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DYNAMIC COMPRESSION OF A BRITTLE SPHERICAL SPECIMEN

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The dynamic compression process of a spherical specimen made of a brittle material in the split Hopkinson pressure bar (SHPB) is described. The aim of the present paper is to determine the stress distribution over a spherical surface as a result of passage of a wave carrying compressive stresses. The specimen is treated as a quasi-rigid body, the wave process in the SHPB system being taken into account, however. Solution of such a problem is essential for the description of the disintegrating (crushing) process of rock materials, for instance.

1. Introduction

The dynamic properties of a material, that is the surface $(\sigma, \varepsilon, \dot{\varepsilon}) = 0$ can be determined by means of Hopkinson's classical theory [9]. There is also another interesting problem to be solved, however, namely that of determining the dynamic behaviour of the material depending on its form. We shall consider the simplest case, that is the case of compression of a spherical specimen in the SHPB system. In such a case the strain, the strain rate and the stress cannot be determined by using average values which are assumed with the classical measurement method based on the technique of Hopkinson bars. The distribution of those quantities along the compression axis should not be assumed to be uniform, in view of the form of the specimen considered. Due to the spatial type of the loading, the complicated geometry of the problem and the contact conditions between the specimen and the bars, the problem of wave solution due to dynamic compression of a spherical specimen is complicated. We shall propose a simplified method for determining the dynamic behaviour of the material, depending on the form of the specimen. Solution of this problem is essential for the description of the disintegration (spalling) process of a rock material.

There are three principal stages of solving the problem of dynamic destruction of a spherical specimen (see Fig. 1).

Stage 1: determination of the stress distribution over the surface of the spherical specimen as a result of passage of a compression wave through

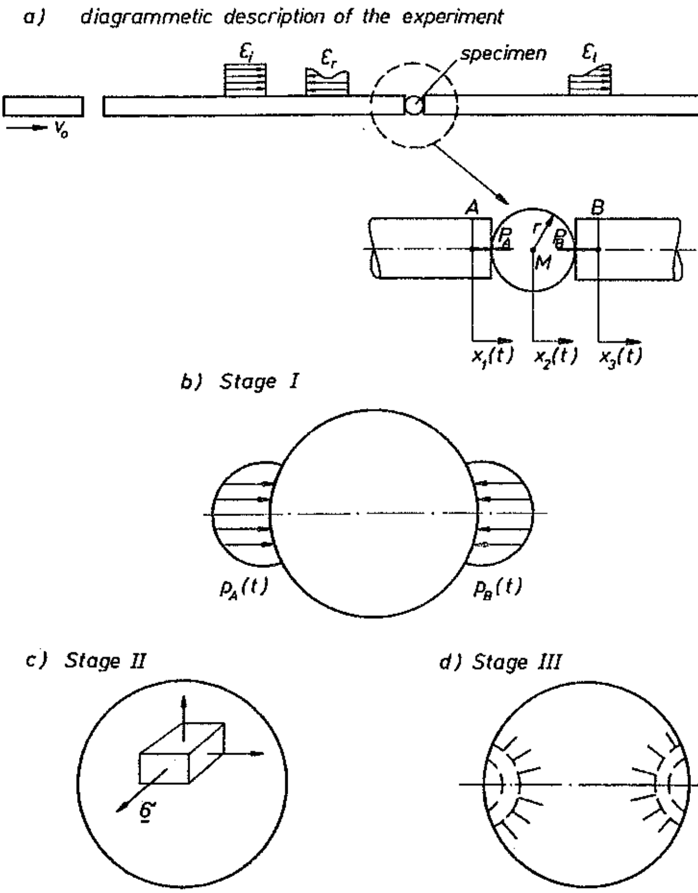

FIG. 1. Dynamic compression of a spherical specimen.

the system composed of the Hopkinson bars and the specimen considered (Fig. 1a and b),

Stage 2: determination of internal stresses in the spherical specimen due to local stresses at the contact between the bars and the specimen (Fig. 1c),

Stage 3: determination of the form of destruction due to internal stresses (Fig. 1d).

The present considerations will be confined to the first stage.

The impact of a missile in the form of a bar, l_0 in length, at a velocity v_0, on the transmitting bar produces in it an elastic wave. Let us assume, for simplicity, that the strain pulse propagating through that bar has a constant amplitude $\varepsilon_i(t) = -v_0/2c_0 = \text{const}$, where $c_0 = (E/\rho)^{1/2}$ is the velocity of propagation of an elastic wave through the bar. The duration of the pulse is $t_i = 2l_0/c_0$. The velocity of the transmitting bar is $v_1 = 0.5v_0$ assuming that the bar constituting the missile is made of the same material as the transmitting bar, and both bars have identical cross-sections. This is an idealised form of wave generating pulse. Examples of real stress pulses obtained using the Hopkinson bar technique are shown in Fig. 2a, b, c. In the first case we observe, for a pulse of 24.8 μs – cf. [13], the occurrence of a second, distinct

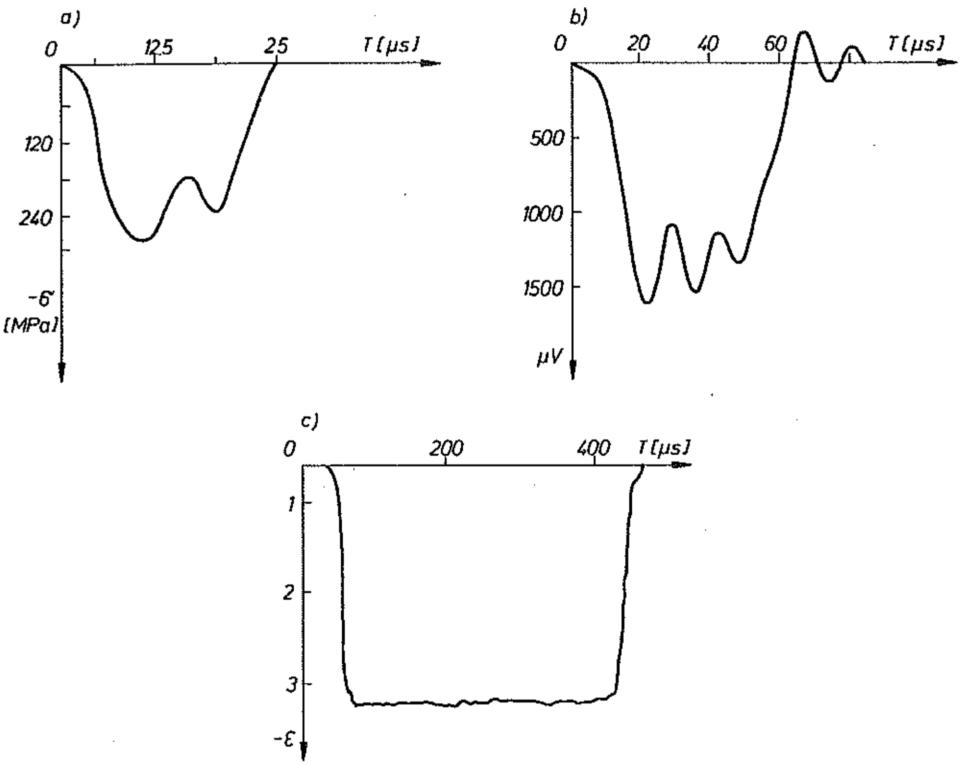

FIG. 2. The form of the pulse load acting on a specimen: a) very short pulse [13], b) short pulse [3], c) relatively long pulse [5, 6].

local maximum (of about 19 μs). Consecutive maxima of this type may occur, if a dynamic experiment is made in a system of Hopkinson bars using missiles which have the form of short bars, the length of which is decisive for the duration of the pulse – see also Fig. 2b [3] – a pulse 60 μs. For a relatively long missile-bar producing a pulse of 400 μs, its form approaches that of a trapeze, Fig. 2c [5, 6]. In view of geometrical dispersion the pulses must be corrected in the measuring bars, see for instance [2], making use of approximations based on the Pochhammer - Chree analysis. To this aim the "DAVID" (Depouillement Automatique et VIsualisation pour compression Dynamique) system of data processing may be used, see [5]. This correction is indispensable, if a long transmitting measurement bar is used.

2. Determination of stress distribution over the surface of a spherical specimen

First, the stress magnitude and distribution (the contact conditions between the specimen and the Hopkinson bars) must be determined. This can be done in an analytical manner by combining the solution of the problem of dynamic collision of a deformable body with the static solution, to determine the pressure between two bodies in contact, similarly to the case of collision of two spherical bodies as described by S. Timoshenko [16].

On the basis of the observation which was made by Rayleigh who stated that the duration of the contact between two colliding bodies is very long as compared with their natural vibration period, those vibrations may be neglected. In the case under consideration the duration t_i of the pulse is very long as compared with the time of passage of a wave through the specimen t_p ($t_3 - t_1 \gg t_p$ because $r \ll 2l_1$). In a time $t_1 \leq t \leq t_3$, in which an elastic wave would travel a distance twice as long as the measuring bar, the number of passages of that wave through the specimen would be several hundred, as a result of which the specimen would attain a state of internal quasi-equilibrium. In this connection the process of wave propagation through the specimen may be neglected by treating the latter as a quasi-rigid body. This is a principle assumed in the classical technique of Hopkinsons bars for compression testing of thin cylindrical specimens, and also in what is termed "Brazilian test", cf. [8]. The wave process will be taken into account, however, in bars of the Hopkinson system. In this connection, to determine the contact forces between the measuring bars and the specimen, it may be assumed that the Hertz law for static conditions (cf. [16], for instance) may also be used for the dynamic problem under consideration. Our task will be to determine the force of collision, the local strain and the duration of the contact. The essence of the idea of applying the Hertz static theory of contact consists in assuming that the distribution of stress and strain, not their values, over the region of contact during collision is the same as in the case of static load. This assumption has been confirmed experimentally. Satisfactory results have been obtained within the range of elastic strain. The method just described has been generalized to the case of elastic-plastic impact (cf. [1], for instance).

Let us consider the case of dynamic compression of a spherical specimen as represented in Fig. 1, and let us select two points A and B on the transmitting and receiving bar, respectively, sufficiently near their ends but at a distance at which the stress distribution over the cross-sections of the bars is uniform. The authors of [2] assume that the strain gauges recording the signals of the pulses are located at a distance of $10r$ (r – radius of a measuring bar) from the end of the bar. They assume that the stress pulse is, at such a distance, uniform over the entire cross-section of the measuring bar. This is a very safe assumption. Usually it is assumed that a longitudinal wave moves through the bar already at a distance of six times the radius of the bar (cf. [5, 9], for instance). The motion of the point A and B will be described by $x_1(t)$ and $x_3(t)$. The motion of a spherical specimen of radius r will be described by $x_2(t)$. The forces acting at the points A and B will be denoted by P_A and P_B, respectively. From the solution of the wave equation in the receiving and transmitting bar of the SHPB system we obtain the following relations (cf. [14], for instance):

• the relation along a positive characteristic in the region VI (see Fig. 3):

$$\sigma_A = \rho c_0 (v_A - v_0), \tag{2.1}$$

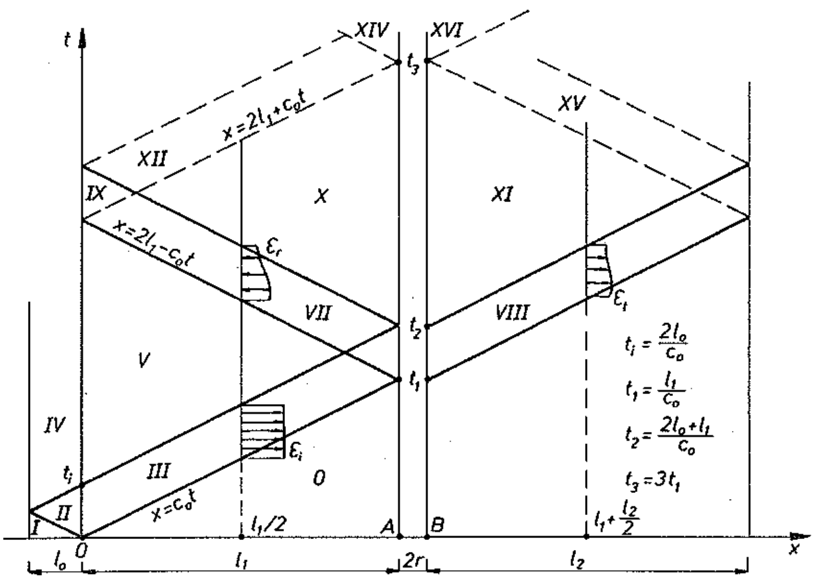

FIG. 3. An image of the solution in the (x, t)–plane.

• the relation along a negative characteristic in the region VIII assuming that the transmitting bar was at rest at the initial instant of time

$$\sigma_B = -\rho c_0 v_B . \tag{2.2}$$

The equation of dynamic equilibrium of the specimen has the form

$$P_B - P_A + M\, \ddot{x}_2\,(t) = 0, \tag{2.3}$$

where M is the mass of the specimen, $P_A = -F_0\sigma_A$, $P_B = -F_0\sigma_B$, and F_0 the cross-section area of the transmitting and receiving bar. The velocities of the points A and B are

$$v_A = \dot{x}_1(t), \qquad v_B = \dot{x}_3(t), \tag{2.4}$$

respectively.

Let us denote by α_1 and α_2 the lengths by which the specimen and the bars approach each other due to local compression in the planes of contact

$$\alpha_1(t) = x_1(t) - x_2(t), \qquad \alpha_2(t) = x_2(t) - x_3(t). \tag{2.5}$$

Hertz's law of static contact between two elastic bodies of revolution has the form

$$P = \beta\alpha^n, \tag{2.6}$$

where n and β depend on the geometry of the bodies and their physical properties. In the case under consideration we have

$$P_A = \beta(x_1 - x_2)^n, \qquad P_B = \beta(x_2 - x_3)^n \tag{2.7}$$

assuming that both Hopkinson bars are made of the same material.

From the Eq. (2.3) we have

$$\ddot{x}_2(t) = \beta(x_1 - x_2)^n / M - \beta(x_2 - x_2)^n / M. \tag{2.8}$$

On differentiating (2.1) and (2.2) with respect to time, the relations (2.5), (2.7) and (2.8) being taken into consideration, we obtain the following set of two ordinary differential equations of second order for the two quantities α_1 and α_2

$$\begin{aligned} &\ddot{\alpha}_1 + \lambda_1\alpha_1^{n-1}\dot{\alpha}_1 + \lambda_2(\alpha_1^n - \alpha_2^n) = 0, \\ &\ddot{\alpha}_2 + \lambda_1\alpha_2^{n-1}\dot{\alpha}_2 - \lambda_2(\alpha_1^n - \alpha_2^n) = 0, \end{aligned} \tag{2.9}$$

where $\lambda_1 = \beta n/\rho c_0 F_0$, $\lambda_2 = \beta/M$. It is a set of nonlinear differential equations to be solved for α_1 and α_2 with the following initial conditions for $t = t_1$:

$$\begin{aligned} &\alpha_1(t_1) = 0, \qquad \dot{\alpha}_1(t_1) = v_0, \\ &\alpha_2(t_1) = 0, \qquad \dot{\alpha}_2(t_1) = 0. \end{aligned} \tag{2.10}$$

3. The particular case of $M = 0$

Let us study a particular case of the problem of dynamic contact formulated above. We assume that the mass of the specimen is negligibly small as compared with the mass of the bars $(4\rho_p \pi r^3/3 \ll \rho\pi r_0^2(l_1 + l_2)$, where r_0 is the radius of the Hopkinson bars), that is $M \to 0$. Then, from the Eq. (2.3) we find that $P_A = P_B$. On adding the Eqs. (2.1) and (2.2), we obtain

$$\rho c_0 \left[\dot{x}_1(t) - x_3(t) - v_0\right] = 2\sigma, \tag{3.1}$$

where $\sigma_A = \sigma_B = \sigma$. From the Eq. (2.7), for $P_A = P_B$, we have $\alpha_1 = \alpha_2 = \alpha$ and $x_2 = 0.5(x_1 + x_3)$. Hence $x_1 - x_3 = 2a$. Equation (3.1) leads to the relation

$$\dot{\alpha} = 0.5 v_0 + \sigma(\alpha)/\rho c_0 . \tag{3.2}$$

Then

$$T = t_1 + \int_0^{\alpha} \frac{1}{0.5 v_0 + \sigma(\overline{\alpha})/\rho c_0} d\overline{\alpha}, \tag{3.3}$$

with the condition of $\alpha = 0$ for $t = t_1$ which has been used. According to Hertz's law (2.6), the stress $\sigma(\alpha)$ at the contact is determined by the formula $\sigma(\alpha) = -\beta\alpha^n/F_0$. By denoting the integral involved in (3.3) by the symbol $\psi(\alpha)$, the relative displacement of the ends of the bars can thus be expressed by

$$\alpha = \psi^{-1}(t). \tag{3.4}$$

The solution of this simple problem will play the role of a limiting case of the fundamental problem described by the set of equations (2.9) for $M \neq 0$. Let us assume temporarily that the ends of the bars in the Hopkinson system are slightly rounded (that is spherical with a large radius). From Hertz's theory for elastic bodies or the more general Steuermann theory [15] it follows for $n = 1$, where n is the degree of closeness of contact between the two surfaces, that the relation (2.6) takes the form

$$P(\alpha) = \beta\alpha^{3/2} \leftrightarrow \sigma(\alpha) = -\beta\alpha^{3/2}/F_0. \tag{3.5}$$

The magnitude of approach is maximum $(\alpha_{\max})$ if $\dot{\alpha} = 0$, therefore by virtue of (3.2) and bearing in mind (3.5), we have

$$\alpha_{\max} = \left(\frac{v_0 F_0 \rho c_0}{2\beta}\right)^{2/3}. \tag{3.6}$$

The relevant stress is

$$\sigma_{(\alpha_{\max})} = -v_0 \rho c_0/2 . \tag{3.7}$$

Denoting $x = \alpha/\alpha_{\max}$ $(0 \leq x \leq 1)$, we find from (3.3) the duration of the contact

$$t_k = t_1 + 2\frac{2\alpha_{\max}}{v_0}\int\limits_0^1 \frac{dx}{1 - x3/2} = t_1 + 6.715\frac{\alpha_{\max}}{v_0} . \tag{3.8}$$

The factor 2 follows from the fact that, assuming the elastic character of the impact, the duration of the contact is equal to the sum of duration of the two phases, that is the phase of loading and that of unloading, which are of equal duration. It should be observed that the duration of contact is in inverse proportion to the velocity of impact v_0 of the missile-bar.

If we assume the real changes occurring during the loading process, such as those which are shown as an example in Fig. 2, the velocity v_0 will be a function of time. On differentiating the relation along the positive characteristic (2.1) we shall find $\dot{\sigma}_A = \rho c_0(\dot{v}_A - \dot{v}_0(t)) = \rho c_0\, \ddot{x}_1(t) - f(t)$. Equation $(2.9)_1$ will become non-homogeneous, with a right-hand term $f(t)$. Then, the way of solving particular problems will undergo slight modification. The solution will be completed with particular integrals.

Figure 4 shows the variation of $x(\overline{t}) = \alpha(\overline{t})/\alpha_{\max}$ as a function of the dimensionless time $\overline{t} = t/t_k$. The curve OCD illustrates the case of elastic collision under the assumption of an excitation pulse in the form $v_0 H(t)$, where $H(t)$ is the Heaviside function. This curve can be approximated with a fairly good accuracy by the equation

$$\alpha(t) = \alpha_{\max}\sin(\pi t/t_k). \tag{3.9}$$

The variation of the stress at the points A and B as a function of time will be described by the equation

$$\begin{aligned} \sigma|_{x=l_1}(t) &= \frac{\beta}{F_0}\sigma_{(\alpha_{\max})}\,[\sin[\varphi(t-t_1)]] \qquad \text{for} \quad t_1 \leq t \leq t_k, \\ \sigma|_{x=l_1}(t) &= 0 \qquad \text{for} \quad t \geq t_k \quad \text{and for} \quad t < t_1, \end{aligned} \tag{3.10}$$

where $\varphi = \pi/(t_k - t_1)$.

Knowing the variation of the stress at the point A and B (Fig. 3), we can easily find the solution in the remaining regions of the phase plane (x,t). From the relation (2.1) we shall obtain the following expression of the variation of velocity of the end of the bar $x = l_1$ at the point A:

$$v_A = v_{6|x=l_1} = v_0 - a[\sin[\varphi(t-t_1)]]^{3/2}. \tag{3.11}$$

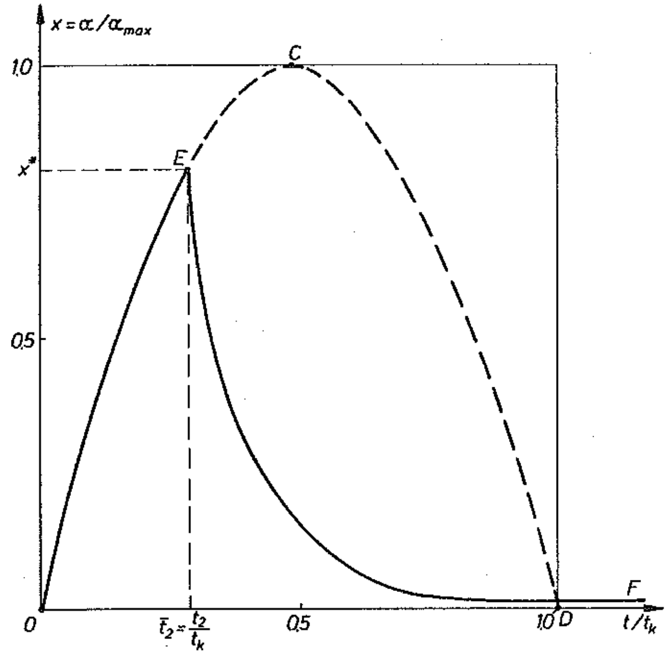

FIG. 4. The case of $M = 0$.

For the point B we have, from (2.2),

$$v_B = v_{8|x=l_1} = a[\sin[\varphi(t - t_1)]]^{3/2}, \tag{3.12}$$

where $a = \beta v_0/2F_0$.

Making use of the relations along the positive and negative characteristics we can determine the stresses in the regions VI and VIII

$$\begin{aligned} \sigma_6(x,t) &= -a\rho c_0 \left[\sin[\varphi(t - 2t_1 + x/c_0)]\right]^{3/2}, \\ \sigma_8(x,t) &= -a\rho c_0 \left[\sin[\varphi(t - x/c_0)]\right]^{3/2}. \end{aligned} \tag{3.13}$$

The boundary condition for $x = 0$ being $\sigma_5(0,t) = 0$ for $t > t_i$, the solution in the region V is $\sigma_5(x,t) = 0$, $v_5(x,t) = 0$.

In the region VII the stress is expressed thus:

$$\sigma_7(x,t) = 0.5\rho c_0 v_0 - a\rho c_0 \left[\sin[\varphi(t - 2t_1 + x/c_0)]\right]^{3/2}. \tag{3.14}$$

The regions VII and VIII are bounded from above by characteristics, the equations of which are $x = l_1 - c_0(t - t_2)$ and $x = l_1 + c_0(t - t_2)$, respectively. In the case considered we have $t_k > t_2 = (2l_0 + l_1)/c_0$. The solutions in the regions X and XI (Fig. 4) are constructed in the same manner as it was done for the regions VI and VII, there being a difference, however, consisting in the fact that the relation (2.1) is replaced in the region X by $\sigma_A = \rho c_0 v_A$. Equation (3.2) takes the form $\dot{\alpha} = \sigma(t)/\rho c_0$. Hence $t - t_2 = \rho c_0 \int_{\alpha^*}^{\alpha} \frac{d\overline{\alpha}}{\sigma(\overline{\alpha})}$ where α^* is equal to the magnitude of approach at the time $t = t_2$ (Fig. 4). Taking into account (3.5) and integrating, we find

$$t = t_2 + \frac{F_0 \rho c_0}{\beta}. \tag{3.15}$$

From this equation it follows that the magnitude of approach α decreases with increasing time t. Starting from $\bar{t}_2$ the unloading process proceeds along the path EF (Fig. 4). The approach α tends to zero for $\bar{t} \to \infty$.

The variation of the approach $\alpha(t)$ in the regions X and XI is determined by the following prescription

$$\begin{aligned} \alpha(t) &= \alpha_{\max} \sin[\varphi(t - t_1)] \qquad &&\text{for} \quad t_1 \le t \le t_2, \\ \alpha(t) &= \left[\frac{1}{\sqrt{\alpha^*}} + \frac{\beta(t - t_2)}{F_0 \rho c_0}\right]^{-2} &&\text{for} \quad t > t_2, \quad \text{where} \quad \alpha^* = \alpha(t_2). \end{aligned} \tag{3.16}$$

The function $x(t) = \alpha(t)/\alpha_{\max}$ is shown in Fig. 4. The loading process is represented by the curve OC, that of unloading – by the curve OEF (it being assumed that $\bar{t}_2 \le 0.5$).

Let us assume that the strain gauges in the system of Hopkinson bars are glued at the middle points of bars (which are of equal length). The strain pulses to be measured (in the case of elastic impact) by that system of strain gauges should vary in time as results from the computation which have been made for the simplified scheme, that is for $M = 0$

$$\begin{aligned} \varepsilon_i(t) &= -\frac{v_0}{2c_0} = \text{const} &&\text{for } 0.5t_1 \le t \le 0.5t_1 + t_i, \\ \varepsilon_r(t) &= \frac{v_0}{2c_0} - \frac{a}{c_0}\left\{\sin\left[\varphi\left(t - \frac{3}{2}t_1\right)\right]\right\}^{3/2} &&\text{for } t_1 \le t \le t_2 + 0.5t_1, \\ \varepsilon_r(t) &= -\frac{\beta}{\rho c_0^2}\left\{\frac{1}{\sqrt{\alpha^*}} + \frac{\beta\left[t - \frac{3}{2}(t_1 + t_i)\right]}{F_0 \rho c_0}\right\}^{-3} &&\text{for } t \ge t_2 + 0.5t_1, \\ \varepsilon_t(t) &= -\frac{a}{\rho c_0}\left\{\sin\left[\varphi\left(t - \frac{3}{2}t_1\right)\right]\right\}^{3/2} &&\text{for } 1.5t_1 \le t \le t_2 + 0.5t_1, \\ \varepsilon_t(t) &= -\frac{\beta}{\rho c_0^2}\left\{\frac{1}{\sqrt{\alpha^*}} + \frac{\beta\left[t - \frac{3}{2}(t_1 + t_i)\right]}{F_0 \rho c_0}\right\}^{-3} &&\text{for } t \ge t_2 + 0.5t_1. \end{aligned} \tag{3.17}$$

For correct determination of $\varepsilon_r(t)$ and $\varepsilon_t(t)$ it would be advisable to construct a solution for further regions of the phase plane, that is in the regions IX and XII to XVI. In the regions XII and XIV tensile stresses will occur, which will result in a loss of contact between the bars and the specimen for $t > t_3 = 3t_1$. The solution in the regions XV and XVI will depend on the mechanical characteristics of the damper at the end of the reception bar. Because the bar-missiles are short ($l_0 \ll l_1$), in general, the duration of a pulse t_i is relatively short as compared with the time t_1. Hence, the duration $\varepsilon_r(t)$ calculated for the middle point of the transmission bar (region X) will be reduced to a very small value before the arrival of the wave $x = 2l_1 + c_0 t$; it decreases in time according to the expression $1/t^3$.

4. The case of $M \neq 0$. Linearized equation of motion

Let us return to our fundamental problem described by the system of Eqs. (2.9), that is to the case of $M \neq 0$.

Our argumentation will now be different. Let us linearize the Eqs. (2.9). To this aim let us assume that $n = 1$. We find

(4.1)
$$\begin{aligned} \ddot{\alpha}_1 + \lambda_1 \dot{\alpha}_1 + \lambda_2(\alpha_1 - \alpha_2) &= 0, \\ \ddot{\alpha}_2 + \lambda_1 \dot{\alpha}_2 - \lambda_2(\alpha_1 - \alpha_2) &= 0. \end{aligned}$$

On introducing the new variables

(4.2)
$$\xi(t) = \alpha_1(t) + \alpha_2(t), \qquad \eta(t) = \alpha_1(t) - \alpha_2(t)$$

we obtain, by subtracting and adding the equations (4.1), the following set of uncoupled equations

(4.3)
$$\ddot{\xi} + \lambda_1 \dot{\xi} = 0, \qquad \ddot{\eta} + \lambda_1 \dot{\eta} + 2\lambda_2 \eta = 0$$

with the initial conditions

(4.4)
$$\xi = \eta = 0, \qquad \dot{\xi} = v_0, \qquad \dot{\eta} = v_0$$

for $t = 0$.

On integrating twice Eq. (4.3)$_1$ we find

(4.5)
$$\xi(t) = \frac{v_0}{\lambda_1}\left[1 - e^{-\lambda_1 t}\right].$$

Figure 5 shows the variation of $\xi = \alpha_1 + \alpha_2 = x_1 - x_3$ as a function of the time t. This is a monotonic function increasing from zero and tending to the asymptote v_0/λ_1.

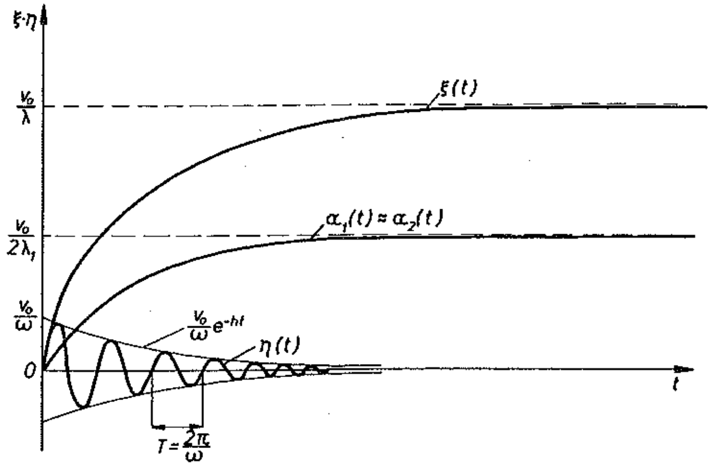

FIG. 5. The case of $M \neq 0$ and the linearized equations of motion.

Equation $(4.3)_2$ describes the damped vibrations. Its generalized equation has the form

$$\eta(t) = e^{-ht}(C_1 \cos \omega t + C_2 \sin \omega t), \tag{4.6}$$

where

$$h = \lambda_1/2, \qquad \omega = \left\{ \frac{2\beta}{M} - (\lambda_1/2)^2 \right\}^{1/2}. \tag{4.7}$$

Taking into consideration the initial conditions (4.4), we find

$$\eta(t) = e^{-ht} \frac{v_0}{\omega} \sin \omega t. \tag{4.8}$$

Figure 5 represents the function $\eta(t)$ on a much greater scale ($\omega \gg \eta_1$). It is seen that damped vibrations of very high frequency ω and very small initial amplitude, equal to about v_0/ω, die out rapidly.

In the problem under consideration we are interested in the approaches $\alpha_1(t)$ and $\alpha_2(t)$, therefore we find

$$\begin{aligned} \alpha_1(t) &= \frac{v_0}{2\lambda_1} \left\{ 1 - e^{-\lambda_1 t} + \frac{\lambda_1}{\omega} \sin \omega t \right\}, \\ \alpha_2(t) &= \frac{v_0}{2\lambda_1} \left\{ 1 - e^{-\lambda_1 t} - \frac{\lambda_1}{\omega} \sin \omega t \right\}. \end{aligned} \tag{4.9}$$

Figure 5 represents the variations of α_1 and α_2 as functions of t. Because $\omega \gg \eta_1$, the last term in those expressions is negligibly small as compared with the first two, therefore

$$\alpha_1(t) \approx \alpha_2(t) \approx \frac{v_0}{\lambda_1} \left\{ 1 - e^{-\lambda_1 t} \right\}. \tag{4.10}$$

Numerical solution of the set of equations (2.9) yields, if it is assumed that $n = 1$ (for the purpose of linearization), the results which are identical with those obtained by rigorous methods. The rigorous solution has however a shortcoming of considerable numerical instability resulting from the fact of superposition of damped vibrations of very high frequency. This instability vanishes, if it is assumed that $\lambda_1 = 0 \quad (h = 0)$.

Hertz's law of static contact between two spherical elastic bodies with radii r_0 and r_1 has the form (3.5), the constant β being determined by the formula [16]

$$\beta = \frac{4}{3\pi(k_0 + k)}\sqrt{\frac{r_0 r_1}{r_0 + r_1}}, \tag{4.11}$$

where $k_0 = (1 - v_0^2)/\pi E_0$, $k = (1 - v^2)/\pi E$; E_0, E and ν_0 and ν are Young's modulus and the Poisson ratio for the bars and the specimen, respectively. In our case we are interested in the problem of contact between a sphere and a cylindrical bar. The curvature of the latter at the point of contact is zero, therefore $r_0 = \infty$. For the constant β we obtain the expression

$$\beta = 4\sqrt{r}/3\pi(k_0 + k). \tag{4.12}$$

The set of Eqs. (2.9) can be solved numerically by the Runge–Kutta method, for instance. It can be seen that in the case of a specimen of very small mass as compared to that of the measuring bars, we find $\alpha_1 \approx \alpha_2 \approx \alpha$ where $\alpha(t)$ is the solution of the auxiliary problem for $M = 0$, described by the Eqs. (3.16).

5. Determination of the forces acting on the spherical specimen

We shall now determine the forces acting on the spherical specimen. In the case of Hertz's problem of interaction force P between a spherical specimen and the plane surface, the region of contact is a circle. The radius r_k of that circle for the maximum magnitude of approach $\alpha_{\max}$ as obtained by solving the wave problem (for $\alpha_{\max} \ll r_k$) can be found from the formula

$$r_k = (\alpha_{\max} r_r)^{1/2}, \tag{5.1}$$

where r_r is the reduced radius of curvature of the surface of contact between a spherical body and a plane $r_r = r$. Bearing in mind (3.6) and (4.12) we obtain the following expression for the variation of the displacement r_k during the process of collision

$$r_k = \left[\frac{3\pi P}{4}(k_0 + k_1)r\right]^{1/3}. \tag{5.2}$$

The maximum force of collision $R_A = R_B = P_{\max}$ will be calculated from the formula

$$P_{\max} = \frac{4\alpha_{\max} r_k}{3\pi(k_0 + k_1)}. \tag{5.3}$$

The maximum value of the approach $\alpha_{\max}$ will be read from Fig. 5. The value of r_k has been determined by the formula (5.1).

The value of maximum pressure of contact $p_{\max}$ between the specimen and the bars will be obtained by comparison of the sum of pressures acting on the surface of contact to the compression forces $P_{\max}$. For a semicircular pressure distribution we find

$$p_{\max} = \frac{3P_{\max}}{2\pi r_k^2}. \tag{5.4}$$

The pressure on the surface of contact at a point located at a distance R from the centre of the circle of contact is determined by the formula [7]

$$p_{\max}(R) = \frac{3P_{\max}}{2\pi r_k^2}\left[1 - (R/r_k)^2\right]^{1/2}. \tag{5.5}$$

At the boundary of the region of contact ($R = r_k$) it is equal to zero: $p_{\max}(r_k) = 0$. At the centre of that region it is determined by the formula (5.4). The mean pressure on the surface of contact between the bar and the spherical specimen will be found from the formula

$$\overline{p} = \frac{1}{\pi r_k^2}\int_0^{r_k} 2\pi R p_{\max}(R)\, dR. \tag{5.6}$$

Let us determine the variation in time of the quantities $P_{\max}$, r_k, $p_{\max}$ and $\overline{p}$. We assume that the variation of the approach $\alpha(t)$ is determined by the function (cf. (3.16))

$$\alpha(t) = \alpha_{\max} f(t). \tag{5.7}$$

Then, the variation of the pressure of contact $p(t)$ is determined by the formula

$$p(t) = p_{\max}(R) f(t) = \frac{3P_{\max}}{2\pi r_k^2}\left[1 - (R/r_k)^2\right]^{1/2} f(t) \tag{5.8}$$

and the variation of the mean pressure acting on the specimen is

$$\overline{p}(t) = \frac{1}{\pi r_k^2} \int_0^{r_k} 2\pi R p_{\max}(R)\, dR f(t). \tag{5.9}$$

It should be stated, on the basis of the results of our own experimental studies of brittle photoelastic materials [10] and on the basis of the literature data ([1] and [7], for instance), that, in the case of collision of brittle bodies (if at least one of them is brittle), the Hertz theory is valid over the entire loading interval until the moment of destruction of the specimen.

6. Conclusion. Some remarks on the solution for the stages 2 and 3

The problem of dynamic destruction of brittle bodies has been studied in many works. Thus, for instance, a theoretical description of that phenomenon has been proposed, on an experimental basis, in [4]. The total strain in that model is a sum of the strain in the material without micro-defects and the mean strain of the damage, the inelastic deformations of the existing micro-cracks and their growth being taken into consideration. A criterion for dynamic growth of cracks was proposed and the evolution of a micro-damage has been studied. The case of decay of a stress pulse due to the development of damages in the material has been analysed. Compression stress waves propagating in a brittle material undergo decay as a result of inelastic deformations and development of damages. Detailed analysis has been made in the case of uni-axial state of stress – problem of wave propagation along a semi-infinite bar. The problem of propagation of tension waves along a bar and the associate problem of formation of material fragments (spalling) were not considered. This problem is studied in [13]. The first part of that work is devoted to a discussion of the author's experiment on the dynamic formation of fragmentation of the material. The second part is devoted to the estimation of the numerical procedure which is that of three-stage approximation: i) uni-axial elastic (linear) estimation, ii) correction of the effects of geometrical dispersion and iii) correction of the nonlinearity of the brittle material tested, due to phenômena of damage in the neighbourhood of the plane of spall. The considerations are based on the model proposed in [12]. It is assumed that the damage of the material develops by activation of the initial sources of damage (such as micro-cracks or pores) during the action of tension load.

Both works ([12] and [13]) may be used for determining the internal state of stress in the specimen as well as its form of destruction. In the case considered we are concerned with a complex state of stress, namely a two-dimensional problem (of rotational symmetry). Following the reference [10] we can describe the behaviour of the material of the specimen as an elastic-brittle material, by introducing a parameter of damage, for which an evolution equation is constructed. The set of equations describing a process of damage to the material is of a semi-linear hyperbolic type. The determination of stress distribution in a spherical body requires numerical solutions using programs of the method of finite elements.

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