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ANALYSIS OF PRESTRESSED VISCO-ELASTIC STRUCTURES BY THE VIRTUAL DISTORTION METHOD

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New method of modeling of visco-plastic structures is presented. A fictitious virtual (initial) distortions (controlled in time in a prescribed way) introduced to elastic structure can cause the same resultant strains and stresses as in a visco-elastic structure without virtual distortions but under the same loading. The presented method can describe structures with nonhomogeneously distributed mechanical properties. It allows also to take into account the states of prestressing. Therefore, it can be a helpful tool to solve problems of optimal prestress for visco-elastic structures.

1. Introduction – the concept of the virtual distortion method

The aim of this paper is to discuss an application of the Virtual Distortion Method (VDM - cf.[7]) to the analysis of prestressed, visco-elastic structures. The VDM method is based on the concept of the *virtual* distortions ε^0 (incompatible in general) causing a self-equilibrated state of *initial* stresses σ^R and a state of compatible initial deformations ε^R. The idea of modeling of nonlinear structural behavior by superposing self-equilibrated stresses is not new (cf. eg.[8]). However, introducing a new quantity of virtual distortions causing these self-equilibrated stresses allows to formulate effective rules of control of modelling processes.

An appropriate state of fictitious virtual distortions can cause the initial states simulating the behavior of the structure with a modified (in a prescribed way) physical or geometrical properties (cf.[4, 5]). On the other hand, however, the virtual distortions can describe a real *prestress* distortions causing prescribed modifications of the stress distribution. Therefore, the most natural way to describe the problem of prestress of visco-elastic structures is to consider a combined virtual distortion field composed of two components; the first one – denoting the *real prestress distortions*, and the second – denoting *fictitious distortions* simulating the rheological behavior of the structure (cf. [6]).

Any state of virtual distortions $\boldsymbol{\varepsilon}^0$ induces in the structure a self-equilibrated state of initial stresses $\boldsymbol{\sigma}^R$ satisfying the equilibrium equations:

$$\begin{aligned} \operatorname{div}\boldsymbol{\sigma}^R &= \mathbf{0} \ \text{ in } V, \\ \boldsymbol{\sigma}^R\mathbf{n} &= \mathbf{0} \ \text{ on } \mathcal{A}_p, \end{aligned} \tag{1.1}$$

inside the volume V of the body and on the part $\mathcal{A}_p$ of the boundary with defined loading conditions, respectively. On the other hand, the state of virtual distortions causes a compatible state of deformations $\boldsymbol{\varepsilon}^R$ that can be expressed by a field of displacements $\mathbf{u}^R$:

$$\begin{aligned} \boldsymbol{\varepsilon}^R &= \operatorname{grad}^s \mathbf{u}^r \quad \text{ in } V, \\ \mathbf{u}^R &= \mathbf{0} \qquad\quad\ \text{ on } \mathcal{A}_u, \end{aligned} \tag{1.2}$$

where $\mathcal{A}_u$ denotes the part of the boundary with a fixed displacements and $\operatorname{grad}^s(..)$ denotes the symmetric part of the gradient. The complete set of equations describing the states caused by virtual distortions is composed of the above equations (Eqs.(1.1), (1.2)) and the following constitutive relation:

$$\boldsymbol{\sigma}^R = \mathbf{A}(\boldsymbol{\varepsilon}^R - \boldsymbol{\varepsilon}^0), \tag{1.3}$$

where $\mathbf{A}$ denotes the tensor of elasticity.

Superposing the initial states with deformations $\boldsymbol{\varepsilon}^L$ and stresses $\boldsymbol{\sigma}^L$ caused by the external load:

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}^L + \boldsymbol{\sigma}^R, \\ \boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}^L + \boldsymbol{\varepsilon}^R, \end{aligned} \tag{1.4}$$

the set of equations describing the structure loaded by the external loads ($\mathbf{f}$ in the area V and $\mathbf{p}$ on the part $\mathcal{A}_p$ of the boundary) and virtual distortions take the following form:

$$\begin{aligned} &\operatorname{div}\boldsymbol{\sigma}\mathbf{f} = \mathbf{0} && \text{in } V, \\ &\boldsymbol{\sigma}\mathbf{n} = \mathbf{p} && \text{on } \mathcal{A}_p, \\ &\boldsymbol{\varepsilon} = \operatorname{grad}^s \mathbf{u} && \text{in } V, \\ &\mathbf{u} = \mathbf{0} && \text{on } \mathcal{A}_u, \\ &\boldsymbol{\sigma} = \mathbf{A}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0) && \text{in } V. \end{aligned} \tag{1.5}$$

The concept of the VDM method applied in this paper for description as well prestress distortions (β^0) as the distortions simulating the rheological behavior of truss structures will be applied in a separate paper to the problem of plates prestressed by a family of fibers.

2. Formulation of the problem

Following the description of visco-elastic structures proposed by Świtka and Husiar [1,2], the constitutive equations for all particular cases of visco-elastic materials can be formulated as the general rule:

$$a_0\sigma + a_1\dot{\sigma} + a_2\ddot{\sigma} = b_0\varepsilon + b_1\dot{\varepsilon} + b_2\ddot{\varepsilon}. \tag{2.1}$$

Let us confine our considerations to Zener's constitutive law (cf. Fig.1). In

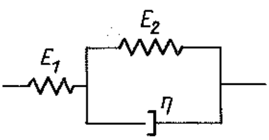


Fig. 1.

this case the constitutive rule (2.1) takes the following simpler form:

$$a_0\sigma + a_1\dot{\sigma} = b_0\varepsilon + b_1\dot{\varepsilon}. \tag{2.2$_1$}$$

The general constitutive equation $(1.5)_5$ can now be specified to simulate the visco-elastic case (cf. Eq. $(2.2)_1$, due to the material properties described by tensors $\mathbf{A}'$, $\mathbf{B}$, $\mathbf{C}$, respectively:

$$\sigma = \mathbf{A}'\varepsilon - \mathbf{B}\,\dot{\sigma} + \mathbf{C}\,\dot{\varepsilon} \tag{2.2$_2$}$$

or, (rearranging this formula) in the following form:

$$\sigma = \mathbf{A}'(\varepsilon - \varepsilon^0), \tag{2.3$_1$}$$

where

$$\varepsilon^0 = (\mathbf{A}')^{-1}(\mathbf{B}\,\dot{\sigma} - \mathbf{C}\,\dot{\varepsilon}). \tag{2.3$_2$}$$

Applying the considered model (Fig.1) to truss structure, the above tensors take the form of diagonal tensors with the following vectors describing the diagonal elements (where i denotes the number of the corresponding element):

$$A'_i = E^i_1 E^i_2/(E^i_1 + E^i_2), \quad B_i = \eta_i/(E^i_1 + E^i_2), \quad C_i = \eta_i E^i_1/(E^i_1 + E^i_2).$$

Then, the expression determining the virtual distortions (Eq. $(2.3)_2$) takes the form:

$$\varepsilon^0 = (\dot{\sigma}_i/E^i_1 - \dot{\varepsilon}_i)\eta_i/E^i_2. \tag{2.4}$$

Replacing the constitutive equation $(1.5)_5$ by the above relation, adding the equilibrium equation $(1.5)_{1,2}$ and the compatibility conditions formulated for the velocities of deformations (cf. Eqs. $(1.5)_{3,4}$)

$$\begin{aligned} \dot{\varepsilon} &= \operatorname{grad}^s \dot{\mathbf{u}} && \text{in } V, \\ \dot{\mathbf{u}} &= \mathbf{0} && \text{on } A_u \end{aligned} \tag{2.5}$$

and adding the initial conditions, the following simulation rule can be formulated.

The evolution of strains and stresses for the visco-elastic structure with the Zener's material properties (Eq. (2.2)) *is the same as for the linear elastic structure with the virtual distortions generated in a controlled way (it means that $\sigma = \mathbf{A}'(\varepsilon - \varepsilon^0)$, where $\mathbf{A}'$ and the virtual distortions ε^0 are defined by* Eqs.$(2.2)_2$ and $(2.3)_2$, *respectively).*

Assuming that, additionally, the prestress distortions β^0_i can be generated in the structure (cf. Fig. 2), the definition of distortions (2.4) in the constitutive relation (2.3) should be modified as follows:

$$\varepsilon^0_i = (\dot{\sigma}_i/E^i_1 - \dot{\varepsilon}_i + \dot{\beta}^0_i)\eta_i/E^i_2 + \beta^0_i. \tag{2.6}$$

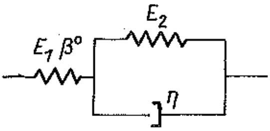


FIG. 2.

It can be noted that, in the case of homogeneous (E^i_1 =const, E^i_2 =const, η_i =const) and unprestressed structure ($\beta^0_i \equiv 0$) loaded by constant forces, the problem of evolution of the visco-elastic deformations $(1.5)_{1-4}$, (2.5), (2.3), (2.6) leads to the following differential equation

$$(2.7)_1 \qquad \varepsilon_i + \dot{\varepsilon}_i \eta_i / E_2^i = \sigma_i (E_1^i + E_2^i)/E_1^i E_2^i = \text{const},$$

with ε_i and $\dot{\varepsilon}_i$ satisfying the compatibility conditions $(1.5)_{3,4}$ and (2.5), respectively, and σ_i satisfying the equilibrium conditions $(1.5)_{1,2}$. The deformations are growing up while the stresses are constant in time in this case.

On the other hand, in the case of a homogeneous structure loaded only by a constant prestress distortions, the problem of evolution of the prestress effect $(1.5)_{3,4}$, (2.5), (2.3), (2.6) leads to the following differential equation

$$(2.7)_2 \qquad \sigma_i + \dot{\sigma}_i \eta_i /(E_1^i + E_2^i) = (\varepsilon_i - \beta_i^0)/E_1^i E_2^i/(E_1^i + E_2^i) = \text{const},$$

where ε_i satisfies $(1.5)_{3,4}$ and $\sigma_i, \dot{\sigma}_i$ are self-equilibrated states. The stresses are decreasing while the deformations are constant in time in this case.

3. Numerical algorithm for rheological analysis

Let us restrict our further discussion to the case of truss structures. The numerical procedure based on the VDM method combined with the FEM method requires the composition of the global stiffness matrix and calculation of its inverse matrix. The first matrix $(\mathbf{K}_1)$ is the matrix corresponding to the linear elastic properties of elements described by the elasticity vector E_1^i. The second one $(\mathbf{K}_2)$ is the matrix corresponding to the final elastic properties reached after infinite time (described by $E_i^f = E_1^i E_2^i/(E_1^i + E_2^i)$) for all elements.

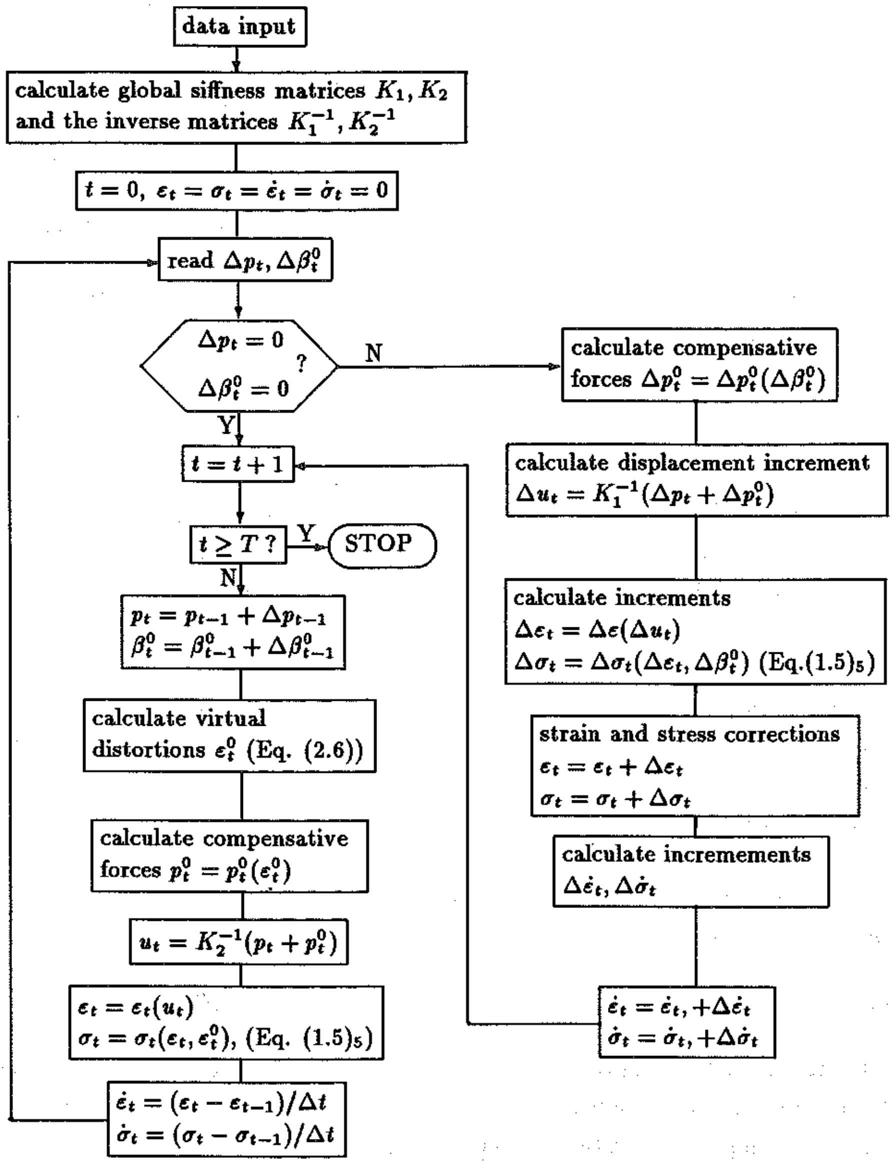
The response of the structure to the load increment $\Delta p_i, \Delta\varepsilon_i^0$ is due to the elastic solution related to the stiffness matrix $\mathbf{K}_1$. The corresponding increment of time rates of stresses and strains can be calculated from the constitutive equations (2.3) and (2.4)

$$(3.1) \qquad \eta_i E_1^i \Delta\dot{\varepsilon}_i - \eta_i \Delta\dot{\sigma}_i = (E_1^i + E_2^i)\Delta\sigma_i - E_1^i E_2^i \Delta\varepsilon_i,$$

where the proportion between $\Delta\dot{\varepsilon}_i$ and $\Delta\dot{\sigma}_i$ corresponds to the proportion between the part of stress increment $\Delta\sigma_i^1$ caused by the force loading ($\Delta\sigma_i^1 = E_1^i \Delta\varepsilon_i^1$) and the self-equilibrated part of stress increment $\Delta\sigma_i^2 = \Delta\sigma_i - \Delta\sigma_i^2$ caused by prestressing. After calculation of the response of the structure to the load increment, the further evolution of strains and stresses is computed according to the constitutive rule (2.3) and (2.4).

The algorithm of the method (assuming $\Delta\dot{p}_i = 0, \Delta\dot{\varepsilon}^0_i = 0$) is shown in the Table 1.

Table 1.



Let us note that the deformations caused by virtual distortions are calculated by the algorithm making use of so-called compensative forces, defined in the case of general formulation in the following form:

$$
(3.2)\qquad
\begin{aligned}
\mathbf{f}^0 &= -\mathrm{div}\,\dot{\mathbf{A}}\boldsymbol{\varepsilon}^0 && \text{in } V,\\
\mathbf{p}^0 &= \mathbf{A}\boldsymbol{\varepsilon}^0\mathbf{n} && \text{on } \mathcal{A}_u.
\end{aligned}
$$

However, in the case of a truss structure, the distinction between the boundary and the body forces can be neglected; therefore, let us denote all loading forces as $\mathbf{p}$.

The algorithm for calculation of viscous evolution of the states allows to compute (for all time steps) the stresses $\boldsymbol{\sigma}_i$ and the deformations $\boldsymbol{\varepsilon}_i$ at the time step t knowing the velocities of deformations $\dot{\boldsymbol{\varepsilon}}_{t-1}$ and stresses $\dot{\boldsymbol{\sigma}}_{t-1}$ at the previous time step $t-1$. Then, the calculation of the velocities of stresses and deformations at the time step t is performed due to the following approximative rule:

$$
(3.3)\qquad
\begin{aligned}
\dot{\boldsymbol{\sigma}}_t &= (\boldsymbol{\sigma}_t - \boldsymbol{\sigma}_{t-1})/\Delta t\,,\\
\dot{\boldsymbol{\varepsilon}}_t &= (\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}_{t-1})/\Delta t\,.
\end{aligned}
$$

As to the numerical cost of computation, the proposed method is comparable with the method proposed by Świtka [2] and applied by Olejniczak [3], where the influence of the previous time steps is taken into account not through the virtual distortions, but through the so-called transfer matrix.

4. Example of truss structure

Let us illustrate our considerations by the example of simple truss structure shown in Fig. 3. The truss is loaded by the constant force P and by the prestress distortion β^0 generated in the element No. 4 at the initial moment t_0.

Assuming the following material properties for all members: $E_1 = 10^6$ MPa, $E_2 = 0.5\cdot 10^6$ MPa, $\eta = 5\cdot 10^6$MPah, the cross-sectional areas for elements $A_1 = A_2 = A_3 = 0.0289\,\mathrm{cm}^2$, $A_4 = A_5 = 0.0201\,\mathrm{cm}^2$ and the load value $P = 1\,\mathrm{KN}$, the elastic response of the structure gives the following

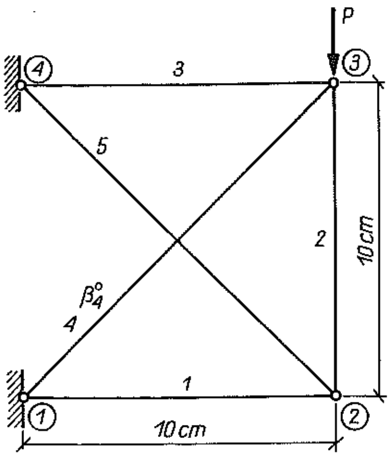


FIG. 3.

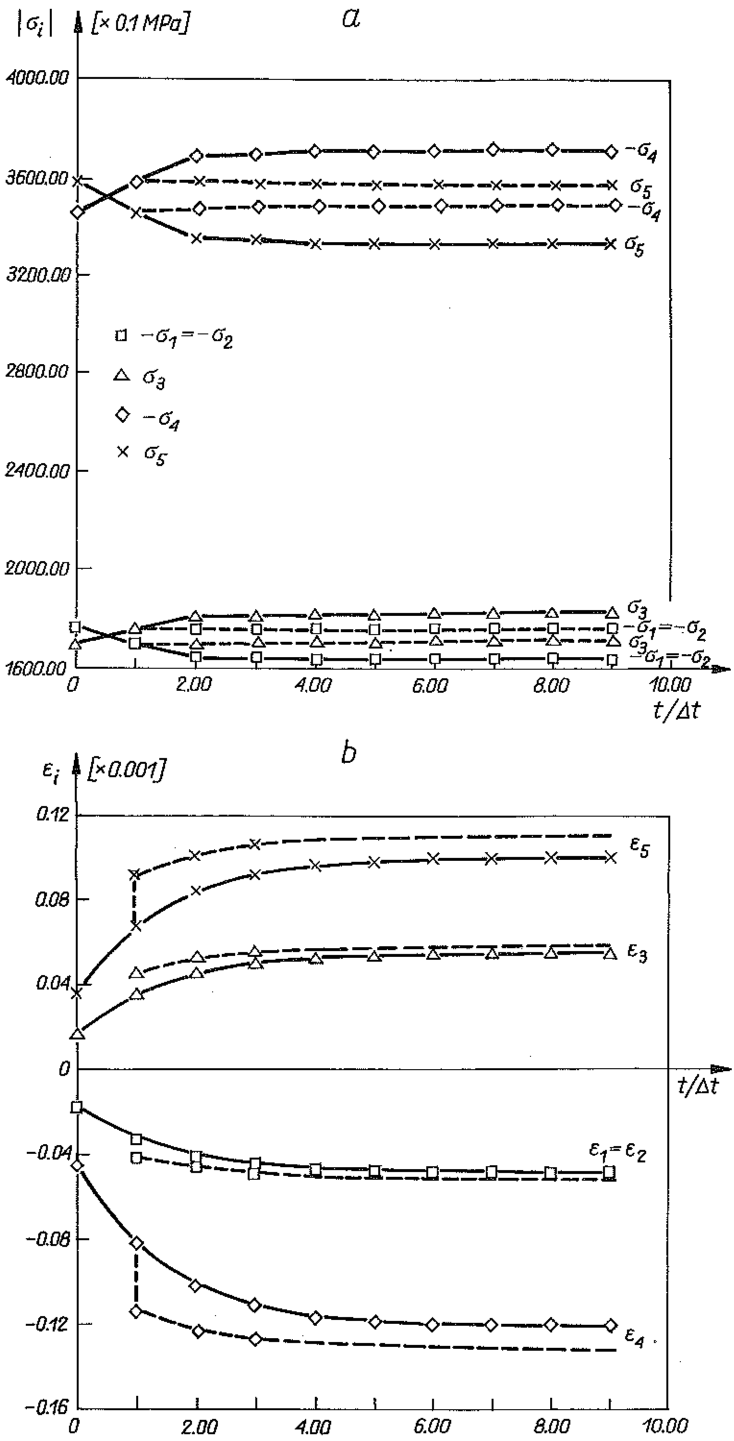
distribution of stresses and strains:

$$
\begin{aligned}
\sigma_1^L &= -157.45\,\mathrm{MPa}, & \varepsilon_1^L &= -0.0001575,\\
\sigma_2^L &= -157.45\,\mathrm{MPa}, & \varepsilon_2^L &= -0.0001575,\\
\sigma_3^L &= -188.53\,\mathrm{MPa}, & \varepsilon_3^L &= -0.0001885,\\
\sigma_4^L &= -383.28\,\mathrm{MPa}, & \varepsilon_4^L &= -0.0003833,\\
\sigma_5^L &= 320.09\,\mathrm{MPa}, & \varepsilon_5^L &= 0.0003201,
\end{aligned}
\tag{4.1}
$$

The numerical algorithm is general and can solve problems with non-uniformly distributed material properties. However, as the main considered problem is concentrated on prestressing, it is assumed that the material properties are uniformly distributed, what makes the problem of discussion of influence of the prestress effect on final strain and stresses easier.

Applying the prestress distortion in the element No.4 $\beta_4^0 = -0.0001038$ the initial, elastic response of the structure gives the following, modified stress $\sigma = \sigma^L + \sigma^R$ and strain $\varepsilon = \varepsilon^L + \varepsilon^R$ distribution:

$$
\begin{aligned}
\sigma_1 &= -176.10\,\mathrm{MPa}, & \varepsilon_1 &= -0.000176,\\
\sigma_2 &= -176.10\,\mathrm{MPa}, & \varepsilon_2 &= -0.000176,\\
\sigma_3 &= 169.89\,\mathrm{MPa}, & \varepsilon_3 &= 0.000170,\\
\sigma_4 &= -345.37\,\mathrm{MPa}, & \varepsilon_4 &= -0.000449,\\
\sigma_5 &= 358.00\,\mathrm{MPa}, & \varepsilon_5 &= 0.000358,
\end{aligned}
\tag{4.2}
$$



a
$|\sigma_i|$ [×0.1 MPa]
4000.00
3600.00
3200.00
2800.00
2400.00
2000.00
1600.00
0
2.00
4.00
6.00
8.00
10.00
$t/\Delta t$
$-\sigma_1=-\sigma_2$
σ_3
$-\sigma_4$
σ_5
b
ε_i [×0.001]
0.12
0.08
0.04
0
−0.04
−0.08
−0.12
−0.16
ε_5
ε_3
$\varepsilon_1=\varepsilon_2$
ε_4

FIG. 4.

The stress concentration in element No.4 ($\sigma_4^L = -383.28\,\mathrm{MPa}$) has been reduced to the maximal stress value in the element No.5 ($\sigma_5 = \bar{\sigma} = 358.00$-MPa) as the instantaneous effect of the prestressing.

Applying the time step $\Delta t = 20\,\mathrm{h}$, the VDM simulation algorithm leads to the solution shown in Fig. 4. After 10 time steps the solution very close to the elastic solution for the structure composed of elements made from material with the Young's modulus $E' = E_1 E_2/(E_1 + E_2)$ has been reached. The evolution of stresses and strains in truss elements is shown in Fig. 4a and Fig. 4b, respectively (continuous lines).

5. Discussion

The simple example discussed above shows the necessity of corrections of prestress in real visco-elastic structures. The truss prestressed in the initial moment to satisfy the constraints:

$$|\sigma_i| \le \bar{\sigma} \tag{5.1}$$

usually needs corrections of this prestress after several time steps. For example, if $\bar{\sigma} = 358.00\,\mathrm{MPa}$, the constraint (5.1) is violated in the second time step of the rheological process for the above truss structure and, therefore, the distortion β_4^0 has to be corrected. Generally speaking (generating time-dependent states of distortions $\beta^0 = \beta(t)$), the problem of the optimal prestress of visco-elastic truss structures can be formulated as the following rule for each time step:

$$\min \sum_i (\Delta\beta_i^0)^2 \tag{5.2}$$

subject to the constraint (5.1) and the set of Eqs. $(1.5)_{3,4}$, (2.3), (2.5) and (2.6).

According to this rule, the solution of the problem discussed in Sect. 4 would require the determination of the increase of the distortions $\Delta\beta_4^0$ (solving the condition $\sigma_4 = \bar{\sigma}$) in each time step greater than 1. We can expect the evolution of the prestress distortion β^0 resembling the evolution of plastic distortions generated in the visco-elastic-plastic structure. Therefore, analogously to the solution of the optimal prestress problem for elastic structures (cf. [7]) we can expect an analogy between the solution of optimal prestress problem for visco-elastic structure and the analysis of visco-elasto-plastic structures. The further discussion of this problem will be presented in a separate paper.

However, the second, and probably more interesting (from the practical point of view) way of formulation of the optimal prestress problem is the following. Design the time-dependent process of prestressing of the structure with stresses constrained by formula (5.1) applying as small number of inspections (corrections of prestressing) and as long time intervals between them as possible. This heuristic formulation can be replaced by the following optimization routine:

$$\min \max \left[\sigma_i \operatorname{sgn}(\dot{\sigma}_i)\right] \tag{5.3}$$

subject to Eq.(5.1), resolved whenever constraint (5.1) is violated in the rheological process. The optimal prestress designed according to this formulation for the truss example discussed in Sect.4 requires one inspection after the first time step. The prestress distortion is increased to the value $\beta_4^0 = -0.0002936$ modifying the stress distribution in such a way that other inspections are not necessary (cf.Figs. 4a and 4b, broken lines).

Prestress due to the formulation (5.2) leads to the solution with the smallest quantities of distortion corrections but it requires inspections in each time step, similarly to the plastic distortions growing up in a rheological process for visco-elasto-plastic structure.

As we have discussed above, the prestress for visco-elastic structures requires new formulations of the problem and the VDM simulation technique can be helpful in solving these design examples.

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