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BRIEF NOTES

ALIGNED MAGNETIC EFFECTS ON THE FLOW DOWN AN OPEN INCLINED CHANNEL WITH HIGHLY POROUS BED

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The aligned magnetic effects on a steady, laminar, viscous, incompressible, conducting fluid flow down an open inclined rectangular channel, bounded below by a highly porous bed, have been studied. The Brinkman equation modified for MHD is assumed to govern the flow in the porous bed and a set of modified boundary conditions, taking into account the effective viscosity considerations, have been applied at the fluid-porous medium interface. The expressions for velocity distribution, magnetic strength and flux along the channel are obtained and discussed.

1. Introduction

The viscous flow in an inclined channel with free surface has several scientific and engineering applications, such as those in hydraulic engineering, chemical engineering, in coating on paper rolls, and in the designs of the drainage canals. Several researchers [1 – 7] investigated such flows down an inclined channel with a rigid bottom, or in a channel whose bottom is a porous medium where the flow is governed by the Darcy's law, with or without the magnetic field. Chauhan and Shekhawat [8] also investigated this problem in rectangular inclined channel using Brinkman model for the porous medium bed.

In the present investigation, the effect of aligned magnetic field in a laminar flow of viscous incompressible conducting fluid down an open, inclined, rectangular channel with a porous bed of a highly permeable material, is studied. In the coupled-flow problem the Brinkman effective viscosity in the porous bed is considered, taking into account the effective medium theory of Freed and Muthu Kumar [9], which predicts the effective viscosity $\overline{\mu}$, for dilute arrays as

$$(\bar{\mu}/\mu) = 1 + \frac{5}{2}C - 9\left(\frac{C}{2}\right)^{3/2}, \tag{1.1}$$

where C is the volume fraction.

The set of modified boundary conditions, discussed by KIM and RUSSEL [10] at the porous interface, together with some suitable conditions for the aligned magnetic field, has been applied. The expressions for the velocity profiles, magnetic strength and the flux across the cross-section of the channel are obtained and discussed.

2. FORMULATION OF THE PROBLEM

A steady laminar flow of a viscous, incompressible, electrically conducting fluid, down an open inclined channel of depth h and width $2b$, under the influence of an aligned magnetic field, has been considered. The rigid side walls of the channel are perpendicular to the surface of the porous bed. The surface of the porous bed is taken to be inclined at an angle β $(0 \le \beta \le \pi/2)$ to the horizontal. The thickness of the porous bed is a, with an impermeable bottom. A Cartesian coordinate system is used with the x-axis along the central line in the flow direction at the free surface, the y-axis normal to it in the porous bed direction, and the z-axis directed along the width of the channel. The fluid motion is considered under the action of gravity, and simultaneously a constant pressure gradient is applied at the mouth of the channel in both the free fluid and porous regions. The length of the porous bed is assumed to be long enough to make the derivative $\partial/\partial x$ of all physical quantites (except pressure) zero. A magnetic field of strength H_0 is applied at an angle θ to the porous bed.

The governing equations are:

In the free fluid region $(0 \le y \le h)$,

$$\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g \sin\beta - \frac{\partial p}{\partial x} + \mu_e H_y \frac{\partial H_x}{\partial y} = 0, \tag{2.1}$$

and the magnetic field equation,

$$\sigma_e \mu_e H_y \frac{\partial u}{\partial y} + \left(\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2}\right) = 0. \tag{2.2}$$

The y-component H_y of the induced magnetic field becomes constant, that is

$$H_y = -H_0 \sin\theta. \tag{2.3}$$

The governing equation in the porous region ($h \le y \le h+a$) is given by

$$\overline{\mu}\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) + \rho g \sin\beta - \frac{\partial p}{\partial x} - \frac{\mu}{K_0} U - \sigma_e \mu_e^2 H_0^2 \sin^2\theta U = 0. \tag{2.4}$$

Here, u and U are velocities in free fluid and porous region, respectively; p – the pressure, ρ – the density, σ_e – the electrical conductivity, μ_e – the magnetic permeability, and K_0 – the permeability of the porous medium.

The boundary conditions are:

at $z = \pm b$, $u = 0 = U$, $H_x = 0$;

at $y = 0$, $\dfrac{\partial u}{\partial y} = 0$, $H_x = 0$;

at $y = h$, $u = U$, $\mu \dfrac{\partial u}{\partial y} = \overline{\mu}\dfrac{\partial U}{\partial y}$, $H_x = -H_0 \cos\theta$;

and

$$\text{at} \quad y = h + a, \qquad U = 0. \tag{2.5}$$

Let the mean flow velocity U_0 be the characteristic velocity. Introducing the following dimensionless quantities:

$$\begin{aligned} u^* &= \frac{u}{U_0}, & U^* &= \frac{U}{U_0}, & x^* &= \frac{x}{h}, \\ y^* &= \frac{y}{h}, & z^* &= \frac{z}{h}, & p^* &= \frac{p}{\rho U_0^2}, \\ K_0^* &= \frac{K_0}{h^2}, & H_x^* &= \frac{H_x}{U_0(\sigma_e \mu)^{1/2}}, & a^* &= \frac{a}{h}, \end{aligned} \tag{2.6}$$

and using Eq. (2.3), Eqs. (2.1), (2.2) and (2.4), after dropping the asterisks, are transformed to the following form:

$$\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - M \sin\theta \frac{\partial H_x}{\partial y} = \left(R\frac{dp}{dx} - \frac{R}{F}\sin\beta\right), \tag{2.7}$$

$$\left(\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2}\right) - M \sin\theta \frac{\partial u}{\partial y} = 0, \tag{2.8}$$

$$\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) - \phi^{-1}\left(\frac{1}{K_0} + M^2 \sin^2\theta\right) U = \phi^{-1}\left(R\frac{dp}{dx} - \frac{R}{F}\sin\beta\right), \tag{2.9}$$

where

$$M = \left(\frac{\sigma_e}{\mu}\right)^{1/2} \mu_e H_0 h, \qquad R = \frac{\rho U_0 h}{\mu},$$

$$F = \frac{U_0^2}{gh}, \qquad \phi^{-1} = \mu/\overline{\mu}.$$

The dimensionless boundary conditions are

$$\text{at} \quad z = \pm d, \qquad u = 0 = U, \qquad H_x = 0;$$

$$\text{at} \quad y = 0, \qquad \frac{\partial u}{\partial y} = 0, \qquad H_x = 0;$$

$$\text{at} \quad y = 1, \qquad u = U, \qquad \phi^{-1}\frac{\partial u}{\partial y} = \frac{\partial U}{\partial y}, \qquad H_x = -\frac{M}{R_m}\cos\theta;$$

and

(2.10) $$\text{at} \quad y = d_1, \qquad U = 0,$$

where

$d = \dfrac{b}{h}$, $d_1 = 1+a$, and $R_m = \dfrac{U_0 h}{(1/\sigma_e \mu_e)}$ (magnetic Reynolds number).

3. Method of solution

Equations (2.7) and (2.8) are coupled. They can be uncoupled by the transformations

(3.1) $$v = u + H_x \qquad \text{and} \qquad w = u - H_x.$$

Using these transformations, we obtain

(3.2) $$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} - M\sin\theta\frac{\partial v}{\partial y} = Q,$$

(3.3) $$\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + M\sin\theta\frac{\partial w}{\partial y} = Q,$$

where

$$Q = \left(R\frac{dp}{dx} - \frac{R}{F}\sin\beta\right).$$

Substituting $z = 2d\xi/\pi - d$ in Eqs. (3.2), (3.3) and (2.9), we get

(3.4) $$\frac{\partial^2 v}{\partial y^2} + \frac{\pi^2}{4d^2}\frac{\partial^2 v}{\partial \xi^2} - M\sin\theta\frac{\partial v}{\partial y} = Q,$$

(3.5) $$\frac{\partial^2 w}{\partial y^2} + \frac{\pi^2}{4d^2}\frac{\partial^2 w}{\partial \xi^2} + M\sin\theta\frac{\partial w}{\partial y} = Q,$$

and

$$\frac{\partial^2 U}{\partial y^2} + \frac{\pi^2}{4d^2}\frac{\partial^2 U}{\partial \xi^2} - \phi^{-1}\left(\frac{1}{K_0} + M^2\sin^2\theta\right)U = \phi^{-1}Q. \tag{3.6}$$

The boundary conditions are

$$\begin{aligned}
&\text{at}\quad \xi = 0 \quad\text{and}\quad \xi = \pi, \qquad u = 0 = U, \qquad H_x = 0;\\
&\text{at}\quad y = 0, \qquad \frac{\partial u}{\partial y} = 0, \qquad H_x = 0;\\
&\text{at}\quad y = 1, \qquad u = U, \qquad \phi^{-1}\frac{\partial u}{\partial y} = \frac{\partial U}{\partial y}, \qquad H_x = -\frac{M}{R_m}\cos\theta;
\end{aligned}$$

and

$$\text{at}\quad y = d_1, \qquad U = 0. \tag{3.7}$$

Applying finite sine-transform to Eqs. (3.4)–(3.7) we get

$$\frac{d^2\overline{v}}{dy^2} - M\sin\theta\frac{d\overline{v}}{dy} - \frac{\pi^2 n^2}{4d^2}\overline{v} = \left(\frac{1-\cos n\pi}{n}\right)Q, \tag{3.8}$$

$$\frac{d^2\overline{w}}{dy^2} + M\sin\theta\frac{d\overline{w}}{dy} - \frac{\pi^2 n^2}{4d^2}\overline{w} = \left(\frac{1-\cos n\pi}{n}\right)Q, \tag{3.9}$$

$$\frac{d^2\overline{U}}{dy^2} - \left[\phi^{-1}\left(\frac{1}{K_0} + M^2\sin^2\theta\right) + \frac{\pi^2 n^2}{4d^2}\right]\overline{U} = \phi^{-1}\left(\frac{1-\cos n\pi}{n}\right)Q, \tag{3.10}$$

and the boundary conditions

$$\begin{aligned}
&\text{at}\quad y = 0, \qquad \frac{d\overline{u}}{dy} = 0, \qquad \overline{H}_x = 0,\\
&\text{at}\quad y = 1, \qquad \overline{u} = \overline{U}, \qquad \phi^{-1}\frac{d\overline{u}}{dy} = \frac{d\overline{U}}{dy}, \qquad \overline{H}_x = -\frac{M}{R_m}\left(\frac{1-\cos n\pi}{n}\right)\cos\theta,\\
&\text{at}\quad y = d_1, \qquad \overline{U} = 0,
\end{aligned} \tag{3.11}$$

where

$$\overline{v} = \int_0^\pi v\sin n\xi\, d\xi, \qquad \overline{w} = \int_0^\pi w\sin n\xi\, d\xi,$$

$$\overline{U} = \int_0^\pi U\sin n\xi\, d\xi, \qquad \overline{H}_x = \int_0^\pi H_x\sin n\xi\, d\xi,$$

and n is a positive integer.

4. Solutions

On solving Eqs. (3.8) to (3.10) under the boundary conditions (3.11), and using the inversion formula for finite sine-transform, we get

$$u = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(C_1\cosh m_1 y + C_2\cosh m_2 y - \frac{NQ}{Q_1}\right)\sin n\xi, \tag{4.1}$$

$$H_x = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(C_1\sinh m_1 y + C_2\sinh m_2 y\right)\sin n\xi, \tag{4.2}$$

and

$$U = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(C_3 e^{\alpha_1 y} + C_4 e^{-\alpha_1 y} - \frac{\phi^{-1}QN}{\alpha_1^2}\right)\sin n\xi, \tag{4.3}$$

where

$$\begin{aligned}
Q &= \left(R\frac{dp}{dx} - \frac{R}{F}\sin\beta\right),\\
Q_1 &= \frac{\pi^2 n^2}{4d^2},\\
N &= \left(\frac{1-\cos n\pi}{n}\right),\\
\alpha_1^2 &= \left[\phi^{-1}\left(\frac{1}{K_0} + M^2\sin^2\theta\right) + \frac{\pi^2 n^2}{4d^2}\right],\\
A &= NQ\left(\frac{1}{Q_1} - \frac{\phi^{-1}}{\alpha_1^2}\right),\\
m_1 &= \frac{1}{2}\left[M\sin\theta + \left(M^2\sin^2\theta + 4Q_1\right)^{1/2}\right],\\
m_2 &= \frac{1}{2}\left[M\sin\theta - \left(M^2\sin^2\theta + 4Q_1\right)^{1/2}\right],\\
A_1 &= (\cosh m_1 - \coth m_2 \sinh m_1),\\
A_2 &= \left(e^{\alpha_1(2d_1-1)} - e^{\alpha_1}\right),\\
A_3 &= \phi^{-1}(m_1 - m_2)\sinh m_1,\\
A_4 &= -\alpha_1\left(e^{\alpha_1} + e^{\alpha_1(2d_1-1)}\right),\\
D_1 &= \left(A + \frac{MN}{R_m}\cos\theta\coth m_2 + \frac{\phi^{-1}QN}{\alpha_1}e^{\alpha_1(d_1-1)}\right),\\
D_2 &= \left(\frac{MN}{R_m}\phi^{-1}m_2\cos\theta - \frac{\phi^{-1}QN}{\alpha_1}e^{\alpha_1(d_1-1)}\right),
\end{aligned} \tag{4.4}$$

(4.4) [cont.]

$$
\begin{aligned}
C_1 &= (D_1 - C_3A_2)/A_1,\\
C_2 &= -\left(\frac{MN}{R_m}\cos\theta + C_1\sinh m_1\right)/\sinh m_2,\\
C_3 &= (D_1A_3 - D_2A_1)/(A_2A_3 - A_1A_4),\\
C_4 &= \left(\frac{\phi^{-1}QN}{\alpha_1^2} - C_3e^{\alpha_1 d_1}\right)e^{\alpha_1 d_1}.
\end{aligned}
$$

5. Flux

The flux across the cross-section of the channel (perpendicular to the x-axis) is given by

$$
\begin{aligned}
V &= \int_{-d}^{d}\int_{0}^{1} u\,dydz,\\
&= \frac{4d}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{NC_1}{m_1}\sinh m_1 + \frac{NC_2}{m_2}\sinh m_2 - \frac{N^2Q}{Q_1}\right).
\end{aligned}
\tag{5.1}
$$

6. Coefficient of skin-friction

The coefficient of skin-friction at the porous bed is given by

$$
\begin{aligned}
(C_f)_{y=1} &= \frac{-\mu\dfrac{du}{dy}}{(\mu U_0/h)}\\
&= -\frac{2}{\pi}\sum_{n=1}^{\infty}(m_1C_1\sinh m_1 + m_2C_2\sinh m_2)\sin n\xi.
\end{aligned}
\tag{6.1}
$$

7. Discussion

Figure 1 illustrates the velocity profiles at the central transverse section ($\xi = \pi/2$) of the channel flow for $\beta = \pi/6$, $d = \pi/2$, $\phi^{-1} = 0.8$, $-dp/dx = R = R_m = F = 1$ and for different values of the permeability parameter K_0, the Hartmann number M, the thickness of the porous bottom a and the angle of inclination θ of the applied magnetic field. It is observed that the velocity in the channel increases by increasing K_0 or a. However, when the angle θ of inclination of the applied magnetic field is equal to $\pi/6$ or to $\pi/4$, the flow in the channel increases by increasing M, whereas it decreases when θ becomes $\pi/2$ by increasing M.

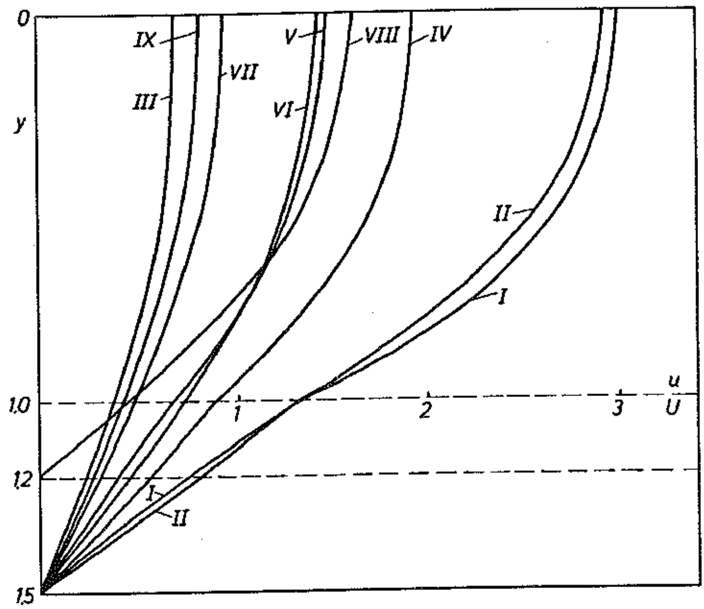

FIG. 1. Velocity profiles for $-\dfrac{dp}{dx} = R = R_m = F = 1$,

$$\beta = \frac{\pi}{6},\ d = \frac{\pi}{2},\ \phi^{-1} = 0.8,$$
$$\xi = \frac{\pi}{2},\ a = 0.2 \text{ and } 0.5.$$

	M	K_0	θ		M	K_0	θ		M	K_0	θ
I	3.0	1.5	$\pi/4$	IV	2.0	1.5	$\pi/4$	VII	0.0	1.5	$\pi/4$
II	3.0	1.5	$\pi/6$	V	1.5	1.5	$\pi/6$	VIII	2.0	1.5	$\pi/4$
III	3.0	1.5	$\pi/2$	VI	1.5	0.5	$\pi/6$	IX	2.0	1.5	$\pi/2$

Figure 2 and 3 show the variation of flux across the cross-section of the channel. It is observed that the flux increases by increasing K_0. It also increases by increasing M when θ is $\pi/6$ to $\pi/4$, whereas it decreases by increasing M when θ becomes $\pi/2$. In Fig. 4 it is observed that the coefficient of skin-friction C_f increases by increasing M, while it decreases by increasing K_0. It increases when θ is increased from $\pi/6$ to $\pi/4$ and decreases when θ becomes $\pi/2$.

In Fig. 5 H_x is plotted against y, for different angles of inclinations θ, K_0 and M. It is found that the magnetic strength H_x decreases from negative value toward zero, and becomes positive when θ varies from $\pi/6$ to $\pi/2$. It increases numerically by increasing M. When the magnetic field is in

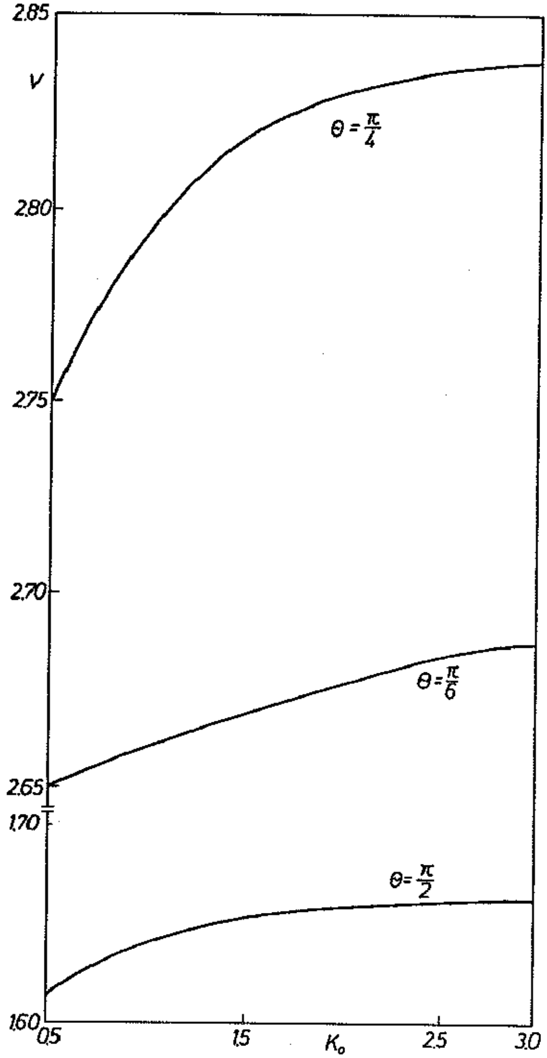

FIG. 2. Flux for $M = 1.5$, $\beta = \frac{\pi}{6}$, $d = \frac{\pi}{2}$,

$-\frac{dp}{dx} = R = R_m = F = 1$, $\xi = \frac{\pi}{2}$,

$\phi^{-1} = 0.8$ and $a = 0.5$.

transverse direction, H_x is positive and increases by increasing M. However, by increasing K_0, H_x increases in the negative side when θ is from $\pi/6$ to $\pi/4$, but when θ is $\pi/2$, H_x is positive and decreases by increasing K_0.

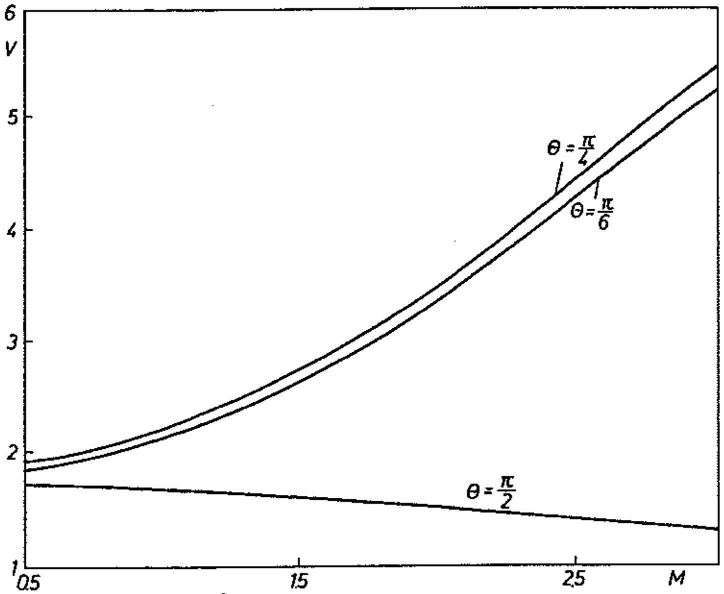

FIG. 3. Flux for $K_0 = 1.5$, $-\dfrac{dp}{dx} = R = R_m = F = 1$, $\phi^{-1} = 0.8$, $\beta = \dfrac{\pi}{6}$, $d = \dfrac{\pi}{2}$, $\xi = \dfrac{\pi}{2}$ and $a = 0.5$.

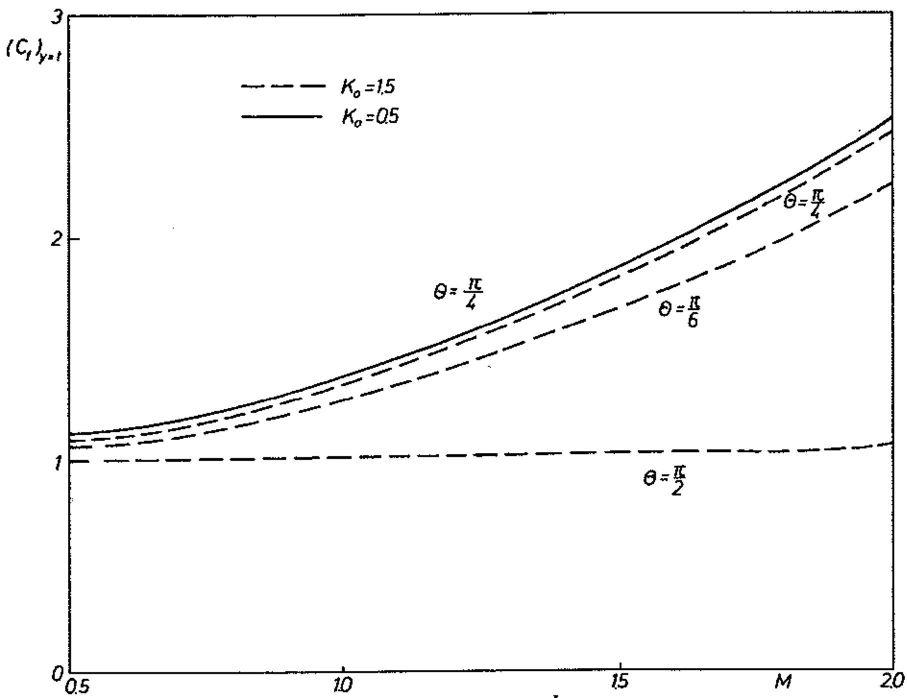

FIG. 4. Coefficient of skin friction $(Cf)_{y=1}$ for $-\dfrac{dp}{dx} = R = R_m = F = 1.0$, $\beta = \dfrac{\pi}{6}$, $d = \xi = \dfrac{\pi}{2}$, $\phi^{-1} = 0.8$ and $a = 0.5$.

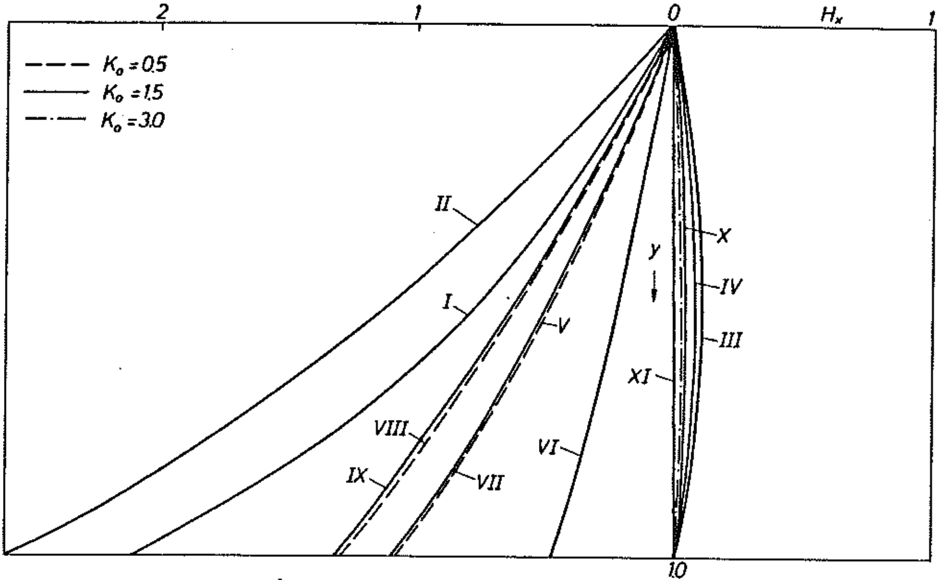

FIG. 5. H_x vs. y for $-\frac{dp}{dx} = R = R_m = F = 1.0$, $\beta = \frac{\pi}{6}$, $\phi^{-1} = 0.8$, $\xi = d = \frac{\pi}{2}$ and $a = 0.5$.

	M	θ		M	θ		M	θ
I	3.0	$\pi/4$	V	1.5	$\pi/4$	IX	1.5	$\pi/6$
II	3.0	$\pi/6$	VI	0.5	$\pi/6$	X	1.5	$\pi/2$
III	3.0	$\pi/2$	VII	1.5	$\pi/4$	XI	1.5	$\pi/2$
IV	1.5	$\pi/2$	VIII	1.5	$\pi/6$			

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