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ANALYSIS OF COMPLEX NONLINEAR DYNAMIC SYSTEMS BY MEANS OF PARTIAL MODELS. CONVERGENCE OF ITERATIVE PROCEDURES

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Problems of iterative convergence in the analysis of complex dynamic systems with the use of partial models are dealt with. Nonlinear systems are considered. An example of some nonlinear system and the suitable iteration process are presented. The time analysis is also dealt with. On the assumption that the solutions of the system can by expanded in power series with respect to a small parameter, the general condition of convergence is formulated. Limitations for the successive derivatives of nonlinear stiffness characteristics are determined as a consequence of the convergence conditions of the procedures.

1. Introduction

Growing demand to lower the level of noise, vibrations and to increase the durability of the machinery make the researchers create and analyze some more and more complex dynamic models of these machines. Current computational techniques, used both to calculate and to measure, make it possible to construct models of several scores or even hundreds of degrees of freedom and take into account various nonlinearities of the system under consideration. However, the use of such complicated models creates certain difficulties. Identification of a large number of parameters entering the models is both cumbersome and expensive. Interpretation of the results obtained is not an easy task. The calculation procedure may appear difficult and probability increases of making "unfriendly" errors in the computational procedures that are very difficult to discover. Those difficulties vanish when some simpler models are employed.

In the paper [2] a concept of an analysis of complex dynamic systems with the use of partial models was put forward. It makes it possible to preserve a high degree of complexity of vibration phenomena and, at the same time, to use relatively simple computational means. The main idea behind this method is as follows:

- a full model (a complex one) is split up into partial (simpler) models by using an analysis of coupling of partial models,
- analysis of the full model by means of partial ones their couplings are considered to be weak, and thus they can be treated as perturbations of successive iterations.

In general, two types of partial models can be constructed:

- describing a certain group of vibrations caused by various structural parts of a given machine,
- describing vibrations of particular parts.

Proper choice of partial models depends on the type of particular situation and the aim of analysis [3].

A characteristic feature of the proposed method is its iterative manner of calculations. Thus the convergence of computational procedures constitutes, from the applicability point of view, the main problem to be dealt with. Such analysis with respect to linear system was made in [4] where the conditions for, and domains of, convergence of iterative procedures were determined, both for conservative and dissipative systems.

In this paper a problem of convergence in the case of nonlinear systems is considered.

2. Formulation of the problem

To focus our attention on the analysis of the influence of nonlinearity on the convergence of the iterative computational procedure let us assume a simple, two-mass model. Elastic characteristic of an element connecting two masses is assumed to be nonlinear, i.e. the nonlinearity is present along the decomposion line of the element. For the sake of generality, no constraints will be imposed on the type of this nonlinearity. Assume that the decomposition is made in the same manner as in the case of weak associations between masses in the complex model [2]. Let us sepatare from a nonlinear element (in an arbitrary manner) a linear component with the stiffness k_2, Fig. 1.

Equations for vibrations of the model have the form

$$
\begin{aligned}
m_1\ddot{z}_1 + z_1(k_1 + k_2) &= k_1 z_0 + k_2 z_2 + \varphi(z_2 - z_1), \\
m_2\ddot{z}_2 + z_2 k_2 &= k_2 z_1 - \varphi(z_2 - z_1).
\end{aligned}
\tag{2.1}
$$

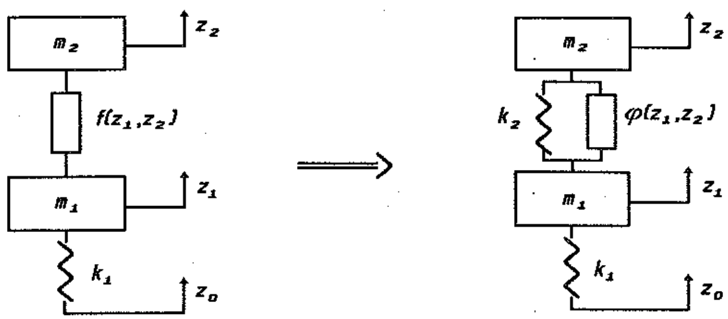

FIG. 1. Initial and transformed forms of a model.

This model will be split up into two partial models, the vibrations of the first one being expressed by Eq. $(2.1)_1$, and of the other one by Eq. $(2.1)_2$.

Assume that the solutions z_1, z_2 and the frequencies ω_1^2, ω_2^2 can be represented by power series with respect to a parameter ε

$$(2.2)\qquad \begin{aligned} z_1 &= y_{01} + \varepsilon y_{11} + \varepsilon^2 y_{21} + \ldots + \varepsilon^i y_{i1}\,, \\ z_2 &= y_{02} + \varepsilon y_{12} + \varepsilon^2 y_{22} + \ldots + \varepsilon^i y_{i2}\,; \end{aligned}$$

$$(2.3)\qquad \begin{aligned} \omega_1^2 &= \omega_{01}^2 + \varepsilon\omega_{11}^2 + \varepsilon^2\omega_{21}^2 + \ldots + \varepsilon^i\omega_{i1}^2\,, \\ \omega_2^2 &= \omega_{02}^2 + \varepsilon\omega_{12}^2 + \varepsilon^2\omega_{22}^2 + \ldots + \varepsilon^i\omega_{i2}^2\,, \end{aligned}$$

and that the parameter ε is also present in the coupling terms of the first equation and in the nonlinear terms. Let us rearrange the system (2.1) as follows:

$$(2.4)\qquad \begin{aligned} \ddot{z}_1 + \omega_{01}^2 z_1 &= \frac{k_1}{m_1} z_0 + \underline{\frac{\varepsilon K}{m_1} z_2 + \frac{\varepsilon}{m_1} g(z_2 - z_1)}, \\ \ddot{z}_2 + \omega_{02}^2 z_2 &= \frac{k_2}{m_2} z_1 - \underline{\frac{\varepsilon}{m_2} g(z_2 - z_1)}, \end{aligned}$$

where

$$(2.5)\qquad \varphi(z_2 - z_1) = \varepsilon g(z_2 - z_1)$$

and

$$(2.6)\qquad k_2 = \varepsilon K\,.$$

The underlined terms in Eqs. (2.4) will be treated as weak couplings.

If the function $g(z_1, z_2)$ has, in the neighbourhood of zero solutions y_{01}, y_{02}, continuous partial derivatives up to an arbitrary order, it can be expanded into Taylor's series,

$$
\begin{aligned}
(2.7)\qquad g(z_1,z_2) = g(y_{01},y_{02}) &+ \frac{\partial g}{\partial z_1}(\varepsilon y_{11} + \varepsilon^2 y_{21} + \varepsilon^3 y_{31} + \ldots) \\
&+ \frac{\partial g}{\partial z_2}(\varepsilon y_{12} + \varepsilon^2 y_{22} + \ldots) + \frac{1}{2}\left[\frac{\partial^2 g}{\partial z_1^2}(\varepsilon y_{11} + \varepsilon^2 y_{21} + \ldots)^2\right. \\
&+ \frac{\partial^2 g}{\partial z_2^2}(\varepsilon y_{12} + \varepsilon^2 y_{22} + \ldots)^2 + 2\frac{\partial^2 g}{\partial z_1 \partial z_2}(\varepsilon y_{11} + \varepsilon^2 y_{21} + \ldots) \\
&\left.\times (\varepsilon y_{12} + \varepsilon^2 y_{22} + \ldots)\right] + \frac{1}{6}\left[\frac{\partial^3 g}{\partial z_1^3}(\varepsilon y_{11} + \varepsilon^2 y_{21} + \ldots)^3\right. \\
&+ \frac{\partial^3 g}{\partial z_2^3}(\varepsilon y_{12} + \varepsilon^2 y_{22} + \ldots)^3 + 3\frac{\partial^3 g}{\partial z_1 \partial z_2^2}(\varepsilon y_{11} + \varepsilon^2 y_{21} + \ldots) \\
&\times(\varepsilon y_{12} + \varepsilon^2 y_{22} + \ldots)^2 + 3\frac{\partial^3 g}{\partial z_1^2 \partial z_2}(\varepsilon y_{11} + \varepsilon^2 y_{21} + \ldots)^2 \\
&\left.\times (\varepsilon y_{12} + \varepsilon^2 y_{22} + \ldots)\right] + \ldots .
\end{aligned}
$$

On rearranging with respect to the powers of ε, the above expression takes the form

$$
(2.8)\qquad g(z_1,z_2) = g(y_{01},y_{02}) + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \varepsilon^4 A_4 + \ldots,
$$

where

$$
(2.9)\qquad A_1 = \frac{\partial g}{\partial z_1} y_{11} + \frac{\partial g}{\partial z_2} y_{12},
$$

$$
(2.10)\qquad A_2 = \frac{\partial g}{\partial z_1} y_{21} + \frac{\partial g}{\partial z_2} y_{22} + \frac{1}{2}\frac{\partial^2 g}{\partial z_1^2} y_{11}^2 + \frac{1}{2}\frac{\partial^2 g}{\partial z_2^2} y_{12}^2 + \frac{\partial^2 g}{\partial z_1 \partial z_2} y_{11} y_{12},
$$

$$
\begin{aligned}
(2.11)\qquad A_3 = \frac{\partial g}{\partial z_1} y_{31} &+ \frac{\partial g}{\partial z_2} y_{32} + \frac{\partial^2 g}{\partial z_1^2} y_{11} y_{21} + \frac{\partial^2 g}{\partial z_2^2} y_{12} y_{22} \\
&+ \frac{\partial^2 g}{\partial z_1 \partial z_2}(y_{11} y_{22} + y_{12} y_{21}) + \frac{1}{6}\frac{\partial^3 g}{\partial z_1^3} y_{11}^3 + \frac{1}{6}\frac{\partial^3 g}{\partial z_2^3} y_{12}^3 \\
&+ \frac{1}{2}\frac{\partial^3 g}{\partial z_1 \partial z_2^2} y_{11} y_{12}^2 + \frac{1}{2}\frac{\partial^3 g}{\partial z_1^2 \partial z_2} y_{11}^2 y_{12},
\end{aligned}
$$

$$
\begin{aligned}
(2.12)\qquad A_4 = \frac{\partial g}{\partial z_1} y_{41} &+ \frac{\partial g}{\partial z_2} y_{42} + \frac{\partial^2 g}{\partial z_1^2}\left(y_{11} y_{31} + \frac{1}{2} y_{21}^2\right) \\
&+ \frac{\partial^2 g}{\partial z_2^2}\left(y_{12} y_{32} + \frac{1}{2} y_{22}^2\right) + \frac{\partial^2 g}{\partial z_1 \partial z_2}(y_{11} y_{32} + y_{12} y_{31} + y_{21} y_{22}) \\
&+ \frac{1}{2}\frac{\partial^3 g}{\partial z_1^3} y_{11}^2 y_{21} + \frac{1}{2}\frac{\partial^3 g}{\partial z_2^3} y_{12}^2 y_{22} + \frac{1}{2}\frac{\partial^3 g}{\partial z_1 \partial z_2^2}(2 y_{11} y_{12} y_{22} + y_{12}^2 y_{21})
\end{aligned}
$$

(2.12) [cont.]
$$+\frac{1}{2}\frac{\partial^3 g}{\partial z_1^2 \partial z_2}(y_{11}^2 y_{22} + 2y_{11}y_{21}y_{12}) + \frac{1}{24}\frac{\partial^4 g}{\partial z_1^4}y_{11}^4 + \frac{1}{24}\frac{\partial^4 g}{\partial z_2^4}y_{12}^4 + \frac{1}{6}\frac{\partial^4 g}{\partial z_1^3 \partial z_2}y_{11}^3 y_{12} + \frac{1}{4}\frac{\partial^4 g}{\partial z_1^2 \partial z_2^2}y_{11}^2 y_{12}^2 + \frac{1}{6}\frac{\partial^4 g}{\partial z_1 \partial z_2^3}y_{11}y_{12}^3 .$$

Assume that the system is acted upon by a harmonic excitation in the form

$$z_0 = Z \sin \nu t.$$

The steady vibrations of the system will be considered. Periodicity of the solution is also assumed, i.e. no secular terms of the type $t \sin \nu t$ should appear in the solutions.

3. Realization of the iterative procedure

Assume that in the i-th iteration the terms of the expansion $g(z_1, z_2)$ into Taylor's series up to the i-th power of the parameter ε will be included, and that the i-th iteration yields an approximate solution up to the i-th power of the parameter ε. The course of the iterative procedure will be the following:

0-th iteration

(3.1)
$$\begin{aligned} \ddot{z}_1^{(0)} + \omega_{01}^2 z_1^{(0)} &= \frac{k_1}{m_1} z_0 \longrightarrow z_1^{(0)} = y_{01}; \quad \omega_1^2 = \omega_{01}^2 , \\ \ddot{z}_2^{(0)} + \omega_{02}^2 z_2^{(0)} &= \frac{k_2}{m_2} z_1^{(0)} \longrightarrow z_2^{(0)} = y_{02}; \quad \omega_2^2 = \omega_{02}^2 . \end{aligned}$$

1-st iteration

(3.2)
$$\begin{aligned} \ddot{z}_1^{(1)} + \omega_{01}^2 z_1^{(1)} &= \frac{k_1}{m_1} z_0 + \frac{\varepsilon K}{m_1} z_2^{(0)} + \frac{\varepsilon}{m_1} g(y_{01}, y_{02}) \\ &\hookrightarrow \quad z_1^{(1)} = y_{01} + \varepsilon y_{11}, \quad \omega_1^2 = \omega_{01}^2 + \varepsilon \omega_{11}^2 , \\ \ddot{z}_2^{(1)} + \omega_{02}^2 z_2^{(1)} &= \frac{k_2}{m_2} z_1^{(1)} - \frac{\varepsilon}{m_2} g(y_{01}, y_{02}) \\ &\hookrightarrow \quad z_2^{(1)} = y_{02} + \varepsilon y_{12}, \quad \omega_2^2 = \omega_{02}^2 + \varepsilon \omega_{12}^2 , \end{aligned}$$

2-nd iteration

(3.3)
$$\begin{aligned} \ddot{z}_1^{(2)} + \omega_{01}^2 z_1^{(2)} &= \frac{k_1}{m_1} z_0 + \frac{\varepsilon K}{m_1} z_2^{(1)} + \frac{\varepsilon^2}{m_1} A_1 \\ &\hookrightarrow \quad z_1^{(2)} = y_{01} + \varepsilon y_{11} + \varepsilon^2 y_{21}, \quad \omega_1^2 = \omega_{01}^2 + \varepsilon \omega_{11}^2 + \varepsilon^2 \omega_{21}^2, \\ \ddot{z}_2^{(2)} + \omega_{02}^2 z_2^{(2)} &= \frac{k_2}{m_2} z_1^{(2)} - \frac{\varepsilon}{m_2} A_1 \\ &\hookrightarrow \quad z_2^{(2)} = y_{02} + \varepsilon y_{12} + \varepsilon^2 y_{22}, \quad \omega_2^2 = \omega_{02}^2 + \varepsilon \omega_{12}^2 + \varepsilon^2 \omega_{22}^2. \end{aligned}$$

When the solutions appearing on the right-hand side are substituted to the system of differential equations belonging to a given iteration and are ordered according to the powers of ε, a condition of satisfaction of these equations for an arbitrary value of the parameter can be replaced by a condition to fulfil the equations at suitable powers of the parameter ε.

For example, consideration of the 2-nd iteration leads so the conclusion that the system obtained for the 0-th power of ε is identical with all the other iterations starting from the 1-st one, and so on. These systems have the form:

at ε°

$$L_0 = \begin{cases} \ddot{y}_{01} + \omega_1^2 y_{01} = \dfrac{k_1}{m_1} z_0 \longrightarrow y_{01}, \quad \omega_{01}^2, \\ \ddot{y}_{02} + \omega_2^2 y_{02} = \dfrac{k_2}{m_2} y_{01} \longrightarrow y_{02}, \quad \omega_{02}^2; \end{cases} \tag{3.4}$$

at ε^1

$$L_1 = \begin{cases} \ddot{y}_{11} + \omega_1^2 y_{11} = \omega_{11}^2 y_{01} + \dfrac{K}{m_1} y_{02} + \dfrac{1}{m_1} g(y_{01}, y_{02}) \rightarrow y_{11}, \ \omega_{11}^2, \\ \ddot{y}_{12} + \omega_2^2 y_{12} = \omega_{12}^2 y_{02} + \dfrac{k_2}{m_2} y_{11} - \dfrac{1}{m_2} g(y_{01}, y_{02}) \rightarrow y_{12}, \ \omega_{12}^2; \end{cases} \tag{3.5}$$

generally,

at ε^i

$$L_i = \begin{cases} \ddot{y}_{i1} + \omega_1^2 y_{i1} = \dfrac{K}{m_1} y_{i-1,2} + \displaystyle\sum_{k=1}^{i} \omega_{k1}^2 y_{i-k,1} + \dfrac{1}{m_1} A_{i-1} \rightarrow y_{i1}, \ \omega_{i1}^2, \\ \ddot{y}_{i2} + \omega_2^2 y_{i2} = \dfrac{k_2}{m_2} y_{i1} + \displaystyle\sum_{k=1}^{i} \omega_{k2}^2 y_{i-k,2} - \dfrac{1}{m_2} A_{i-1} \rightarrow y_{i2}, \ \omega_{i2}^2. \end{cases} \tag{3.6}$$

In addition, in each iteration a sequence of residuals will be obtained in the form

$$R_{i+p}^{(i)} = \begin{cases} \displaystyle\sum_{k=p}^{i} \omega_{k1}^2 y_{i+p-k,1}, \\ \displaystyle\sum_{k=p}^{i} \omega_{k2}^2 y_{i+p-k,2}. \end{cases} \tag{3.7}$$

They appear by the powers ε^{i+p}, where $p = 1, 2, \dots, i$.

The systems of equations to be solved at consecutive steps of the iterative procedure can be expressed as follows:

0-th iteration

$$L_0 = 0; \tag{3.8}$$

1-th iteration

$$L_0 + \varepsilon L_1 + \varepsilon^2 R_2^{(1)} = 0; \tag{3.9}$$

2-nd iteration

$$L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \varepsilon^3 R_3^{(2)} + \varepsilon^4 R_4^{(2)} = 0; \tag{3.10}$$

i-th iteration

$$L_0 + \varepsilon L_1 + \ldots + \varepsilon^i L_i + \varepsilon^{i+1} R_{i+1}^{(i)} + \varepsilon^{i+2} R_{i+2}^{(i)} + \ldots + \varepsilon^{2i} R_{2i}^{(i)} = 0. \tag{3.11}$$

The above relationships are satisfied for an arbitrary value of the parameter if

$$L_i = 0 \qquad \text{for} \qquad i = 0, 1, 2, \ldots . \tag{3.12}$$

Assume the constant term in Taylor's expansion (2.8) to vanish,

$$g(y_{01}, y_{02}) = 0 \tag{3.13}$$

and the characteristic of the nonlinear element to be a function of the displacement difference $(z_2 - z_1)$, i.e.

$$\frac{\partial^n g}{\partial z_1^n} = (-1)^n \frac{\partial^n g}{\partial z_2^n} . \tag{3.14}$$

Introduce the following notation:

$$T = \frac{Zk_1}{m_1}, \qquad U = \frac{Zk_1k_2}{m_1m_2}, \qquad W = \frac{Kk_2}{m_1m_2}, \tag{3.15}$$

and

$$P = \omega_2^2 - \omega_1^2, \qquad R = \omega_1^2 - \nu^2, \qquad S = \omega_2^2 - \nu^2 . \tag{3.16}$$

Now the solution to the system L_0 (relationship (3.4)) has the form

$$\begin{aligned} y_{01} &= e_{01} \sin \nu t = \frac{T}{R} \sin \nu t, \qquad \omega_{01}^2 = \frac{k_1 + k_2}{m_1}, \\ y_{02} &= e_{02} \sin \nu t = \frac{U}{RS} \sin \nu t, \qquad \omega_{02}^2 = \frac{k_2}{m_2} . \end{aligned} \tag{3.17}$$

Solution of the system L_1 (3.5) leads to

$$\begin{aligned} y_{11} &= e_{11}\sin\nu t = -\frac{TW}{PRS}\sin\nu t\,, \qquad & \omega_{11}^2 = -\frac{W}{P}\,,\\ y_{12} &= e_{12}\sin\nu t = 0\,, & \omega_{12}^2 = \frac{W}{P}\,. \end{aligned} \tag{3.18}$$

Introducing the notation

$$\frac{\partial g}{\partial z_2} = g' \tag{3.19}$$

to solve the system L_2, remembering the relation (3.14) and adopting a coefficient N_1 (in the case $N_1 = 1$), we can, in accordance with Eq. (2.9), write down

$$A_1 = N_1\, g'\,(y_{12} - y_{11}). \tag{3.20}$$

The solution to the system L_2 is

$$\begin{aligned} y_{21} &= e_{21}\sin\nu t = -\frac{TW}{P^2RS}X_{11}\sin\nu t\,, \qquad & \omega_{21}^2 = -\frac{W}{P^2}X_{11},\\ y_{22} &= e_{22}\sin\nu t = 0\,, & \omega_{22}^2 = \frac{W}{P^2}X_{12}, \end{aligned} \tag{3.21}$$

where

$$X_{11} = \frac{W}{P} + \frac{N_1 g'}{m_1}\,, \tag{3.22}$$

$$X_{12} = \frac{N_1 g' P}{k_2} + X_{11}. \tag{3.23}$$

Accounting for Eqs. (3.14), (3.19) and denoting

$$g'' = \frac{\partial^2 g}{\partial z_2^2} = \frac{\partial^2 g}{\partial z_1^2} = -\frac{\partial^2 g}{\partial z_1 \partial z_2} \tag{3.24}$$

the relationship (2.10) can be written down in the form

$$A_2 = g'\,(y_{22} - y_{21}) + \frac{1}{2}g''(y_{12} - y_{11})^2\,. \tag{3.25}$$

Assume that

$$\frac{1}{2}g''(y_{12} - y_{11})^2 \le (N_2 - 1)\,g'\,(y_{22} - y_{21}) \tag{3.26}$$

what means that

$$A_2 \le N_2\, g'\,(y_{22} - y_{21}) \tag{3.27}$$

and, instead of A_2 let us insert into the system L_3 the right-hand side of the above inequality. This means that a system with stronger perturbation will be analysed. When such a procedure turns out to be convergent, the procedure for the system L_3 with weaker perturbation A_2 will also be convergent. The value of the coefficient N_2 in (3.27) will be derived from the convergence condition of the procedure.

On replacing the expression A_2 by the right-hand side of the inequality (3.27), the solution of the system L_3 yields

$$\begin{aligned} y_{31} &= e_{31}\sin\nu t = -\frac{TW}{P^3RS}X_{11}X_{21}\sin\nu t\,, \quad & \omega_{31}^2 &= -\frac{W}{P^3}X_{11}X_{21}, \\ y_{32} &= e_{32}\sin\nu t = 0\,, & \omega_{32}^2 &= \frac{W}{P^3}X_{11}X_{22}, \end{aligned} \tag{3.28}$$

where

$$X_{21} = \frac{2W}{P} + \frac{N_2 g'}{m_1}\,, \tag{3.29}$$

$$X_{22} = \frac{N_2 g' P}{k_2} + X_{21}. \tag{3.30}$$

Remembering Eqs. (3.14), (3.19), (3.24) and denoting

$$g''' = \frac{\partial^3 g}{\partial z_2^3} = -\frac{\partial^3 g}{\partial z_1^3} = \frac{\partial^3 g}{\partial z_1^2 \partial z_2} = -\frac{\partial^3 g}{\partial z_1 \partial z_2^2}\,, \tag{3.31}$$

the relationship (2.11) for A_3 takes the form

$$A_3 = g'(y_{32} - y_{31}) + g''(y_{22} - y_{21})(y_{12} - y_{11}) + g'''(y_{12} - y_{11})^3\,. \tag{3.32}$$

Assume that

$$g''(y_{22} - y_{21})(y_{12} - y_{11}) + g'''(y_{12} - y_{11})^3 \le (N_3 - 1)g'(y_{32} - y_{31}), \tag{3.33}$$

that is

$$A_3 \le N_3\, g'(y_{32} - y_{31}), \tag{3.34}$$

and, similarly as before, instead of A_3 let us insert to the system L_4 the right-hand side of the above inequality. This means that the considered system will also be subject to some stronger perturbations.

Solving the succesive systems L_5, $L_6, \ldots$, we proceed in an analogous manner, i.e. instead of the expression A_{i-1} entering the system L_i we substitute the expression

$$A_{i-1} = N_{i-1}\, g'(y_{i-1,2} - y_{i-1,1}). \tag{3.35}$$

Finally, the solution in the i-th iteration has the form

$$
\begin{aligned}
z_1^{(i)} &= (e_{01} + \varepsilon e_{11} + \varepsilon^2 e_{21} + \ldots + \varepsilon^i e_{i1}) \sin \nu t, \\
z_2^{(i)} &= (e_{02} + \varepsilon e_{12} + \varepsilon^2 e_{22} + \ldots + \varepsilon^i e_{i2}) \sin \nu t,
\end{aligned}
\tag{3.36}
$$

and the frequencies ω_1^2, ω_2^2 are given by Eq. (2.3).

The values of the expressions e_{i1}, e_{i2}, ω_{i1}^2, ω_{i2}^2 and the values of the ratios of the two consecutive expressions for a number of initial iterations are given in Table 1.

Table 1.

i	ω_{i1}^2	e_{i1}	$\frac{\omega_{i+1,1}^2}{\omega_{i1}^2} = \frac{e_{i+1,1}}{e_{i1}}$	ω_{i2}^2	e_{i2}	$\frac{\omega_{i+1,2}}{\omega_{i2}^2}$
0	$\frac{k_1+k_2}{m_1}$	$\frac{T}{R}$	–	$\frac{k_2}{m_2}$	$\frac{U}{RS}$	–
1	$-\frac{W}{P}$	$-\frac{TW}{PRS}$	–	$\frac{W}{P}$	0	–
2	$-\frac{W}{P^2}X_{11}$	$-\frac{TW}{P^2RS}X_{11}$	$\frac{X_{11}}{P}$	$\frac{W}{P^2}X_{12}$	0	$\frac{X_{12}}{P}$
3	$-\frac{W}{P^3}X_{11}X_{21}$	$-\frac{TW}{P^3RS}X_{11}X_{21}$	$\frac{X_{21}}{P}$	$\frac{W}{P^3}X_{11}X_{22}$	0	$\frac{X_{11}X_{22}}{PX_{12}}$
4	$-\frac{W}{P^4}X_{11}X_{21}X_{31}$	$-\frac{TW}{P^4RS}X_{11}X_{21}X_{31}$	$\frac{X_{31}}{P}$	$\frac{W}{P^4}X_{11}X_{21}X_{32}$	0	$\frac{X_{21}X_{32}}{PX_{22}}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮

The magnitudes T, U, W, P, R, S in Table 1 are determined by the relations (3.15) and (3.16); X_{11}, X_{21}, X_{12}, X_{22} by the relations (3.22), (3.23), (3.29), (3.30), respectively.

In addition

$$
X_{31} = \frac{2W}{P} + \frac{WX_{11}}{PX_{21}} + \frac{N_3 g'}{m_1}, \qquad X_{32} = \frac{N_3 g' P}{k_2} + X_{31}, \tag{3.37}
$$

$$
X_{41} = \frac{2W}{P} + \frac{2WX_{11}}{PX_{31}} + \frac{N_4 g'}{m_1}, \qquad X_{42} = \frac{N_4 g' P}{k_2} + X_{41}. \tag{3.38}
$$

4. Convergence conditions for the procedure

4.1. *General criteria of convergence*

The iterative procedure for solwing the system (2.4) will be convergent provided the series of coefficients shown in Table 1 will also be convergent. It can be seen that for e_{i1} and ω_{i1}^2 the ratio of the next term to the given one is the same, so that the convergence condition of these series will be also the same. Identical series, but shifted by one iteration, appears in the expressions for ω_{i2}^2.

In order to prove the convergence of those solutions let us assume that there exists such an X_1 that

$$\left|\frac{X_1}{P}\right| \leq 1 \tag{4.1}$$

and

$$X_1 \geq X_{i1} \qquad \text{for} \qquad i = 1, 2, 3, \ldots . \tag{4.2}$$

When the coefficients N_i will be equal to

$$N_1 = \frac{1}{1}, \quad N_2 = \frac{2}{1}, \quad N_3 = \frac{5}{2}, \quad N_4 = \frac{14}{5}, \quad N_5 = \frac{42}{14}, \ldots \tag{4.3}$$

then the expressions X_{k1} will attain the values given in Table 2.

Table 2.

N_k	X_{k1}	X_{k2}
$N_1 = \frac{1}{1}$	$X_{11} = \frac{W}{P} + \frac{g'}{m_1}$	$X_{12} = \frac{g'P}{k_2} + X_{11}$
$N_2 = \frac{2}{1}$	$X_{21} = 2X_{11}$	$X_{22} = 2X_{12}$
$N_3 = \frac{5}{2}$	$X_{31} = \frac{5}{2}X_{11}$	$X_{32} = \frac{5}{2}X_{12}$
$N_4 = \frac{14}{5}$	$X_{41} = \frac{14}{5}X_{11}$	$X_{42} = \frac{14}{5}X_{12}$
$\vdots$	$\vdots$	$\vdots$

The ratios of the two consecutive coefficients appearing in the solutions (Table 1) will have in the case considered the values

$$\frac{1X_{11}}{1P}, \quad \frac{2X_{11}}{1P}, \quad \frac{5X_{11}}{2P}, \quad \frac{14X_{11}}{5P}, \quad \frac{42X_{11}}{14P}, \ \ldots . \tag{4.4}$$

They constitute a series whose terms are products of X_{11}/P and the numerical series N_i (relationship (4.3)). Convergence of the iteration procedure depends on the properties of this series. If it turns out to be divergent, then the ratios of the two consecutive coefficients appearing in the solutions would be larger and larger, thus the series constituting the solutions (relations (3.36) and (2.3)) would be divergent.

A pattern of creation of this series is presented in Table 3.

Table 3.

iter.		$\sum$	i
2	1×1	1	1
3	1×1 1×1	2	2
4	2×1 1×1 1×2	5	3
5	5×1 2×1 1×2 1×5	14	4
6	14×1 5×1 2×2 1×5 1×14	42	5
7	42×1 14×1 5×2 2×5 1×14 1×42	132	6
8	132×1 42×1 14×2 5×5 2×14 1×42 1×132	429	7
⋮	⋮	⋮	⋮

The values of the terms N_i are determined by the following recurrence formula:

$$N_i = \frac{a_i}{a_{i-1}}\,, \qquad i = 1,2,3,\ldots, \tag{4.5}$$

where

$$a_0 = 1, \qquad a_i = \sum_{k=0}^{i-1} a_k\, a_{i-k-1}, \qquad i = 1,2,\ldots\,. \tag{4.6}$$

This series is finite – see Fig. 2 – and its terms satisfy the condition

$$N_i < 4, \qquad i = 1,2,\ldots\,. \tag{4.7}$$

Thus the series (4.4) possesses a majorant in the form of a geometric series with the quotient

$$q = \frac{4X_{11}}{P}\,. \tag{4.8}$$

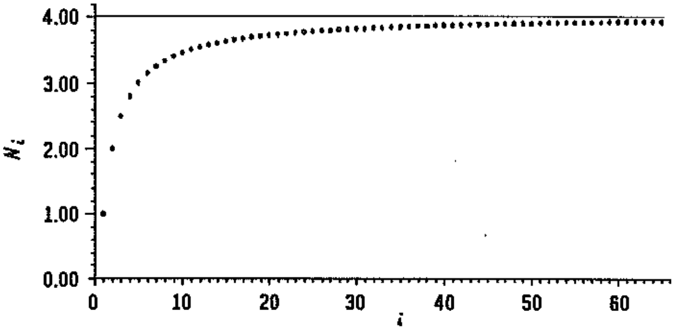

FIG. 2. Series N_i (relation (4.5)).

This means that the sought expression X_1 (relation (4.1)) is determined as

$$X_1 = 4X_{11}\,. \tag{4.9}$$

For the series of coefficients obtained in the i-th iteration of the solution z_1 (relation (3.36)), the majorant has the form

$$e_{01} + \varepsilon e_{11} + \varepsilon e_{11}(\varepsilon q) + \varepsilon e_{11}(\varepsilon q)^2 + \ldots + \varepsilon e_{11}(\varepsilon q)^i\,. \tag{4.10}$$

For the frequencies ω_1^2, ω_2^2 (relation (2.3)) similar series can be formed by majoring them.

The convergence condition for all these series is

$$|\varepsilon q| < 1. \tag{4.11}$$

On substituting here Eq. (3.22) and taking account of Eq. (2.5), we get

$$\left|\frac{4\varepsilon W}{P^2} + \frac{4\varphi'}{Pm_1}\right| < 1. \tag{4.12}$$

Making use of Eqs. (3.15) and (3.16) it can be concluded that the first term of the left-hand side of the above expression is recognized as Mandelstam's coefficient, which for the linear part of the system is determined by [1, 5].

$$\frac{4\varepsilon W}{P^2} = \sigma^2\,. \tag{4.13}$$

Thus the condition (4.12) can be finally rewritten to take the form

$$\left|\sigma^2 + \frac{4\varphi'}{Pm_1}\right| < 1. \tag{4.14}$$

The obtained form of the convergence condition for the iterative procedure makes it possible to formulate two important general conclusions:

1. The convergence criterion of the procedure for nonlinear systems contains a term σ^2 which indicates that the satisfaction of the convergence conditions by the linear part of the system is desirable, although not necessary. In addition, a term is present that imposes certain constraints on the value of the first derivative of the nonlinear part.

2. In some cases the convergence of the iterative procedure of the analysis of nonlinear system is possible even if its separated linear part remains divergent, i.e. when the second term in the condition (4.14) causes a decrease in $|\sigma^2|$.

4.2. Conditions referring to the first derivative of the nonlinear function $\varphi(z_1, z_2)$

Let us analyse what conditions for the first derivative of the nonlinear elastic characteristic φ' are imposed by satisfying of the convergence condition of the iterative procedure.

On denoting

$$G_1(S_1, S_3) = \frac{(S_1 - S_3 - S_1S_3)^2 - 4S_1^2S_3}{4S_1S_3(S_1 - S_3 - S_1S_3)}, \tag{4.15}$$

$$G_2(S_1, S_3) = \frac{-(S_1 - S_3 - S_1S_3)^2 - 4S_1^2S_3}{4S_1S_3(S_1 - S_3 - S_1S_3)}, \tag{4.16}$$

where

$$S_1 = \frac{k_2}{k_1}, \qquad S_3 = \frac{m_2}{m_1}, \tag{4.17}$$

and

$$\varphi' = \frac{d\varphi}{d\lambda} \tag{4.18}$$

as well as

$$\lambda = z_2 - z_1, \tag{4.19}$$

the condition (4.14) generates the following constraints on φ', depending on the parameters S_1 and S_3 of the system and on the stiffness k_2 of the separated linear part:

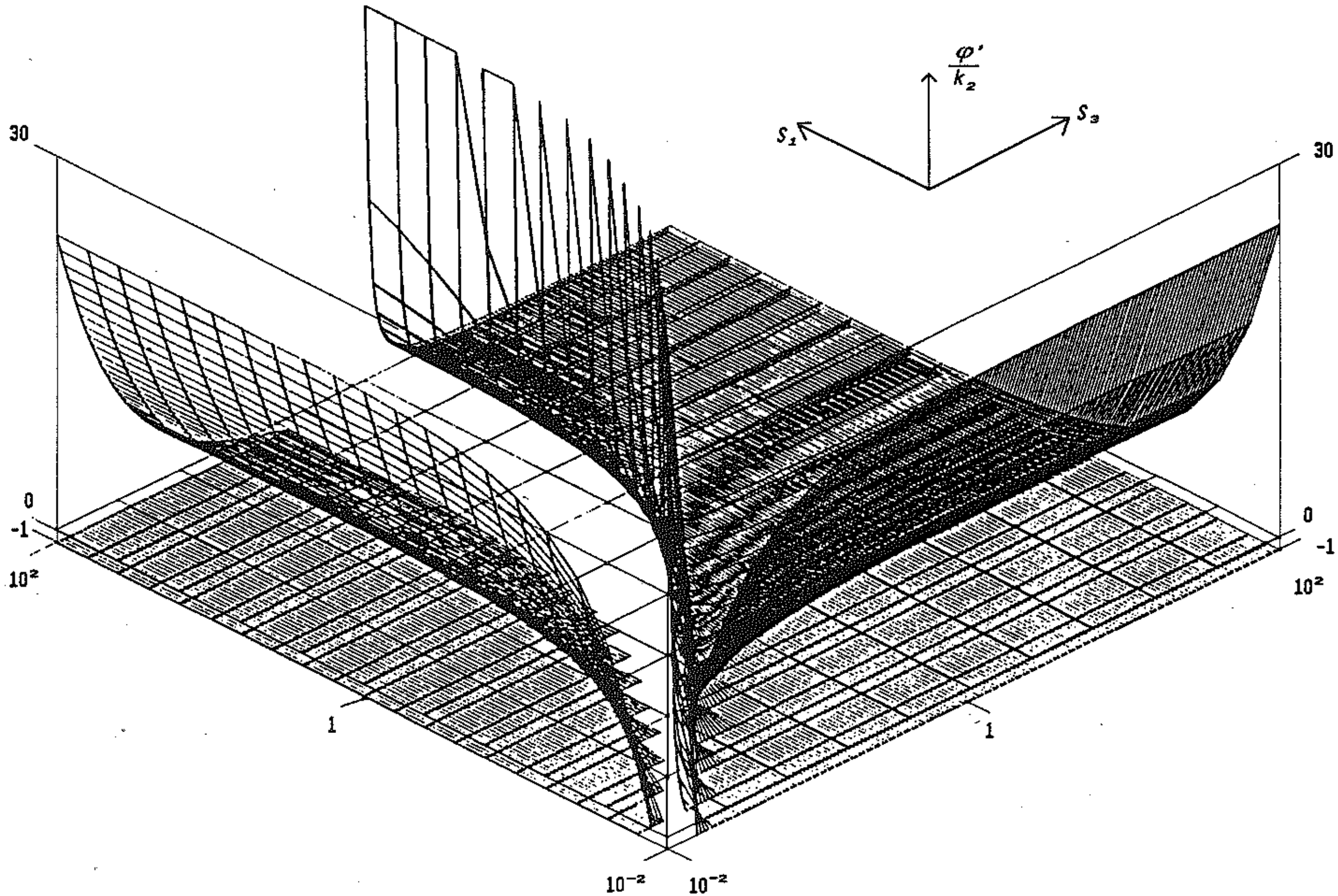

FIG. 3. Admissible domains of variability of the ratio φ'/k_2.

a) for $P < 0$, i.e. $\omega_2 < \omega_1$

$$G_1(S_1, S_3) < \frac{\varphi'}{k_2} < G_2(S_1, S_3), \tag{4.20}$$

b) for $P > 0$, i.e. $\omega_2 > \omega_1$

$$G_2(S_1, S_3) < \frac{\varphi'}{k_2} < G_1(S_1, S_3). \tag{4.21}$$

It should be remembered that the condition

$$\frac{\varphi'}{k_2} > -1 \tag{4.22}$$

has to be met, otherwise a zero or even negative stiffness would exist between the masses m_1 and m_2.

The conditions (4.20)–(4.22) are shown in Fig.3.

4.3. Conditions referring to the second derivative of the function $\varphi(z_1, z_2)$

Although the condition (4.14) analysed above imposes explicitly the constraints on the first derivative of the nonlinear function $\varphi(\lambda)$ only, the values of the higher order derivatives have by no means been ignored in the considerations. These conditions result from certain additional assumptions made in the course of analysis, namely Eqs. (3.26), (3.33) and (3.35).

According to the relationships (3.26) and $(4.3)_2$ we can write

$$g'' \leq 2g' \frac{(y_{22} - y_{21})}{(y_{12} - y_{11})^2} . \tag{4.23}$$

For $\varepsilon > 0$ the above inequality, in accordance with Eq. (2.5), is equivalent to the inequality

$$\varphi'' \leq 2\varphi' \frac{(y_{22} - y_{21})}{(y_{12} - y_{11})^2} . \tag{4.24}$$

Substituting into (4.24) the values of coefficients of the solutions given in Table 1 and rearranging, we get the condition for the second derivative of the function $\varphi(\lambda)$:

$$\varphi'' \leq 2\varphi' \frac{\dfrac{\varepsilon W}{P^2} + \dfrac{\varphi'}{m_1 P}}{\varepsilon y_{11}} . \tag{4.25}$$

The right-hand side of this inequality is a quadratic polynomial with respect to φ' and has the roots:

$$\varphi'_1 = 0, \qquad \varphi'_2 = k_2\, G_3(S_1, S_3), \tag{4.26}$$

where

$$G_3(S_1, S_3) = \frac{S_1}{S_1 S_3 + S_3 - S_1}\,. \tag{4.27}$$

The condition (4.25) is shown diagrammatically in Figs. 4 and 5. The domains of admissible values of the second derivative of φ' are shaded. Moreover, the lines are shown indicating the convergence limits of the procedures: $G_2 k_2$ for $P < 0$ (Fig. 4) and $G_1 k_2$ for $P > 0$ (Fig. 5), in conformity with the relations visualized in Fig. 3. In this figure we can also see that only for very small values of S_1 or S_3 – i.e. for those cases in which the iterative analysis of the linear part would be convergent very rapidly – a larger range

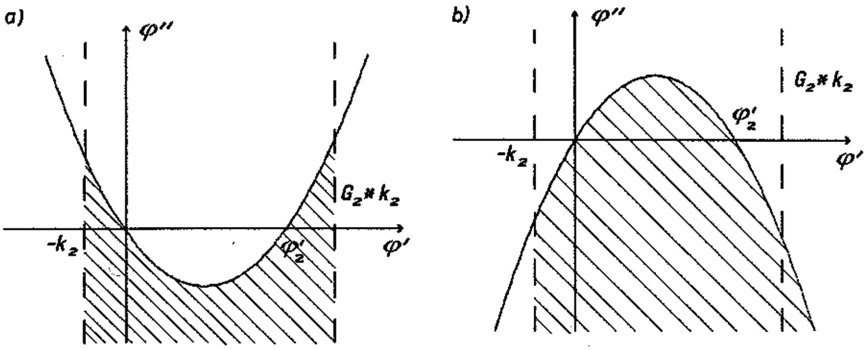

FIG. 4. Diagram of the relation (4.25) for $P < 0$ ($\omega_2 < \omega_1$). a) $y_{11} > 0$; b) $y_{11} < 0$.

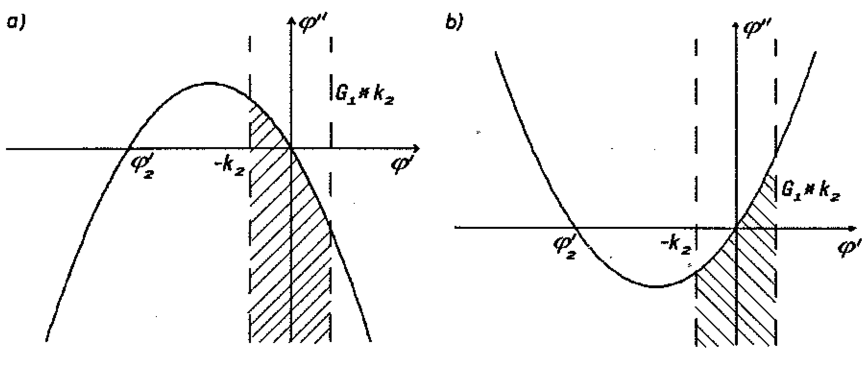

FIG. 5. Diagram of the relation (4.25) for $P > 0$ ($\omega_2 > \omega_1$). a) $y_{11} > 0$; b) $y_{11} < 0$.

of changes in the first derivative φ' is allowable. For the greater part of the remaining cases the intervals $(-1, G_2k_2)$ or $(-1, G_1k_2)$ are very narrow and the values G_1k_2, G_2k_2, G_3k_2 differ very little. In view of this fact, Figs. 4 and 5 show that in the majority of cases the convergence procedure requires the second derivative to be negative (regressive characteristic).

5. Final remarks

The conditions regarding higher order derivatives of the nonlinear characteristic $\varphi(\lambda)$ can be derived in a similar way. For instance, the relationships (3.33), (4.3) and suitable expressions from Table 1 supply, after some rearrangements, the condition for the third derivative

$$\varphi''' y_{11}^3 \geq 9\varphi' y_{31} + 6\varphi'' y_{11} y_{21}.$$

Basing on the relation (3.35), the conditions can be formulated referring to higher order derivatives. However, their form becomes more complex and they are functions of the lower order derivatives, so their analysis is excessively complicated. That is why it is reasonable to terminate the analysis on lower order derivatives only.

It is interesting to realize that the presented results can be used to "synthesize" certain nonlinear characteristic ensuring the iterative convergence of the analysis of the complete system with given parameters.

During derivation of the convergence conditions presented in this paper, an assessment from above was made repeatedly (e.g. the series N_i, use of the majorant and others), thus the actual convergence conditions are less stringent than those presented.

The problem of the convergence of the iterative procedures for nonlinear systems can be approached in two ways:

- without specifying the type of nonlinearity, to analyse the problem of convergence in order to formulate general criteria for the nonlinear characteristics to ensure their convergence;
- assuming certain typical forms of nonlinear characteristics, to examine which of them ensure the convergence of the procedures or analyze what changes are caused in the convergence domains (their narrowing or expanding) for linear systems by the introduction of a given nonlinearity.

The analysis made in this paper belongs to the first type of approach to the convergence problems of nonlinear systems. The other approach will be tackled in a separate paper.

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