ENGINEERING TRANSACTIONS • Engng. Trans. • 41, 1, 97–116, 1993
Polish Academy of Sciences • Institute of Fundamental Technological Research

DIFFRACTION OF TORSIONAL ELASTIC WAVES BY A RIGID ANNULAR DISC AT A BIMATERIAL INTERFACE

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The problem of diffraction of normally incident torsional wave by an annular rigid disc lying at the interface of two bonded dissimilar semi-infinite elastic media is analysed. The three-part mixed boundary value problem is reduced to the solution of a set of integral equations which are solved by using iterative technique for low frequency, assuming that the ratio of the inner and outer radii is small. The stress distribution on the disc, the torque and far-field amplitudes of the displacement in both the media are evaluated. The variation of dynamic stress intensity factors with normalised frequency for various values of the material parameters, and also the variation in far-field amplitude with the polar angles for a fixed radial distance have been shown by means of graphs.

1. Introduction

The study of problems involving diffraction of elastic waves by cracks or inclusions are of considerable importance in view of their extensive applications in mechanical engineering and also in seismology and geophysics. If the cracks or inclusions are located at the interface of layered media, the study becomes more relevant. The extensive use of composite materials in modern technology has evoked interest in the wave propagation problems in layered media with interfacial discontinuities. Onder *et al.* [1] studied the diffraction of plane SH-wave obliquely incident on a rigid half-plane lying at the interface of two dissimilar semi-infinite elastic media. Following Mal [2], problem of interaction of antiplane shear wave by a Griffith crack at the interface of two bonded dissimilar elastic half-spaces has been treated by Srivastava *et al.* [3]. Bostrom [4] also treated the same problem following a procedure similar to that of Krenk and Schmidt [5]. The corresponding problem of diffraction of antiplane shear wave by a finite rigid strip at the bimaterial interface has been treated by Palaiya and Majumder [6]. The problem of diffraction of transient torsional shear waves by a penny-shaped crack at the interface of two bonded dissimilar elastic half-spaces has been investigated by UEDA *et al.* [7]. As regards the dynamic crack or strip problems, research has mainly been confined to the case of a single crack or strip of finite width or circular in shape. These are the mixed boundary value problems which are usually reduced to solutions of dual integral equations. But the solution of the interesting problems involving the diffraction of elastic waves by annular discs or cracks at the bimaterial interface which give rise to three-part mixed boundary value problems are still lacking.

However, recently the problems involving the diffraction of torsional waves at a flat annular crack in an infinite elastic medium have been studied by SHINDO [8,9]; the problems are reduced to those of solving singular integral equations of first kind which were later solved by the technique of ERDOGAN [10,11]. The problem of diffraction of an acoustic wave by a soft annular disc was studied by THOMAS [12]. Following the method of WILLIAMS [13], the three-part mixed boundary value problem was reduced to a set of integral equations which was solved by an iterative procedure for low frequency. The same technique was followed by JAIN and KANWAL [14] to study the problem of torsional oscillations of an elastic half-space due to annular disc.

In this paper we have discussed the problem of diffraction of torsional wave by a rigid annular disc located at the interface of two bonded dissimilar elastic media. Applying the method developed by WILLIAMS [13] and used later by THOMAS [12] and JAIN *et al.* [14], the three-part mixed boundary value problem has been reduced to the solution of a set of integral equations. The solutions of these integral equations are obtained iteratively for low frequency and small values of the ratio of the inner and outer radii of the disc. These solutions are used to determine the jump in stresses across the annular disc and stress intensity factors at both the edges of the disc. Torque and far-field amplitudes in both the media have also been deduced. The effect of normalised frequency, material properties and geometric parameters in stress intensity factors and far-field amplitude are shown graphically.

2. FORMULATION OF THE PROBLEM

Let us consider the torsional vibration of frequency ω of an annular rigid disc of inner and outer radii b and a, respectively, lying at the interface of two bonded dissimilar elastic half-spaces. The region occupied by the annular disc is defined by $z = 0$ and $b \le r \le a$ in a cylindrical polar coordinate system (r, θ, z) as shown in the Fig. 1.

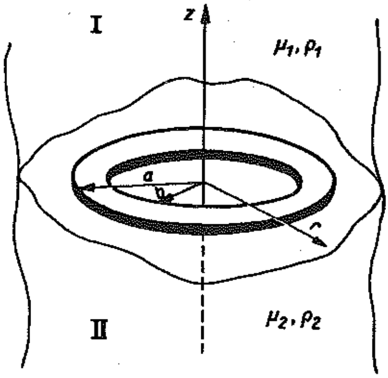

FIG. 1. Geometry of the annular disc.

Let an antiplane shear wave given by $\Omega_2 r e^{ik_2(z-c_2 t)}$, where Ω_2 is a constant, $k_2 = \omega/c_2$, and $c_2 = \sqrt{(\mu_2/\rho_2)}$, the shear wave velocity in medium 2, be incident normally on the disc. Henceforth the time factor $e^{-i\omega t}$ will be suppressed throughout the analysis.

The only non-vanishing θ-component of the displacement V_j and the nonvanishing stresses $\tau_{r\theta}^{(j)}$, $\tau_{z\theta}^{(j)}$ $(j = 1, 2)$ due to the scattered field are independent of θ and are given by

$$V_j = V_j(r, z, t) = v_j(r, z)e^{-i\omega t}, \tag{2.1}$$

$$\begin{aligned} \tau_{r\theta}^{(j)} &= \tau_{r\theta}^{(j)}(r, z) = \mu_j \left(\frac{\partial v_j}{\partial r} - \frac{v_j}{r} \right), \\ \tau_{z\theta}^{(j)} &= \tau_{z\theta}^{(j)}(r, z) = \mu_j \frac{\partial v_j}{\partial z}, \end{aligned} \tag{2.2}$$

where μ_j $(j = 1, 2)$ are the shear moduli of the elastic materials.

The suffices 1 and 2 are used to denote the values of the corresponding quantities in the upper and lower half-spaces, respectively. Without any loss of generality we assume that $c_2 > c_1$.

The displacement V_j satisfies the equation

$$\frac{\partial^2 V_j}{\partial r^2} + \frac{1}{r}\frac{\partial V_j}{\partial r} - \frac{V_j}{r^2} + \frac{\partial^2 V_j}{\partial z^2} = \frac{\rho_j}{\mu_j}\frac{\partial^2 V_j}{\partial t^2}, \tag{2.3}$$

where ρ_j $(j = 1, 2)$ are the densities of the elastic materials.

Putting $V_j = v_j(r, z)e^{-i\omega t}$, Eq. (2.3) and the boundary conditions at the interface $z = 0$, take the form

$$\frac{\partial^2 v_j}{\partial r^2} + \frac{1}{r}\frac{\partial v_j}{\partial r} - \frac{v_j}{r^2} + \frac{\partial^2 v_j}{\partial z^2} + k_j^2 v_j = 0, \tag{2.4}$$

$$v_1(r,0) = v_2(r,0) = -\Omega r, \qquad b \le r \le a, \tag{2.5}$$

$$v_1(r,0) = v_2(r,0), \qquad 0 \le r \le b, \quad a < r < \infty, \tag{2.6}$$

$$\tau_{z\theta}^{(1)}(r,0) = \tau_{z\theta}^{(2)}(r,0), \qquad 0 \le r \le b, \quad a < r < \infty, \tag{2.7}$$

where $k_j = \omega/c_j$, $c_j = \sqrt{(\mu_j/\rho_j)}$ and $\Omega = 2\Omega_2\mu_2 k_2/(\mu_1 k_1 + \mu_2 k_2)$.

The solution of equation (2.4) can be written as

$$v_j(r,z) = \int_0^\infty A_j(\xi)\exp(-\gamma_j|z|)J_1(\xi r)\,d\xi, \tag{2.8}$$

where

$$\begin{aligned} \gamma_j &= \left(\xi^2 - k_j^2\right)^{1/2}, & \xi &> k_j, \\ \gamma_j &= -i\left(k_j^2 - \xi^2\right)^{1/2}, & \xi &< k_j \end{aligned}$$

and $A_j(\xi)$ $(j = 1, 2)$ are functions of ξ to be determined from the boundary conditions.

Therefore, the stress components are

$$\begin{aligned} \tau_{z\theta}^{(1)}(r,z) &= -\mu_1 \int_0^\infty \gamma_1 A_1(\xi)\exp(-\gamma_1|z|)J_1(\xi r)\,d\xi, & z \ge 0, \\ \tau_{z\theta}^{(2)}(r,z) &= \mu_2 \int_0^\infty \gamma_2 A_2(\xi)\exp(-\gamma_2|z|)J_1(\xi r)\,d\xi, & z \le 0. \end{aligned} \tag{2.9}$$

Now, using the boundary conditions (2.5), (2.6) and (2.7) and assuming that $\tau_{z\theta}^{(2)}(r,0) - \tau_{z\theta}^{(1)}(r,0) = f(r)$, $b \le r \le a$, we obtain the integral equation

$$\int_b^a t f(t)\,dt \int_0^\infty \frac{\xi}{(\mu_1\gamma_1 + \mu_2\gamma_2)} J_1(\xi r)J_1(\xi t)\,d\xi = -\Omega r, \quad b \le r \le a, \tag{2.10}$$

where

$$A_1(\xi) = A_2(\xi) = \frac{\xi}{(\mu_1\gamma_1 + \mu_2\gamma_2)} \int_b^a t f(t) J_1(\xi t)\,dt. \tag{2.11}$$

3. Method of solution

In order to solve the integral equation (2.10) we apply the technique developed by Williams [13] for solving integral equations arising in the three-part boundary value problems. The same technique was also applied by Thomas [12] and Jain *et al.* [14] in order to solve the scattering problem by annular disc. Following Kanwal [15], the kernel of the integral equation (2.10) is split into two kernels as follows:

$$(3.1)\qquad (\mu_1+\mu_2)\int_0^\infty \frac{\xi}{(\mu_1\gamma_1+\mu_2\gamma_2)} J_1(\xi r)J_1(\xi t)\,d\xi = K_1(r,t)+K_2(r,t),$$

where

$$(3.2)\qquad K_1(r,t) = \int_0^\infty J_1(\xi r)J_1(\xi t)\,d\xi,$$

$$(3.3)\qquad K_2(r,t) = \int_0^\infty M(\xi,\gamma_1,\gamma_2)J_1(\xi r)J_1(\xi t)\,d\xi,$$

$$(3.4)\qquad M(\xi,\gamma_1,\gamma_2) = \frac{\mu_1(\xi-\gamma_1)+\mu_2(\xi-\gamma_2)}{\mu_1\gamma_1+\mu_2\gamma_2}.$$

Equation (2.10) then takes the form

$$(3.5)\qquad \int_b^a tf(t)K_1(r,t)\,dt = -(\mu_1+\mu_2)\Omega r - \int_b^a tf(t)K_2(r,t)\,dt, \quad b\le r\le a.$$

Next, consider two functions $f_1(r)$ and $f_2(r)$ such that

$$(3.6)\qquad f_1(r)+f_2(r) = \begin{cases} 0, & 0\le r<b,\\ f(r), & b\le r\le a,\\ 0, & a<r<\infty. \end{cases}$$

As a result, Eq. (3.5) reduces to two integral equation given by

$$(3.7)\qquad \int_0^\infty tf_1(t)K_1(r,t)\,dt = -(\mu_1+\mu_2)\Omega r - \int_0^\infty tf_1(t)K_2(r,t)\,dt, \quad 0<r<a$$

and

$$(3.8)\qquad \int_0^\infty tf_2(t)K_1(r,t)\,dt = -\int_0^\infty tf_2(t)K_2(r,t)\,dt, \quad b<r<\infty.$$

The procedure adopted by WILLIAMS [13] and THOMAS [12] is followed to solve these integral equations. Using the results

$$J_n(pr) = \left(\frac{2p}{\pi}\right)^{1/2} \frac{1}{r^n} \int_0^r \frac{J_{n-1/2}(pw)w^{n+1/2}}{(r^2-w^2)^{1/2}} dw$$
$$= \left(\frac{2p}{\pi}\right)^{1/2} r^n \int_r^\infty \frac{J_{n+1/2}(pw)w^{-(n-1/2)}}{(w^2-r^2)^{1/2}} dw,$$

and

$$\int_0^\infty pJ_\mu(pw)J_\mu(pv)\,dp = \delta(w-v)/(wv)^{1/2},$$

we have the following relations:

$$\int_0^\infty K_1(t,r,)tf(t)\,dt = \frac{2}{\pi r}\int_0^r \frac{w^2dw}{(r^2-w^2)^{1/2}}\int_w^\infty \frac{f(t)\,dt}{(t^2-w^2)^{1/2}}, \quad 0<r<a,$$
$$= \frac{2r}{\pi}\int_r^\infty \frac{w^{-2}dw}{(w^2-r^2)^{1/2}}\int_0^w \frac{t^2f(t)\,dt}{(w^2-t^2)^{1/2}}, \quad b<r<\infty, \tag{3.9}$$

and

$$K_2(t,r) = \frac{2}{\pi tr}\int_0^r\int_0^t \frac{L_1(v,w)wv\,dvdw}{(r^2-w^2)^{1/2}(t^2-v^2)^{1/2}}, \quad 0<r<a,$$
$$= \frac{2tr}{\pi}\int_r^\infty\int_t^\infty \frac{L_2(v,w)\,dvdw}{wv(w^2-r^2)^{1/2}(v^2-t^2)^{1/2}}, \quad b<r<\infty, \tag{3.10}$$

where

$$L_1(v,r) = (vr)^{1/2}\int_0^\infty \xi M(\xi,\gamma_1,\gamma_2)J_{1/2}(\xi v)J_{1/2}(\xi r)\,d\xi, \tag{3.11}$$

and

$$L_2(v,r) = (vr)^{1/2}\int_0^\infty \xi M(\xi,\gamma_1,\gamma_2)J_{3/2}(\xi v)J_{3/2}(\xi r)\,d\xi. \tag{3.12}$$

Now, we assume

$$r\int_r^\infty \frac{f_1(t)\,dt}{(t^2-r^2)^{1/2}} = \begin{cases} S_1(r), & 0<r<a, \\ -T_1(r), & a<r<\infty \end{cases} \tag{3.13}$$

and

$$(3.14)\qquad \frac{1}{r}\int_0^r \frac{t^2 f_2(t)\,dt}{(r^2-t^2)^{1/2}} = \begin{cases} -T_2(r), & 0< r <b, \\ S_2(r), & b< r <\infty \end{cases}$$

in which, using the relation (3.6) and Abel's transform, we get the following two integral equations:

$$(3.15)\qquad T_1(r) = l_2(r) + \frac{1}{\sqrt{\pi}\, r\Gamma(5/2)} \int_0^b \frac{u^2 T_2(u)\, {}_2F_1(1/2,1;5/2;u^2/r^2)}{(r^2-u^2)} du,$$

$$a<r<\infty,$$

$$(3.16)\qquad T_2(r) = l_1(r) + \frac{r^2}{\sqrt{\pi}\,\Gamma(5/2)} \int_a^\infty \frac{T_1(u)\, {}_2F_1(1/2,1;5/1;r^2/u^2)}{u(u^2-r^2)} du,$$

$$0<r<b,$$

where

$$(3.17)\qquad l_1(r) = -\frac{2}{\pi r}\int_0^r \frac{t^2 dt}{(r^2-t^2)^{1/2}} \frac{d}{dt}\int_t^a \frac{S_1(u)\,du}{(u^2-t^2)^{1/2}}, \qquad 0<r<b,$$

$$(3.18)\qquad l_2(r) = \frac{2r}{\pi}\int_r^\infty \frac{dt}{t^2(t^2-r^2)^{1/2}} \frac{d}{dt}\int_b^t \frac{u^2 S_2(u)\,du}{(t^2-u^2)^{1/2}}, \qquad a<r<\infty.$$

Further, on substituting the relations (3.9) and (3.10) in (3.7) and (3.8), the resulting equations give rise to the other two integral equations, the relations (3.13) and (3.14) and Abel's transform being used

$$(3.19)\qquad S_1(r)+\int_0^a L_1(v,r)S_1(v)\,dv = -2\Omega(\mu_1+\mu_2)r+\int_a^\infty L_1(v,r)T_1(v)\,dv,$$

$$0<r<a,$$

$$(3.20)\qquad S_2(r)+\int_b^\infty L_2(v,r)S_2(v)\,dv = \int_0^b L_2(v,r)T_2(v)\,dv, \qquad b<r<\infty.$$

Assuming that $\alpha = k_2 a$, $\beta = k_2 b$ and $\lambda = b/a$ are small, the unknown functions $S_1(r)$, $S_2(r)$, $T_1(r)$, $T_2(r)$ which are solutions of integral equations (3.19), (3.20), (3.15) and (3.16) are obtained approximately following the iterative process. Using the result that

$${}_2F_1(1/2,1;5/2;r^2/u^2) = \frac{3u}{4r^3}\left\{2ur-(u^2-r^2)\log\left(\frac{u+r}{u-r}\right)\right\}, \quad r<u,$$

Equations (3.15) and (3.16) become

$$(3.21)\qquad T_1(ar) = l_2(ar) + \frac{1}{\pi}\int_0^1 T_2(bu)\left\{\frac{2\lambda r}{(r^2-\lambda^2u^2)} - \frac{1}{u}\log\left(\frac{r+\lambda u}{r-\lambda u}\right)\right\}du, \qquad 1 < r < \infty,$$

and

$$(3.22)\qquad T_2(br) = l_1(br) + \frac{1}{\pi\lambda r}\int_1^\infty T_1(au)\left\{\frac{2\lambda ur}{(u^2-\lambda^2r^2)} - \log\left(\frac{u+\lambda r}{u-\lambda r}\right)\right\}du, \qquad 0 < r < 1.$$

Next, we assume that $\alpha = O(\lambda)$ so that $\beta = \alpha\lambda = O(\alpha^2)$.

In order to solve Eq. (3.19), we rewrite it as

$$(3.23)\qquad S_1(ar) + a\int_0^1 L_1(av, ar)S_1(av)\,dv = -2\Omega(\mu_1+\mu_2)ar + a\int_1^\infty L_1(av, ar)T_1(av)\,dv, \qquad 0 < r < 1.$$

Now we put

$$(3.24)\qquad S_1(ar) = X(ar) + Y(ar)$$

so that Eq. (3.23) yields a pair of integral equations given by

$$(3.25)\qquad X(ar) = -2\Omega(\mu_1+\mu_2)ar - a\int_0^1 L_1(av, ar)X(av)\,dv, \quad 0 < r < 1$$

and

$$(3.26)\qquad Y(ar) = a\int_1^\infty L_1(av, ar)T_1(av)\,dv - a\int_0^1 L_1(av, ar)Y(av)\,dv, \qquad 0 < r < 1.$$

The kernel $L_1(av, ar)$ given by Eq. (3.11) can be converted to an expression involving finite integrals by the application of the contour integration technique followed by SRIVASTAVA *et al.* [3] and MANDAL *et al.* [16] and is given by

$$aL_1(av, ar) = i(1+\mu)\alpha^2(vr)^{1/2}\left[\int_0^1 \frac{\eta^2 J_{1/2}(\alpha\eta r)H^{(1)}_{1/2}(\alpha\eta v)}{\mu(1-\eta^2)^{1/2}+(\sigma^2-\eta^2)^{1/2}}d\eta + \int_1^\sigma \frac{\eta^2(\sigma^2-\eta^2)^{1/2}J_{1/2}(\alpha\eta r)H^{(1)}_{1/2}(\alpha\eta v)}{\mu^2(\eta^2-1)+(\sigma^2-\eta^2)}d\eta\right], \qquad v > r,$$

where $\sigma = k_1/k_2$, $\mu = \mu_2/\mu_1$. For $v < r$, v and r should be interchanged.

Next, expanding the Bessel and Hankel functions in series for small values of their arguments and integrating, assuming that $\mu > \sigma > 1$, the above expression can be written as

$$\begin{aligned}(3.27)\qquad aL_1(av, ar) &= \alpha^2 r M_1 + i\alpha^3 rvM_2 - \frac{\alpha^4(3v^2r + r^3)}{6}M_3 \\ &\quad - \frac{i\alpha^5(v^3r + r^3v)}{6}M_4 + \frac{\alpha^6(5v^4r + 10r^3v^2 + r^5)}{120}M_5 \\ &\quad + \frac{i\alpha^7(3v^5r + 10r^3v^3 + 3r^5v)}{360}M_6 + O(\alpha^8), \quad v > r, \\ &= \alpha^2 vM_1 + i\alpha^3 rvM_2 - \frac{\alpha^4(3r^2v + v^3)}{6}M_3 \\ &\quad - \frac{i\alpha^5(r^3v + v^3r)}{6}M_4 + \frac{\alpha^6(5r^4v + 10v^3r^2 + v^5)}{120}M_5 \\ &\quad + \frac{i\alpha^7(3r^5v + 10v^3r^3 + 3v^5r)}{360}M_6 + O(\alpha^8), \quad v < r,\end{aligned}$$

where

$$(3.28)\qquad M_1 = \frac{\sigma^2 + \mu}{2(\mu + 1)},$$

$$\begin{aligned}(3.29)\qquad M_2 = \frac{2}{\pi(\mu - 1)}\Bigg[\frac{\sigma^3 - \mu}{3} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1}\Bigg\{(\mu - \sigma) + \mu\sqrt{\frac{\sigma^2 - 1}{\mu^2 - 1}} \\ \times \log\left(\frac{\sigma\sqrt{\mu^2 - 1} + \mu\sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1} + \sqrt{\sigma^2 - 1}}\right)\Bigg\}\Bigg],\end{aligned}$$

$$(3.30)\qquad M_3 = \frac{1}{(\mu - 1)}\left[\frac{1}{8}(\sigma^4 - \mu) + \left(\frac{\mu^2 - \sigma^2}{\mu^2 - 1}\right)\left\{\frac{\sigma^2 - \mu}{2} + \frac{\mu^2 - \sigma^2}{\mu + 1}\right\}\right],$$

$$\begin{aligned}(3.31)\qquad M_4 = \frac{2}{\pi(\mu - 1)}\Bigg[\frac{2(\sigma^5 - \mu)}{15} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1}\Bigg(\frac{\sigma^3 - \mu}{3} + \frac{\mu^2 - \sigma^2}{\mu^2 - 1} \\ \times \Bigg\{(\mu - \sigma) + \mu\sqrt{\frac{\sigma^2 - 1}{\mu^2 - 1}}\log\left(\frac{\sigma\sqrt{\mu^2 - 1} + \mu\sqrt{\sigma^2 - 1}}{\sqrt{\mu^2 - 1} + \sqrt{\sigma^2 - 1}}\right)\Bigg\}\Bigg)\Bigg],\end{aligned}$$

$$\begin{aligned}(3.32)\qquad M_5 = \frac{1}{(\mu - 1)}\Bigg[\frac{\sigma^6 - \mu}{16} + \left(\frac{\mu^2 - \sigma^2}{\mu^2 - 1}\right)\Bigg\{\frac{1}{8}(\sigma^4 - \mu) + \left(\frac{\mu^2 - \sigma^2}{\mu^2 - 1}\right) \\ \times \left(\frac{\sigma^2 - \mu}{2} + \frac{\mu^2 - \sigma^2}{\mu + 1}\right)\Bigg\}\Bigg],\end{aligned}$$

$$(3.33)\quad M_6 = \frac{2}{\pi(\mu-1)}\left[\frac{8}{105}(\sigma^7-\mu)+\frac{\mu^2-\sigma^2}{\mu^2-1}\left\{\frac{2(\sigma^5-\mu)}{15}\right.\right.$$
$$\left.\left.+\frac{\mu^2-\sigma^2}{\mu^2-1}\left(\frac{\sigma^3-\mu}{3}+\frac{\mu^2-\sigma^2}{\mu^2-1}\left\{(\mu-\sigma)+\mu\sqrt{\frac{\sigma^2-1}{\mu^2-1}}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\times\log\left(\frac{\sigma\sqrt{\mu^2-1}+\mu\sqrt{\sigma^2-1}}{\sqrt{\mu^2-1}+\sqrt{\sigma^2-1}}\right)\right\}\right)\right\}\right].$$

Substituting the value of $L_1(av, ar)$ given by Eq. (3.27) in Eq. (3.25) and using the iterative method, an approximate value of $X(ar)$ for low frequency can be derived in the form

$$(3.34)\quad X(ar) = a\Omega(\mu_1+\mu_2)\left[p_1(\alpha)r+p_3(\alpha)r^3+p_5(\alpha)r^5+p_7(\alpha)r^7+O(\alpha^8)\right],$$

where

$$p_1(\alpha) = -2+M_1\alpha^2+\frac{2i}{3}M_2\alpha^3+\frac{1}{12}\left(M_1^2-3M_3\right)\alpha^4$$
$$+\frac{1}{360}\left(94M_1M_3-29M_1^3+80M_2^2+5M_5\right)\alpha^6-\frac{i}{15}\left(9M_1M_2+M_4\right)\alpha^5$$
$$+\frac{i}{1260}\left(269M_1^2M_2+109M_1M_4+249M_2M_3+3M_6\right)\alpha^7,$$
$$p_3(\alpha) = -\frac{1}{3}M_1\alpha^2+\frac{1}{6}\left(M_1^2-M_3\right)\alpha^4+\frac{i}{9}\left(M_1M_2-M_4\right)\alpha^5$$
$$+\frac{1}{72}\left(M_1^3+2M_1M_3+3M_5\right)\alpha^6$$
$$+\frac{i}{90}\left(3M_1M_4-9M_1^2M_2+5M_2M_3+M_6\right)\alpha^7,$$
$$p_5(\alpha) = -\frac{1}{60}\left(M_1^2-M_3\right)\alpha^4+\frac{1}{120}\left(M_1^3-2M_1M_3+M_5\right)\alpha^6$$
$$+\frac{i}{180}\left(M_1^2M_2-M_1M_4-M_2M_3+M_6\right)\alpha^7,$$
$$p_7(\alpha) = \frac{1}{2520}(2M_1M_3-M_1^3-M_5)\alpha^6.$$

Next, replacing r by br, Eq. (3.20) can be written as

$$(3.35)\quad S_2(br)+b\int_1^\infty L_2(bv,br)S_2(bv)\,dv = b\int_0^1 L_2(bv,br)T_2(bv)\,dv,$$
$$1<r<\infty.$$

Following the same procedure as that done for the evaluation of $L_1(av, ar)$, $L_2(bv, br)$ given by Eq. (3.12) can be evaluated to the form

$$bL_2(bv, br) = i(1+\mu)\beta^2(vr)^{1/2}\left[\int_0^1 \frac{\eta^2 J_{3/2}(\beta\eta r)H_{3/2}^{(1)}(\beta\eta v)}{\mu(1-\eta^2)^{1/2}+(\sigma^2-\eta^2)^{1/2}}d\eta + \int_1^\sigma \frac{\eta^2(\sigma^2-\eta^2)^{1/2}J_{3/2}(\beta\eta r)H_{3/2}^{(1)}(\beta\eta v)}{\mu^2(\eta^2-1)+(\sigma^2-\eta^2)}d\eta\right], \quad v > r.$$

For $v < r$, v and r are to be interchanged.

For low frequency $bL_2(bv, br)$ is now reduced to the following form after using the series expansions of Bessel and Hankel functions:

$$\text{(3.36)} \qquad \begin{aligned} bL_2(bv, br) &= \alpha^2\lambda^2\left[\frac{1}{3}M_1\frac{r^2}{v}+O(\alpha^4)\right], \qquad v > r, \\ &= \alpha^2\lambda^2\left[\frac{1}{3}M_1\frac{v^2}{r}+O(\alpha^4)\right], \qquad v < r. \end{aligned}$$

The functions which occur in the integral equations (3.15), (3.16), (3.19) and (3.20) are calculated by an iterative process in the following order:

$$X,\ S_1,\ l_1,\ T_2,\ S_2,\ l_2,\ T_1,\ Y,\ S_1.$$

Iterative procedure is followed in order to obtain the following results sufficiently accurate up to the order of (α^7)

$$\text{(3.37)} \qquad l_1(br) = \frac{8\Omega(\mu_1+\mu_2)ar^2\lambda^2}{3\pi}\left[-1+\frac{2}{3}M_1\alpha^2+\frac{i}{3}M_2\alpha^3 - \frac{2}{5}\lambda^2 r^2 + O(\alpha^4)\right], \qquad 0 < r < 1,$$

$$\text{(3.38)} \qquad T_2(br) = l_1(br) + O(\alpha^7), \qquad 0 < r < 1,$$

$$\text{(3.39)} \qquad S_2(br) = -\frac{8\Omega(\mu_1+\mu_2)aM_1\alpha^2\lambda^4}{45\pi}\left[\frac{1}{r}+O(\alpha^2)\right], \qquad 1 < r < \infty,$$

$$\text{(3.40)} \qquad l_2(ar) = -\frac{16\Omega(\mu_1+\mu_2)aM_1\alpha^2\lambda^5}{45\pi^2}\left[\frac{1}{r}+O(\alpha^2)\right], \qquad 1 < r < \infty,$$

$$\text{(3.41)} \qquad T_1(ar) = \frac{16\Omega(\mu_1+\mu_2)a\lambda^5}{45\pi^2}\left[-\frac{1}{r}M_1\alpha^2 + 2\left\{-1+\frac{2}{3}M_1\alpha^2 - \frac{2}{7}\lambda^2\right\}\frac{1}{r^3} - \frac{12\lambda^2}{7}\frac{1}{r^5} + O(\alpha^3)\right], \qquad 1 < r < \infty,$$

$$(3.42)\qquad Y(ar) = -\frac{16\Omega(\mu_1+\mu_2)aM_1\alpha^2\lambda^5}{45\pi^2}[r+O(\alpha)], \qquad 0<r<1,$$

$$(3.43)\qquad S_1(ar) = X(ar) - \frac{16\Omega(\mu_1+\mu_2)aM_1r\alpha^2\lambda^5}{45\pi^2} + O(\alpha^8), \quad 0<r<1.$$

4. Stress difference across the annular disc, torque and far-field amplitude

The jump of the stresses at the annular disc is given by

$$\tau(r,0,t) = \tau(r)e^{-i\omega t} = \tau^{(2)}_{z\theta}(r,0,t) - \tau^{(1)}_{z\theta}(r,0,t) = f(r)e^{-i\omega t},$$
$$b \le r \le a, \quad z=0,$$
$$= f_1(r) + f_2(r) \qquad \text{(supressing } e^{-i\omega t}\text{)}.$$

Putting the values of $f_1(r)$ and $f_2(r)$ which are the solutions of Abel-type integral equations given by Eqs. (3.13) and (3.14) in the above expression, we obtain

$$(4.1)\qquad \tau(r) = \frac{2}{\pi}\left[\frac{d}{dr}\left\{-\int_r^a \frac{S_1(u)\,du}{(u^2-r^2)^{1/2}} + \int_a^\infty \frac{T_1(u)\,du}{(u^2-r^2)^{1/2}}\right\}\right.$$
$$\left. + \frac{1}{r^2}\frac{d}{dr}\left\{-\int_0^b \frac{u^2T_2(u)\,du}{(r^2-u^2)^{1/2}} + \int_b^r \frac{u^2S_2(u)\,du}{(r^2-u^2)^{1/2}}\right\}\right], \quad b\le r\le a.$$

Finally, substitution of the respective values of $S_1(u)$, $T_1(u)$, $T_2(u)$ and $S_2(u)$ from Eqs. (3.43), (3.41), (3.38) and (3.39) in Eq. (4.1) yields, after integration,

$$(4.2)\qquad \tau(r) = \frac{2(\mu_1+\mu_2)\Omega}{\pi}\left[\frac{\nu_1}{(1-\nu_1^2)^{1/2}}\left\{-2 + \left(\frac{2}{3}M_1\left(1-\nu_1^2\right) + \frac{2}{3}M_1\right)\alpha^2\right.\right.$$
$$+\frac{2i}{3}M_2\alpha^3 + \left(\frac{1}{30}\left(7M_1^2 - 12M_3\right) - \frac{4}{15}\left(M_1^2 - M_3\right)\left(1-\nu_1^2\right)\right.$$
$$\left. -\frac{2}{45}\left(M_1^2 - M_3\right)\left(1-\nu_1^2\right)^2\right)\alpha^4 - i\left(\frac{2}{45}(11M_1M_2 + 4M_4)\right.$$
$$\left. +\frac{2}{9}(M_1M_2 - M_4)\left(1-\nu_1^2\right)\right)\alpha^5 + \left(\frac{1}{1260}\left(386M_1M_3 - 74M_1^3\right.\right.$$
$$\left. +280M_2^2 + 101M_5\right) - \frac{1}{630}\left(37M_1^3 + 80M_1M_3 + 114M_5\right)\left(1-\nu_1^2\right)$$
$$+\frac{1}{1260}\left(24M_1^3 + 78M_1M_3 + 87M_5\right)\left(1-\nu_1^2\right)^2 - \frac{2}{1575}(2M_1M_3$$

(4.2) [cont.]

$$-M_1^3 - M_5\Big)\left(1-\nu_1^2\right)^3\Big)\alpha^6 + i\left(\frac{1}{630}\left(75M_1^2M_2 + 72M_1M_4\right.\right.$$

$$\left.+156M_2M_3 + 12M_6\right) + \frac{2}{135}\left(M_1^2M_2 - M_1M_4 - M_2M_3 + M_6\right)\left(1-\nu_1^2\right)^2$$

$$\left.-\frac{2}{45}\left(M_1M_4 - 4M_1^2M_2 + 2M_2M_3 + M_6\right)\left(1-\nu_1^2\right)\right)\alpha^7 - \frac{16M_1\alpha^2\lambda^5}{45\pi^2}\Bigg\}$$

$$+\frac{16\lambda^5}{45\pi^2}\left\{-\frac{M_1\alpha^2}{\nu_1}\left(\frac{\sin^{-1}\nu_1}{\nu_1} + \left(1-\nu_1^2\right)^{-1/2}\right)\right.$$

$$+\frac{1}{\nu_1^3}\left(-1+\frac{2}{3}M_1\alpha^2 - \frac{2}{7}\lambda^2\right)\left(-\frac{3\sin^{-1}\nu_1}{\nu^1} + \left(1-\nu_1^2\right)^{1/2} + 2\left(1-\nu_1^2\right)^{-1/2}\right)$$

$$\left.-\frac{3\lambda^2}{14\nu_1^5}\left(-\frac{15\sin^{-1}\nu_1}{\nu_1} - 2\left(1-\nu_1^2\right)^{3/2} + 9\left(1-\nu_1^2\right)^{1/2} + 8\left(1-\nu_1^2\right)^{-1/2}\right)\right\}$$

$$-\frac{2\lambda}{3\pi}\left\{2\left(-1+\frac{2}{3}M_1\alpha^2 + \frac{i}{3}M_2\alpha^3\right)\left(3\nu_2\sin^{-1}\left(\frac{1}{\nu_2}\right) - \frac{\left(\nu_2^2-1\right)^{1/2}}{\nu_2}\right.\right.$$

$$\left.-2\nu_2\left(\nu_2^2-1\right)^{-1/2}\right) - \frac{\lambda^2\nu_2^2}{5}\left(15\nu_2\sin^{-1}\left(\frac{1}{\nu_2}\right) + \frac{2\left(\nu_2^2-1\right)^{3/2}}{\nu_3^3}\right.$$

$$\left.\left.-\frac{9\left(\nu_2^2-1\right)^{1/2}}{\nu_2} - 8\nu_2\left(\nu_2^2-1\right)^{-1/2}\right)\right\} - \frac{8M_1\alpha^2\lambda^4}{45\pi\nu_1\left(\nu_2^2-1\right)^{1/2}}O(\alpha^8)\Bigg],$$

$$b \le r \le a,$$

where $\nu_1 = r/a$ and $\nu_2 = r/b$.

Substituting $\alpha = 0$ and $\lambda = 0$ in Eq. (4.2), the jump in the stress across the rigid circular disc of radius a embedded at the bimaterial interface is easily found to be

$$\tau_0(r) = -\frac{4(\mu_1+\mu_2)\Omega}{\pi}\frac{\nu_1}{(1-\nu_1^2)^{1/2}},$$

so that the stress intensity factor at the edge of the circular disc in the statical case is

$$K_0 = \operatorname*{Lt}_{r\to a^-}\left[(1-\nu_1)^{1/2}\tau_0(r)\right] = -\frac{2\sqrt{2}(\mu_1+\mu_2)\Omega}{\pi}.$$

Therefore, in our dynamical problem involving annular disc, stress intensity factors at the outer and inner edges of the disc defined by

$$K_a^* = \frac{K_a}{K_0} = \operatorname*{Lt}_{r\to a^-}\left[\frac{\tau(r)\,(1-\nu_1)^{1/2}}{K_0}\right]$$

and

$$K_b^* = \frac{K_b}{K_0} = \operatorname*{Lt}_{r\to b+}\left[\frac{\tau(r)\,(\nu_2-1)^{1/2}}{K_0}\right]$$

are given by

$$(4.3)\qquad K_a^* = -\frac{1}{2}\Bigg[\Bigg\{-2+\frac{2}{3}M_1\alpha^2+\frac{1}{30}\left(7M_1^2-12M_3\right)\alpha^4 +\frac{1}{1260}\left(386M_1M_3-74M_1^3+280M_2^2+101M_5\right)\alpha^6 -\frac{32\lambda^5}{45\pi^2}\left(1+\frac{1}{3}M_1\alpha^2+\frac{8}{7}\lambda^2\right)\Bigg\} +i\Bigg\{\frac{2}{3}M_2\alpha^3-\frac{2}{45}(11M_1M_2+4M_4)\,\alpha^5 +\frac{1}{630}\left(75M_1^2M_2+72M_1M_4+156M_2M_3+12M_6\right)\alpha^7\Bigg\}\Bigg]$$

and

$$(4.4)\qquad K_b^* = -\frac{1}{2}\left[\left\{\frac{8\lambda}{3\pi}\left(-1+\frac{2}{3}M_1\alpha^2\right)-\frac{16\lambda^3}{15\pi}-\frac{8M_1\alpha^2\lambda^3}{45\pi}\right\}+i\frac{8\lambda}{9\pi}M_2\alpha^3\right].$$

The torque of the shear stress on the annular disc is represented by the expression

$$(4.5)\qquad T = 2\pi\int_b^a r^2\tau(r)\,dr,$$

which can be written, after putting the value of $\tau(r)$ given by Eq. (4.5) and integrating, as follows:

$$(4.6)\qquad T = \frac{4}{3}(\mu_1+\mu_2)\,\Omega a^3\Bigg[-4+\frac{8}{5}M_1\alpha^2+\frac{4i}{3}M_2\alpha^3 +\frac{1}{105}(37M_1-72M_3)\,\alpha^4-\frac{4i}{15}(4M_1M_2+M_4)\,\alpha^5 +\frac{1}{4725}\left(2704M_1M_3+2100M_2^2-650M_1^3+472M_5\right)\alpha^6 +\frac{i}{1575}\left(491M_1^2M_2+328M_1M_4+720M_2M_3+36M_6\right)\alpha^7 +\frac{64\lambda^5}{15\pi^2}\left(1-\frac{4}{3}M_1\alpha^2+\frac{4}{7}\lambda^2\right)+O(\alpha^8)\Bigg].$$

Next, in order to deduce the far-field amplitude of the displacement in both the media, we substitute the value of $A_j(\xi)$ in Eq. (2.8) and obtain

$$v_j(r,z) = \int_b^a tf(t)\,dt \int_0^\infty \frac{\xi}{(\mu_1\gamma_1 + \mu_2\gamma_2)} J_1(\xi r) J_1(\xi t) \exp(-\gamma_j|z|)\,d\xi. \tag{4.7}$$

Evaluating the integral with respect to ξ by the method of steepest descent for large values of $\sqrt{(r^2+z^2)}$, we obtain finally for $z > 0$

$$v_1(r,\theta,z) = F_1(\theta)\frac{e^{ik_1R_1}}{R_1} + O\left(\frac{1}{R_1^2}\right) \qquad \text{as} \quad R_1 \to \infty, \tag{4.8}$$

where $r = R_1\cos\theta$, $|z| = R_1\sin\theta$,

$$\begin{aligned} F_1(\theta) &= -\frac{i\sigma\sin\theta}{\sigma\mu_1\sin\theta + \mu_2\sqrt{(1-\sigma^2\cos^2\theta)}}G_1(\theta), \quad \text{for } |\cos\theta| < \frac{1}{\sigma}, \\ F_1(\theta) &= -\frac{\sigma\sin\theta\left[\mu_2\sqrt{(\sigma^2\cos^2\theta-1)} + i\sigma\mu_1\sin\theta\right]}{\sigma^2\mu_1^2\sin^2\theta + \mu_2(\sigma^2\cos^2\theta - 1)}G_1(\theta), \\ &\qquad\qquad \text{for} \quad |\cos\theta| > \frac{1}{\sigma}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} G_1(\theta) = \frac{2\Omega a^2(\mu_1+\mu_2)\sigma\alpha\cos\theta}{\pi}\Big[&-\frac{2}{3} + \frac{4}{15}M_1\alpha^2 + \frac{2i}{9}M_2\alpha^3 \\ &+\frac{1}{630}\left(37M_1^2 - 72M_3\right)\alpha^4 - \frac{2i}{45}\left(4M_1M_2 + M_4\right)\alpha^5 \\ &+\frac{1}{2835}\left(256M_1M_3 - 65M_1^3 + 210M_2^2 + 40M_5\right)\alpha^6 \\ &+\frac{i}{9450}\left(491M_1^2M_2 + 328M_1M_4 + 720M_2M_3 + 36M_6\right)\alpha^7 \\ -\frac{\sigma^2\alpha^2\cos^2\theta}{6}&\left\{-\frac{2}{5} + \frac{16}{105}M_1\alpha^2 + \frac{2i}{15}M_2\alpha^3 + \frac{1}{1890}\left(73M_1^2 - 136M_3\right)\alpha^4\right. \\ -\frac{2i}{1575}\left(82M_1M_2 + 23M_4\right)\alpha^5&\left.\right\} + \frac{\sigma^4\alpha^4\cos^4\theta}{120}\left\{-\frac{2}{7} + \frac{20}{189}M_1\alpha^2 + \frac{2i}{21}M_2\alpha^3\right\} \\ &+\frac{\sigma^6\alpha^6\cos^6\theta}{22680} + \frac{32\lambda^5}{45\pi^2}\left\{1 - \frac{4}{3}M_1\alpha^2 + \frac{4}{7}\lambda^2 + \frac{1}{6}\sigma^2\alpha^2\cos^2\theta\right\}\Big]. \end{aligned} \tag{4.10}$$

Also for $z < 0$,

$$v_2(r,\phi,z) = F_2(\phi)\frac{e^{ik_2R_2}}{R_2} + O\left(\frac{1}{R_2^2}\right) \quad \text{as} \quad R_2 \to \infty, \tag{4.11}$$

where $r = R_2\cos\phi$, $|z| = R_2\sin\phi$

$$F_2(\phi) = -\frac{i\sin\phi}{\mu_1\sqrt{(\sigma^2-\cos^2\phi)} + \mu_2\sin\phi}G_2(\phi), \tag{4.12}$$

and $G_2(\phi)$ is obtained by replacing θ by ϕ and also σ by 1 in $G_1(\phi)$.

5. Numerical results and discussion

Numerical results have been calculated to study the variations of the dynamic stress intensity factors with the normalized frequency α at both the outer and inner edges of the annular disc, situated at the bimaterial interface, for different values of the ratio of the inner and outer radii of the annular disc, for the following two sets of materials.

First set

Aluminium	$\rho_1 = 2.7\,\text{gm/cm}^3$	$\mu_1 = 2.63 \times 10^{11}\,\text{dyne/cm}^2$
Wrought iron	$\rho_2 = 7.8\,\text{gm/cm}^3$	$\mu_2 = 7.7 \times 10^{11}\,\text{dyne/cm}^2$

Second set

Copper	$\rho_1 = 8.96\,\text{gm/cm}^3$	$\mu_1 = 4.5 \times 10^{11}\,\text{dyne/cm}^3$
Steel	$\rho_2 = 7.6\,\text{gm/cm}^3$	$\mu_2 = 8.32 \times 10^{11}\,\text{dyne/cm}^2$

The dynamic stress intensity factors are normalized by the static solution $K_0 = -2\sqrt{2}\,(\mu_1 + \mu_2)\Omega/\pi$ for the penny-shaped rigid disc.

It is interesting to note that for both the two sets of materials, the stress intensity factor at the outer edge changes appreciably with the normalized frequency α and gradually decreases with the increase of the α; however in the case of the inner edge, the stress intensity factor decrease very slowly with the increase in the values of the normalized frequency. It may further be noted that the changes in the values of the stress intensity factor following the increase in the values of λ is more pronounced at the inner edge than that at the outer edge. We also note from Fig. 2 and Fig. 3 that stress intensity factors for the two sets of materials are nearly the same in case of low frequency, and increase gradually with the increase in frequency.

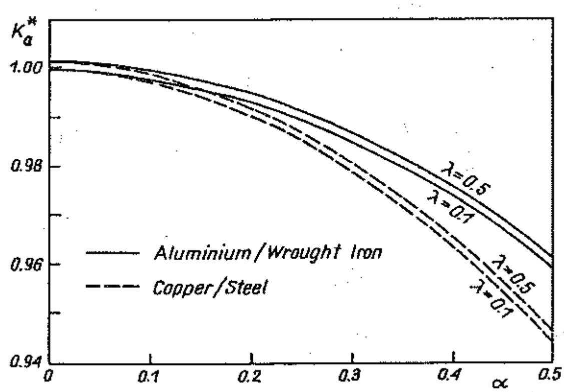

Fig. 2. Stress intensity factor vs. normalized frequency (outer edge).

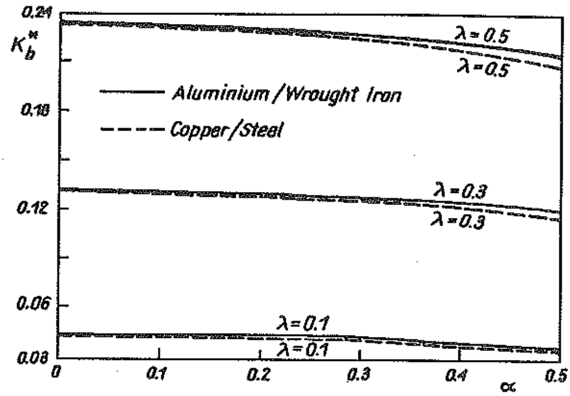

FIG. 3. Stress intensity factor vs. normalized frequency (inner edge).

Far-field amplitudes defined by $F_1(\theta)$ and $F_2(\phi)$ in the upper and lower medium $z > 0$ and $z < 0$, respectively, for a fixed value of R, have been plotted in Fig. 4–Fig. 7 against their arguments for different values of the normalized frequency α and λ, the ratio of the inner and outer radii of the annular disc for two different sets of materials.

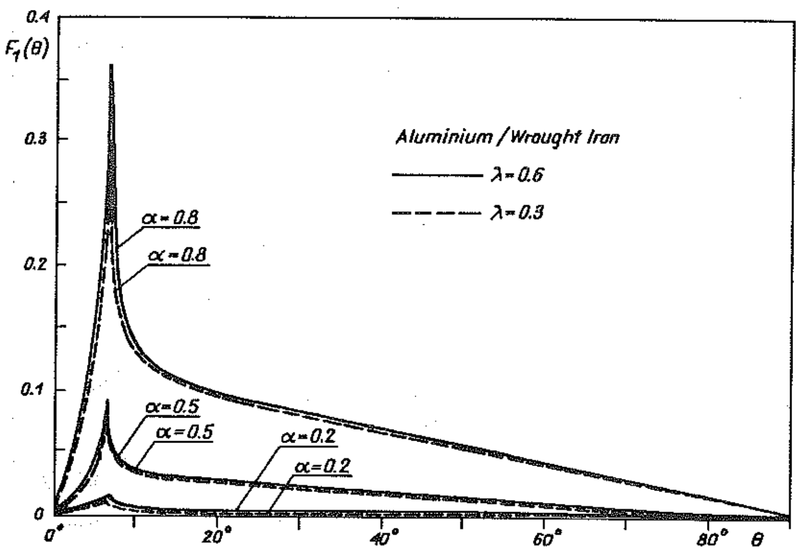

FIG. 4. Far-field amplitude $F_1(\theta)$ vs. argument θ of the amplitude for upper medium ($z > 0$).

It may be noted that both in the upper and lower medium for the two sets of materials, amplitudes $F_1(\theta)$ and $F_2(\phi)$, respectively, increase gradually

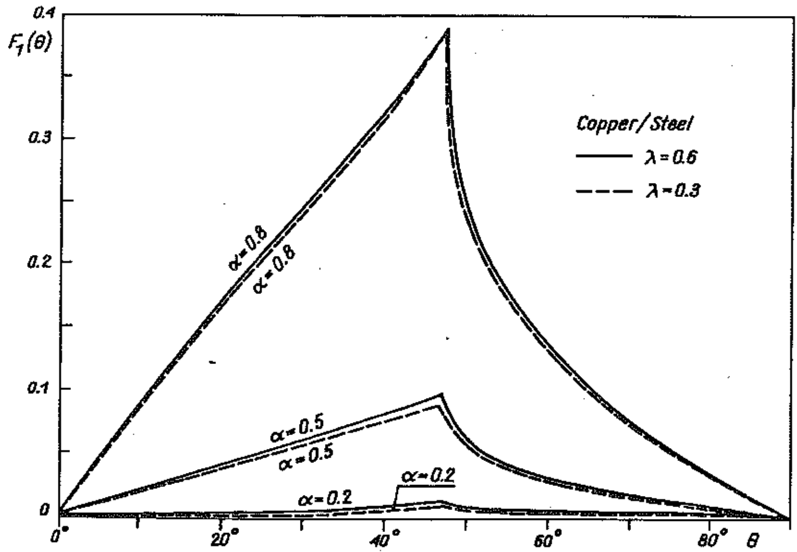

FIG. 5. Far-field amplitude $F_1(\theta)$ vs. argument θ of the amplitude for upper medium ($z > 0$).

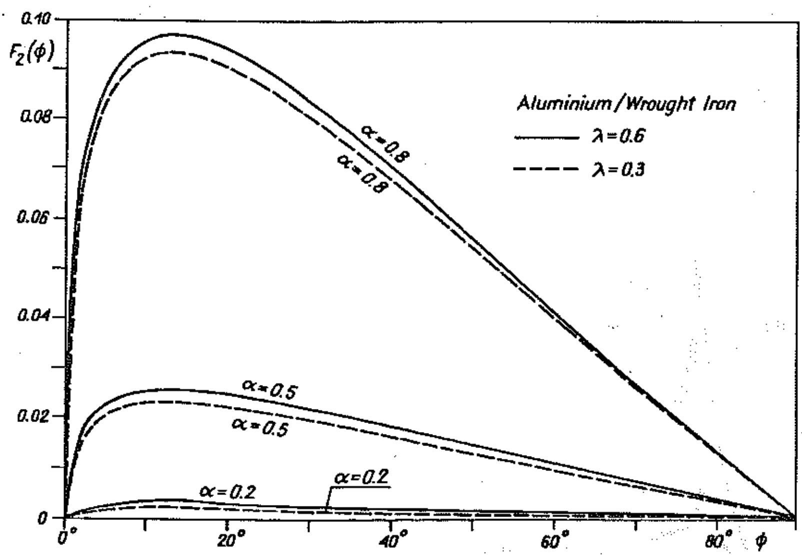

FIG. 6. Far-field amplitude $F_2(\phi)$ vs. argument ϕ of the amplitude for lower medium ($z < 0$).

from θ and ϕ equal to zero, attain maximum values and then gradually decrease to zero at θ and ϕ equal to 90°. The values of the angle at which the maxima attained are found to depend on the material properties and not on the values of the frequency and λ. On the other hand, if the material

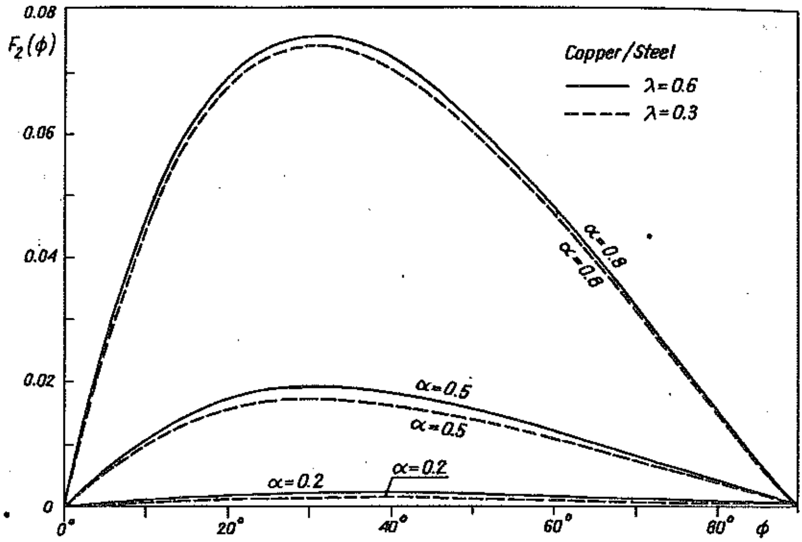

FIG. 7. Far-field amplitude $F_2(\phi)$ vs. argument ϕ of the amplitude for lower medium ($z < 0$).

properties are kept fixed, maximum values of the far-field amplitude are found to depend on the normalized frequency α and λ, which is equal to the ratio of the inner and outer radii of the annular disc.

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Received November 12, 1992.