ENGINEERING TRANSACTIONS • Engng. Trans. • 42, 4, 393–398, 1994
Polish Academy of Sciences • Institute of Fundamental Technological Research

ANALYSIS OF GENERALIZED SELF-EXCITED VIBRATIONS

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The asymptotic nonlinear method is applied to solve the generalized equation of the self-excited vibration. In the last several decades, increased interest in the study of asymptotic analysis is observed, based mainly on application of the exact convergence. The solution describes better the self-excited vibrations of a real mechanical system.

1. Introduction

Analysis of self-excited vibrations [4, 6, 9] in machines and tools with application of the exact convergence has been performed. Nonlinearity of the analysed vibration equations is often connected with negative damping characteristic, input and dissipation of energy in the work cycle. Variable dry friction force [1, 5, 9] is the main feedback cause of such a system.

Power series convergence of the analysed asymptotic solution was proved by CHOMA [2].

Analysis of nonlinear vibrations is complicated and only in a few cases exact solutions exist. The method presented here is more efficient than most of the approximate [3, 8, 10] approaches. The asymptotic analysis [2, 7] represents a progress in the small parameter application. Accuracy of asymptotic nonlinear solution is very high.

2. The nonlinear equation of self-excited vibrations

The general nature of self-excited vibrations is in the reality more complicated than the basic models of Rayleigh and Van der Pol. Power series with good convergence are more suitable to describe the nonlinear effects.

We can write the differential nonlinear equation of self-excited vibrations in the form

$$\ddot{x} + x = \varepsilon\,\dot{x} \sum_{i=1}^{\infty} (-1)^{i-1} C_i x^{2(i-1)}, \tag{2.1}$$

where $\varepsilon = \varepsilon(\omega, \mu)$ – small parameter, ω – angular frequency, μ – friction coefficient, C_i – constant.

The function of friction is symmetric for the velocity of propagation occurring in both directions.

In the asymptotic theory of nonlinear self-excited vibrations the first constants are

$$C_1 = C_2 = 1. \tag{2.1$_1$}$$

Other constants are determined experimentally.

3. The asymptotic solution

General solution of differential nonlinear Eq. (2.1) has the form

$$x = A(t,\varepsilon)\cos[\alpha(t,\varepsilon)], \tag{3.1}$$

$$A(t,\varepsilon) = B + \varepsilon D_1(t,B,h) + \varepsilon^2 D_2(t,B,h) + \ldots, \tag{3.2}$$

$$\alpha(t,\varepsilon) = \omega t + h + \varepsilon\beta_1(t,B,h) + \varepsilon^2\beta_2(t,B,h) + \ldots. \tag{3.3}$$

We analyse the nonlinear part of (2.1) with the values $(2.1)_1$

$$\varepsilon f(x,\dot{x}) = \varepsilon\, \dot{x}\, (1 - x^2 + C_3 x^4 - \ldots). \tag{3.4}$$

The first approximate solution is given by

$$\begin{aligned} x &\approx \sum_{n=1}^{\infty} B_n(t,\varepsilon)\cos\left[\omega_n^{(0)} t + h_n(t,\varepsilon)\right], \\ \dot{x} &\approx -\sum_{n=1}^{\infty} \omega_n^{(0)} B_n(t,\varepsilon)\sin\left[\omega_n^{(0)} t + h_n(t,\varepsilon)\right]. \end{aligned} \tag{3.5}$$

Using the last formulae we obtain

$$\begin{aligned} \varepsilon f \equiv -\varepsilon \sum_{n=1}^{\infty} \omega_n^{(0)} \dot{B}_n(t,\varepsilon)\sin\left[\omega_n^{(0)} t + h_n(t,\varepsilon)\right] \\ \times \left(1 - \left(\sum_{n=1}^{\infty} B_n(t,\varepsilon)\cos\left[\omega_n^{(0)} t + h_n(t,\varepsilon)\right]\right)^2\right. \\ \left. + C_3\left(\sum_{n=1}^{\infty} B_n(t,\varepsilon)\cos\left[\omega_n^{(0)} t + h_n(t,\varepsilon)\right]\right)^4 - \ldots\right). \end{aligned} \tag{3.6}$$

Important are the Fourier series coefficients expressed in the following form:

$$(3.7)\quad \begin{aligned} f_{0n} &= -\omega_n B_n \sin\gamma_\nu^{(0)}\left[1 - B_n^2\cos^2\varphi_n^{(0)} + C_3 B_n^4 \cos^4\varphi_n^{(0)} - \ldots\right],\\ \varphi_n^{(0)} &= \omega_n^{(0)} t + h_n . \end{aligned}$$

From the asymptotic averaging we obtain the results

$$(3.8)\quad \frac{dB_n}{dt} = -\frac{\varepsilon}{\omega} f_{0n}(A,\varphi)\sin\varphi,$$

$$(3.9)\quad \frac{dh_n}{dt} = -\frac{\varepsilon}{\omega} f_{0n}(A,\varphi)\cos\varphi$$

which, on the basis of Eq. $(3.7)_1$, can be expressed in the form

$$(3.10)\quad \frac{dB_n}{dt} = \varepsilon B_n \sin^2\varphi_n^{(0)}\left[1 - B_n^2\cos^2\varphi_n^{(0)} + C_3 B_n^4\cos^4\varphi_n^{(0)} - \ldots\right],$$

$$(3.11)\quad \frac{dh_n}{dt} = \varepsilon B_n \sin\varphi_n^{(0)}\cos\varphi_n^{(0)}\left[1 - B_n^2\cos^2\varphi_n^{(0)} + C_3 B_n^4\cos^4\varphi_n^{(0)} - \ldots\right].$$

Very important part of the generalized analysis is the problem of averaging convergence

$$(3.12)\quad \underset{t}{M}\,(f_0(A,\varphi)\sin\varphi) \equiv \lim_{T\to\infty}\frac{1}{T}\int^{T} f_0(A,\varphi)\sin\varphi\, dt = E_1,$$

$$(3.13)\quad \begin{aligned} &\underset{t}{M}\left(\sin^2\varphi_n^{(0)}\left[1 - B_n^2\cos^2\varphi_n^{(0)} + C_3B_n^4\cos^4\varphi_n^{(0)} - \ldots\right]\right)\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \sin^2\varphi_n^{(0)}\left[1 - B_n^2\cos^2\varphi_n^{(0)} + C_3B_n^4\cos^4\varphi_n^{(0)} - \ldots\right]dt\\ &= \frac{1}{2}\left(1 - \frac{B_n^2}{8} + \frac{1}{16}C_3B_n^4 - \ldots\right), \end{aligned}$$

$$(3.14)\quad \underset{t}{M}\,(f_0(B,\varphi)\cos\varphi) \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^{T} f_0(B,\varphi)\cos\varphi\, dt = E_2,$$

$$(3.15)\quad \begin{aligned} &\underset{t}{M}\left(\sin\varphi_n^{(0)}\cos\varphi_n^{(0)}\left[1 - B_n^2\cos^2\varphi_n^{(0)} + C_3B_n^4\cos^4\varphi_n^{(0)} - \ldots\right]\right)\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \sin\varphi_n^{(0)}\cos\varphi_n^{(0)}\left[1 - B_n^2\cos^2\varphi_n^{(0)}\right.\\ &\qquad \left. + C_3B_n^{(0)}\cos^4\varphi_n^{(0)} \ldots\right]dt = 0. \end{aligned}$$

Using the formulae (3.8) and (3.9) and results (3.13) and (3.15) with the friction damping we can write

$$\frac{dB_n}{dt} = -\frac{1}{2}\varepsilon B_n\left(-1+\frac{B_n^2}{8}-\frac{1}{16}C_3B_n^4-\ldots\right), \tag{3.16}$$

$$\frac{dh_n}{dt} = 0. \tag{3.17}$$

Initial data are

$$t_0 = 0, \tag{3.18}$$

$$B_n = B_{n0}, \tag{3.19}$$

$$h_n = h_{n0} \tag{3.20}$$

and we can integrate the formulae (3.16) and (3.17) to obtain

$$\int\limits_{B_{n0}}^{B_n} \frac{dB_n}{B_n\left(-1+\frac{B_n^2}{8}-\frac{C_3B_n}{16}-\ldots\right)} = -\frac{\varepsilon t}{2}, \tag{3.21}$$

$$B_n = F_0 f_1\left(e^{-\frac{\varepsilon t}{2}}\right), \tag{3.22}$$

$$\frac{dh_n}{dt} = 0, \tag{3.23}$$

$$h_n = h_{n0}. \tag{3.24}$$

Amplitude (3.2) and phase (3.3) depend on the expansion coefficients. They are found from the following equations:

$$\frac{\partial D_1}{\partial t} = -\frac{1}{\omega}\left(f_0(B,\varphi)\sin\varphi - \underset{t}{M}\left(f_0(D,\varphi)\sin\varphi\right)\right), \tag{3.25}$$

$$\frac{\partial \beta_1}{\partial t} = -\frac{1}{B\omega}\left(f_0(B,\varphi)\cos\varphi - \underset{t}{M}\left(f_0(D,\varphi)\cos\varphi\right)\right), \tag{3.26}$$

$$\frac{\partial D_1}{\partial t} = B_n\Bigg(\sin^2\varphi_n^{(0)}\left[1-B_n^2\cos^2\varphi_n^{(0)}+C_3B_n^4\cos^4\varphi_n^{(0)}-\ldots\right] - \frac{1}{2}\left(1-\frac{B_n^2}{8}+\frac{1}{16}B_n^4C_3-\ldots\right)\Bigg), \tag{3.27}$$

$$\frac{\partial \beta_1}{\partial t} = \sin\varphi_n^{(0)}\cos\varphi_n^{(0)}\left[1-B_n^2\cos^2\varphi_n^{(0)}+C_3B_n^4\cos^4\varphi_n^{(0)}-\ldots\right]. \tag{3.28}$$

Phase $\varphi_n^{(0)} = \varphi_n^{(0)}(t)$ is given by the formula $(3.7)_2$, and therefore the integration is possible; we obtain

$$
\begin{aligned}
(3.29) \qquad D_1 = B_n \Big(&\frac{1}{2}\gamma_n^{(0)} - \frac{1}{4}\sin 2\gamma_n^{(0)} - B_n^{(2)}\left[\frac{1}{8}\gamma_n^{(0)} - \frac{1}{32}\sin 4\gamma_n^{(0)}\right] \\
&+ C_3 B_n^4 \left(-\frac{1}{6}\sin^3\gamma_n^{(0)} + \frac{1}{2}\left[\frac{1}{8}\gamma_n^{(0)} - \frac{1}{32}\sin 4\gamma_n^{(0)}\right]\right) - \ldots\Big) \\
&- \frac{1}{2}\left(B_n - \frac{B_n^3}{24} + \frac{C_3 B_n^5}{80} - \ldots\right) t - \frac{1}{2} h_{n0} + \frac{1}{2}\sin 2h_{n0} \\
&+ B_n^2\left(\frac{1}{8}h_{n0} - \frac{1}{32}\sin 4h_{n0}\right) - C_3 B_n^4 \Big[-\frac{1}{6}\sin h_{n0} \\
&+ \frac{1}{2}\left(\frac{1}{8}h_{n0} - \frac{1}{32}\sin 4h_{n0}\right)\Big] - \ldots\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.30) \qquad \beta_1 = &\frac{1}{2}\sin^2\gamma_n^{(0)} + \frac{1}{4}B_n^2\cos^4\gamma_n^{(0)} - \frac{1}{5}C_3 B_n^5\cos^5\gamma_n^{(0)} - \ldots \\
&- \frac{1}{2}\sin^2 h_{n0} - \frac{1}{4}B_n^2\cos^4 h_{n0} + \frac{1}{5}C_3 B_n^5 \cos^5 h_{n0} - \ldots .
\end{aligned}
$$

The solution of the Van der Pol equation is the result of analysis of the generalized equation (2.1) and of formulae (3.21), (3.24), (3.29) and (3.30)

$$
\begin{aligned}
(3.31) \qquad x = \sum_{n=1}^{\infty} \Big\{ &F_0 f_1\left(e^{-\frac{\varepsilon t}{2}}\right) + \varepsilon B_n \Big\{\frac{1}{2}\gamma_n^{(0)} - \frac{1}{4}\sin 2\gamma_n^{(0)} - B_n^2\Big[\frac{1}{8}\gamma_n^{(0)} \\
&- \frac{1}{32}\sin 4\gamma_n^{(0)}\Big] + C_3 B_n^4\Big(-\frac{1}{6}\sin^3\gamma_n^{(0)} + \frac{1}{2}\Big[\frac{1}{8}\gamma_n^{(0)} \\
&- \frac{1}{32}\sin 4\gamma_n^{(0)}\Big]\Big) - \ldots - \frac{1}{2}\left(B_n - \frac{B_n^3}{24} + \frac{C_3 B_n^5}{80} - \ldots\right) t \\
&- \frac{1}{2}h_{n0} + \frac{1}{2}\sin 2h_{n0} + B_n^2\left(\frac{1}{8}h_{n0} - \frac{1}{32}\sin 4h_{n0}\right) \\
&- C_3 B_n^4\Big[-\frac{1}{6}\sin h_{n0} + \frac{1}{2}\left(\frac{1}{8}h_{n0} - \frac{1}{32}\sin 4h_{n0}\right)\Big] - \ldots\Big\} + \varepsilon^2 \ldots\Big\} \\
&\times \cos\Big\{\omega_n t + h_n + \varepsilon\Big[\frac{1}{2}\sin^2\gamma_n^{(0)} + \frac{1}{4}B_n^2\cos^4\gamma_n^{(0)} \\
&- \frac{1}{5}C_3 B_n^5\cos^5\gamma_n^{(0)} - \ldots - \frac{1}{2}\sin^2 h_{n0} - \frac{1}{4}B_n^2\cos^4 h_{n0} \\
&+ \frac{1}{5}C_3 B_n^5 h_{n0} - \ldots\Big] + \varepsilon^2 \ldots\Big\} = x_1 + x_2 + \ldots + x_k + \ldots .
\end{aligned}
$$

Important is the radius of convergence of the solution. The parameter has been found in the form

$$
(3.32) \qquad \rho = \lim_{k\to\infty} \sqrt[k]{x_k}
$$

and

$$\rho < 1. \tag{3.33}$$

4. Concluding remarks

1. Convergence of the decisive part of the equation is good.

2. The radius of convergence of the power series representing the accurate solution is sufficient.

3. Van der Pol equation of self-excited vibrations was expanded and solved by an asymptotic method.

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Received November 19, 1993; new version August 24, 1994.