maxwell (see Appendix 2) in 1856. Maxwell and Huber atttributed the following physical sense to the criterion: the material stays elastic as long as the distortion energy does not reach the critical value. The attempt made by W. Olszak and W. Urbanowski, who tried to generalize the criterion to anisotropic bodies, is not convincing owing to the fact that, in the case of anisotropic media, decomposition of the total elastic energy into the parts connected with the change of volume and the change of shape is impossible.

The notion of "energy-ortogonal" states of stress is introduced in the paper. One state of stress is energy-orthogonal to another state of stress if the first one does not perform any work along the deformations produced by the other. The following theorem is proved: each limit criterion may be represented as a certain condition imposed upon a linear combination of elastic energies corresponding to a uniquely determined (for the given material) pari-wise energy-orthogonal, additive components of the total state of stress. Hence, each quadratic criterion has a definite energy interpretation. Moreover, it is shown that each limit criterion may be written in the form of an ineguality bounding the accumulated elastic energy. Considered are also the problems of possible forms of coupling of elastic properties of materials with the corresponding limit criteria.

INSTYTUT PODSTAWOWYCH
PROBLEMÓW TECHNIKI PAN
UL. ŚWIĘTOKRZYSKA 21
00-049 WARSZAWA

Obtained March 25, 1984

Translator's note

My schoolmate, friend and tutor in science Jan Rychlewski is bodily still among us, but his heavy disease makes not possible the author's supervision over the translation. Keeping this in mind, the translator tried to avoid, if possible, any more important departures from a literal translation. This was a difficult task. Russian phrasing of Jan was extremely rich and colorful. He is in fact a Russian native speaker. Being an ethnic Pole born in USSR, he was in early childhood separated from his family in result of sad events of year 1938. During his mature years spent in Poland, he maintained close connection with the Russian culture and language. Neither English nor Russian is a translator's native tongue (however he believes that his Russian is much better than English). Thus, despite the translator's efforts to make his best, sometimes the results may occur to be odd. The whole responsibility for his lack of competence in the linguistic matter, the translator takes on his shoulders. The only justification can be his will to pay his debt of gratitude to the diseased friend.

Andrzej Blinowski

Received April 7, 2011.