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ON LIMITS OF APPLICATION OF KIRCHHOFF'S HYPOTHESIS IN THE THEORY OF VISCOELASTIC FIBROUS COMPOSITE PLATES

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This paper contains the discussion on the application of Kirchhoff's hypothesis to viscoelastic fibrous composite plates based on the example of buckling of a plate strip. It has been assumed, that the plate is made of a fibrous composite, and a simplified two-phase continual model introduced by HOLNICKI–SZULC [8] and ŚWITKA [9, 10] is used. To describe the displacements of the phase I (matrix), the VLASOV'S [11], Hencky – Bolle's and Kirchhoff's kinematic hypotheses were adopted. The matrix material is viscoelastic, while the fibers (phase II) are made of material which satisfies the Hooke's law. Buckling of a plate strip was analyzed within the range of linear theory of stability, by assuming the equilibrium equations in a classical form, that is, in accordance with Hencky – Bolle's assumptions. The results obtained include closed-form analytical solutions for the immediate and sustained critical loads applied to a simply supported plate which was reinforced by fibrous meshes symmetrically distributed across the plate cross-section. A parametric analysis was performed on the obtained solutions, with respect to the plate strip geometric shape and with respect to the properties of the strip material. The analysis made enabled the author to draw some conclusions pointing to the limits of applicability of the Kirchhoff's hypothesis.

NOTATIONS

$x^\alpha, x^3 = z$	Cartesian coordinate system,
$h, h/a$	plate's thickness and the slenderness parameter,
a	plate strip width,
u_i	displacement vector components for any point of the plate,
$\overset{0}{u}_\alpha, w$	displacement vector components for points of the middle plane,
$\overset{*}{w}$	small initial deflection,
ψ_1, ψ_2	rotation angles around the x^2 and x^1 axis of the normals to the middle plane of the plate,
e_{ij}, σ^{ij}	strain tensor and stress tensor components,
$\varepsilon_{\alpha\beta}, \varepsilon_{\alpha 3}$	components of membrane and transverse shear strains at the middle plane of the plate,
$\overset{0}{\kappa}_{\alpha\beta}$	bending strains of the middle plane,
$\overset{2}{\kappa}_{\alpha\beta}, \overset{2}{\kappa}_{\alpha 3}$	bending and twisting strains causing curvature of the normals to the middle plane of the plate,

E, ν	Young's modulus and Poisson's coefficient for matrix material,
G', H'	immediate and sustained Kirchhoff's modulus (shear modulus) for matrix material,
$R^{1212}(t), R^{1313}(t), R^{2323}(t)$	relaxation kernel of the matrix material,
$\overset{r}{E}_{(\Delta)}, \overset{r}{A}_{(\Delta)}, \overset{r}{b}_{(\Delta)}$	Young's modulus of fibers material, single fibre cross-sectional area, distance between individual fibres,
$\overset{r}{e}_{(\Delta)}, \overset{0r}{\varepsilon}_{(\Delta)}$	relative elongation and initial distorsion of fibres,
$\overset{r}{e}_{(\Delta)\alpha\beta}$	components of the elongation of fibres,
$\overset{r}{S}_{(\Delta)}, \overset{r}{S}{}^{\alpha\beta}_{(\Delta)}$	equivalent distributed membrane force in (Δ) family of fibres and its components,
$\overset{r}{S}{}^{\alpha\beta}$	components of the resultant forces in all families of fibres which constitute the r-th mesh,
$\overset{r}{t}{}^{\alpha}_{(\Delta)}$	fibres directional cosines with respect to the x^α axes,
$\mu_{(\Delta)} = (\overset{r}{A}_{(\Delta)})/(\overset{r}{b}_{(\Delta)}\, h)$	density of fibres distribution in a given family (Δ),
$\overline{D}^{11}$	bending stiffness of fibrous composite plate in direction of the x^1 axis,
$\overset{r}{B}{}^{11}$	membrane (disk) stiffness for r-th fibrous mesh in direction of the x^1 axis,
$M^{\alpha,\beta}, N^{\alpha\beta}, Q^\alpha$	moments, longitudinal and transverse forces in a fibrous composite plate,
$\overset{0}{N}{}^{\alpha\beta}$	subcritical membrane forces in the plate,
p	uniform load which compresses the plate strip,
p^D_{cr}, p^N_{cr}	immediate and sustained critical loads of fibrous composite plate strip,
$p_{cr_{(K)}}$	critical load of fibrous composite plate strip according to Kirchhoff's model.

Latin indices i, j run in the range of 1, 2, 3, while Greek indices α, β, δ, λ assume the values 1, 2. Upper index, r $(r = 1, 2, 3, \ldots)$ refers to the r-th mesh of fibres, and the lower index (Δ) denotes the given family of fibres in a mesh, and it assumes the values $\Delta =$ I, II, III, IV. A subscribed comma was used to denote differentiation with respect to spatial variables x^1, x^2, z, while a superscribed dot denotes a time derivative. The summation convention refers exclusively to Greek indices located at different levels.

1. Introduction

Classical strength analysis of fibrous composite 2-D structures is usually limited to linear elastic problems based on the Kirchhoff's kinematic hypothesis (K). However, since fibrous composite materials find more technical applications, in many cases it is required to take their rheological properties into account, and to reject the assumption of the non-deformable normal to the plate middle plane. Such cases appear nowadays, particularly due to a quick development of modern technology, which to a great extent is based on the application of structural materials in conditions of high loads, which exposes the influence of creep and shear deformation [1, 2].

Fibrous composite materials used in modern 2-D structures, are used e.g. in civil and mechanical engineering, and in the chemical, aviation and shipbuilding industries. When compared with classical materials, such as steel, concrete and wood, they have much better physical-mechanical parameters. Besides other features, they have also higher specific strength (strength/weight ratio), lower weight and higher resistance against action of chemical agents. However, fibrous composites, being plastics, exhibit distinct rheological properties and susceptibility to shear loads [2, 3]. These facts cause that in the strength analysis of fibrous composite plates, more precise physical and kinematical models than those based on Hooke's and Kirchhoff's hypotheses are required. From theoretical and experimental surveys [1, 2, 3, 4, 5, 6, 7] it follows that application of the Kirchhoff's hypothesis considerably reduces the magnitudes of displacements and, at the same time, it results in lower critical loads and in lower magnitudes of the frequencies of free vibrations. For example, due to application of a more accurate plate theory, the reduction of the critical load for an isotropic, elastic plate with dimensions of $a \times b$, simply supported and being subjected to uni-directional compression, at the ratios $a/h = 10$ and $a/b = 0.2$ reaches as much as 73.32% [4]. Even greater discrepancies may appear, if rheological properties of the material are additionally accounted for.

In this paper, the fibrous composite is defined as a material which is obtained by bonding at least two material components, one of which creates the basic phase I (matrix), while the other one creates the fibrous phase II which is immersed in the first one. Phase I (matrix) acts as a bonding agent and ensures monolithic structure and the form of the element, while phase II ensures proper strength and rigidity. The fibrous phase is built of any number of families of long fibres, which are located in planes parallel to the middle plane of the plate. Fibres of a given family are thin, they have constant direction and are distributed with such a density, that a continual model can be used to describe their properties. It is assumed that the displacements in both phases are identical. It is estimated that the peak fibrous phase contents (reinforcement) in plate cross-section can reach as much as 8%.

So, the two-phase continual medium assumed as the theoretical model of the fibrous composite considered in the present paper, consists of the continuous phase II (fibrous phase) which is immersed in the continuum of phase I. The concept of such a model has been taken from papers by Holnicki–Szulc [8] and Świtka [9, 10].

As mentioned above, fibrous composites exhibit distinct rheological properties and high susceptibility to shear, thus more precise plate theories must be used for the analysis of composite plates. Recent research trends in more accurate and appropriate approaches to plate theories are comprehensively presented in publications [1, 12, 13, 14, 15, 16, 17, 18, 19, 20] and in [5, 11, 21, 22, 23, 24, 25, 26, 27] (in Russian), which also include detailed lists of references. Among the above mentioned works the papers by JEMIELITA [18, 19] and REDDY [20] deserve special attention, as they include comparative analysis of many kinematic hypotheses suggested by several authors.

The subject of consideration of the present paper is the estimation of the range of applicability of Kirchhoff's hypothesis. The considerations are based on the example of linear stability of a viscoelastic fibrous composite plate in the form of a plate strip, compressed by uniform load p (see Fig. 1) in the direction perpendicular to the supported edge. To describe the state of strains and stresses in the matrix, the theory based on kinematical assumptions of VLASOV [11] is adopted, for which a variationally coherent expansion for the case of laminated composite plates was presented by REDDY [15]. The theory converges with a generalised, second variant of Timoshenko type plate theory [22, 23]. In the present theory, straight line segments, which are normal to the middle plane of the plate before deformation, do not remain normal and they become curved after deformation (see Fig. 2). The transverse shear stresses have a parabolic distribution pattern across the plate thickness, with zero values on the top and bottom surfaces. The theory assumed in the present study does not require to introduce any shear

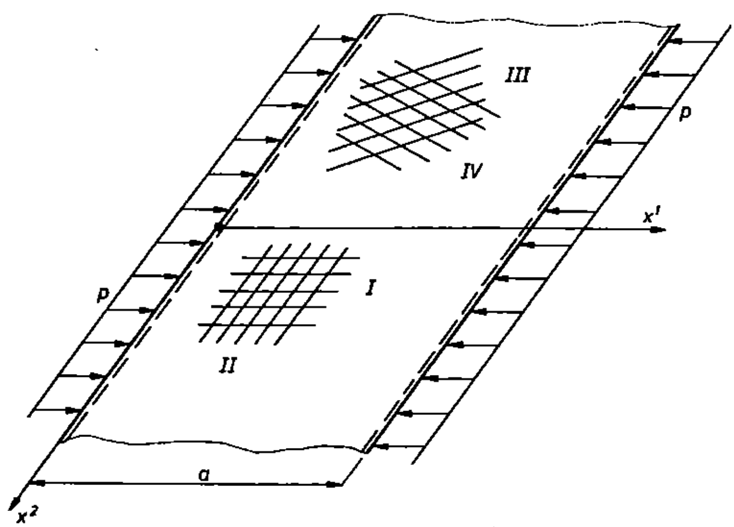


FIG. 1.

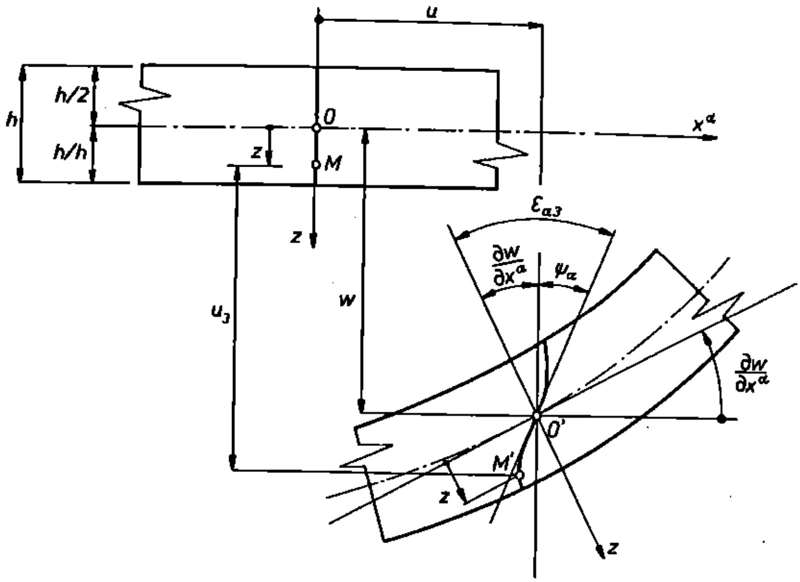


FIG. 2.

correction factor, as it happens in the first order plate shear deformation theory of Hencky - Bolle.

The results were obtained in the form of closed analytical solutions, which allow us to determine the immediate and sustained critical loads for a plate strip, which is reinforced by fibrous meshes located symmetrically in the cross-section and of such arrangement, that the plate orthotropy coincides with the parametric directions $x^\alpha = \text{const}$. Moreover, a parametric analysis was also performed by using the obtained solutions. The parameters investigated were: type of kinematic hypothesis, slenderness of the plate strip, the ratio of longitudinal and transverse elasticity moduli, reinforcement density, reinforcement mesh type and the ratio of Young's modulus of the fibre and matrix materials.

In view of results of the comparative analyses, which are presented in tables and figures, the limits of applicability of the Kirchhoff's hypothesis to the theory of viscoelastic fibrous composite plates become visible.

2. PROBLEM FORMULATION

In the VLASOV'S theory [11], the kinematics of any point $M(x^1, x^2, z \in \langle -(h/2), (h/2) \rangle)$ of the plate is described by the following field of displacements:

$$
(2.1) \qquad \begin{aligned}
u_\alpha(x^1,x^2,z) &= \overset{0}{u}_\alpha + z\left[\psi_\alpha - \frac{4}{3}C\frac{z^2}{h^2}(\psi_\alpha + w_{,\alpha})\right],\\
u_3(x^1,x^2,z) &= w(x^1,x^2),
\end{aligned}
$$

while in constitutive equations the assumption of $\sigma^{33} = 0$ is made. An additional numerical factor, C, can be introduced to generalize the relations (2.1). So, for Vlasov's theory, $C = 1$, for Hencky-Bolle theory, $C = 0$, while for Kirchhoff's theory, $C = 0$ and $\psi_\alpha = -w_{,\alpha}$.

Plate strains resulting from the field of displacements (2.1) have the following form:

$$
(2.2) \qquad \begin{aligned}
e_{\alpha\beta} &= \varepsilon_{\alpha\beta} + z(\overset{0}{\kappa}_{\alpha\beta} + Cz^2\,\overset{2}{\kappa}_{\alpha\beta}),\\
e_{33} &= 0, \qquad e_{\alpha3} = \varepsilon_{\alpha3} + Cz^2\,\overset{2}{\kappa}_{\alpha3},
\end{aligned}
$$

where

$$
(2.3) \qquad \begin{aligned}
&\varepsilon_{\alpha\alpha} = \overset{0}{u}_{\alpha,\alpha}, \qquad \varepsilon_{\alpha\beta} = \overset{0}{u}_{\alpha,\beta} + \overset{0}{u}_{\beta,\alpha} \quad \text{for} \quad \alpha \neq \beta, \quad \varepsilon_{\alpha3} = \psi_\alpha + w_{,\alpha},\\
&\overset{0}{\kappa}_{\alpha\alpha} = \psi_{\alpha,\alpha}, \qquad \overset{0}{\kappa}_{\alpha\beta} = \psi_{\alpha,\beta} + \psi_{\beta,\alpha} \quad \text{for} \quad \alpha \neq \beta, \quad \overset{2}{\kappa}_{\alpha3} = -\frac{4}{h^2}(\psi_\alpha + w_{,\alpha}),\\
&\overset{2}{\kappa}_{\alpha\alpha} = -\frac{4}{3h^2}(\psi_{\alpha,\alpha} + w_{,\alpha\alpha}), \qquad \overset{2}{\kappa}_{\alpha\beta} = -\frac{4}{3h^2}(\psi_{\alpha,\beta} + \psi_{\beta,\alpha} + 2w_{,\alpha\beta}) \quad \text{for} \quad \alpha \neq \beta.
\end{aligned}
$$

The equations of continuity of the strains are satisfied by identity.

It is assumed, that the matrix is made of a transversely isotropic material, which possesses some rheological properties. The matrix material is linearly elastic, though, due to its reinforcement made in the form of dense fibre meshes, which are located in planes parallel to the middle plane of the plate of the directions coinciding with parametric lines $x^\alpha = \text{const}$, the viscoelastic properties are maintained for transverse and in-plane shearing components of appropriate strain and stress tensors [23].

As a result of the assumptions made, constitutive equations for the matrix material have the following form:

$$
(2.4) \qquad \begin{aligned}
\sigma^{11} &= A^{1111}e_{11} + A^{1122}e_{22}, \qquad \sigma^{22} = A^{2222}e_{22} + A^{2211}e_{11},\\
\sigma^{12} &= A^{1212}e_{12} - \int_0^t R^{1212}(t-\tau)e_{12}(\tau)\,d\tau,
\end{aligned}
$$

(2.4) [cont.]

$$\sigma^{13} = A^{1313}e_{13} - \int_0^t R^{1313}(t-\tau)e_{13}(\tau)\,d\tau,$$

$$\sigma^{23} = A^{2323}e_{23} - \int_0^t R^{2323}(t-\tau)e_{23}(\tau)\,d\tau,$$

where

$$A^{1111} = A^{2222} = \frac{E}{1-\nu^2}, \qquad A^{1122} = A^{2211} = \frac{E\nu}{1-\nu^2},$$
$$A^{1212} = G = \frac{E}{2(1+\nu)}, \qquad A^{1313} = A^{2323} = G', \tag{2.5}$$

$R^{1212}(t)$, $R^{1313}(t)$, $R^{2323}(t)$ – relaxation kernel of the matrix material.

Fibrous phase consists of families of straight line fibres (Δ) of constant direction, which create the meshes located in planes $z = z^r$ $(r = 1, 2, 3, \ldots)$ that are parallel to the middle plane of the plate (Fig. 3). A continual

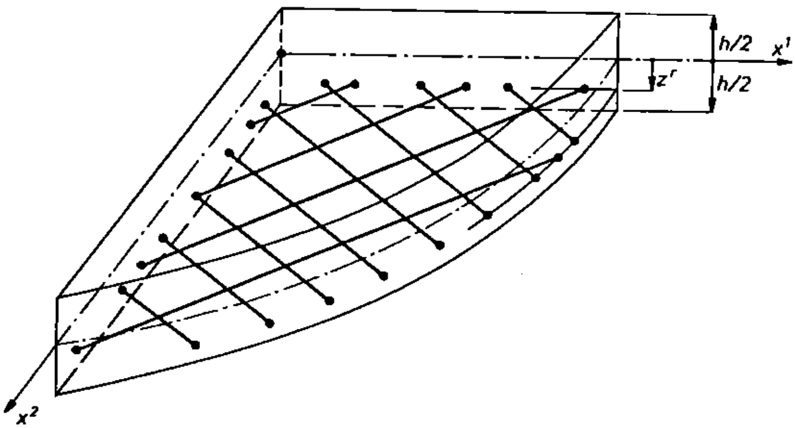
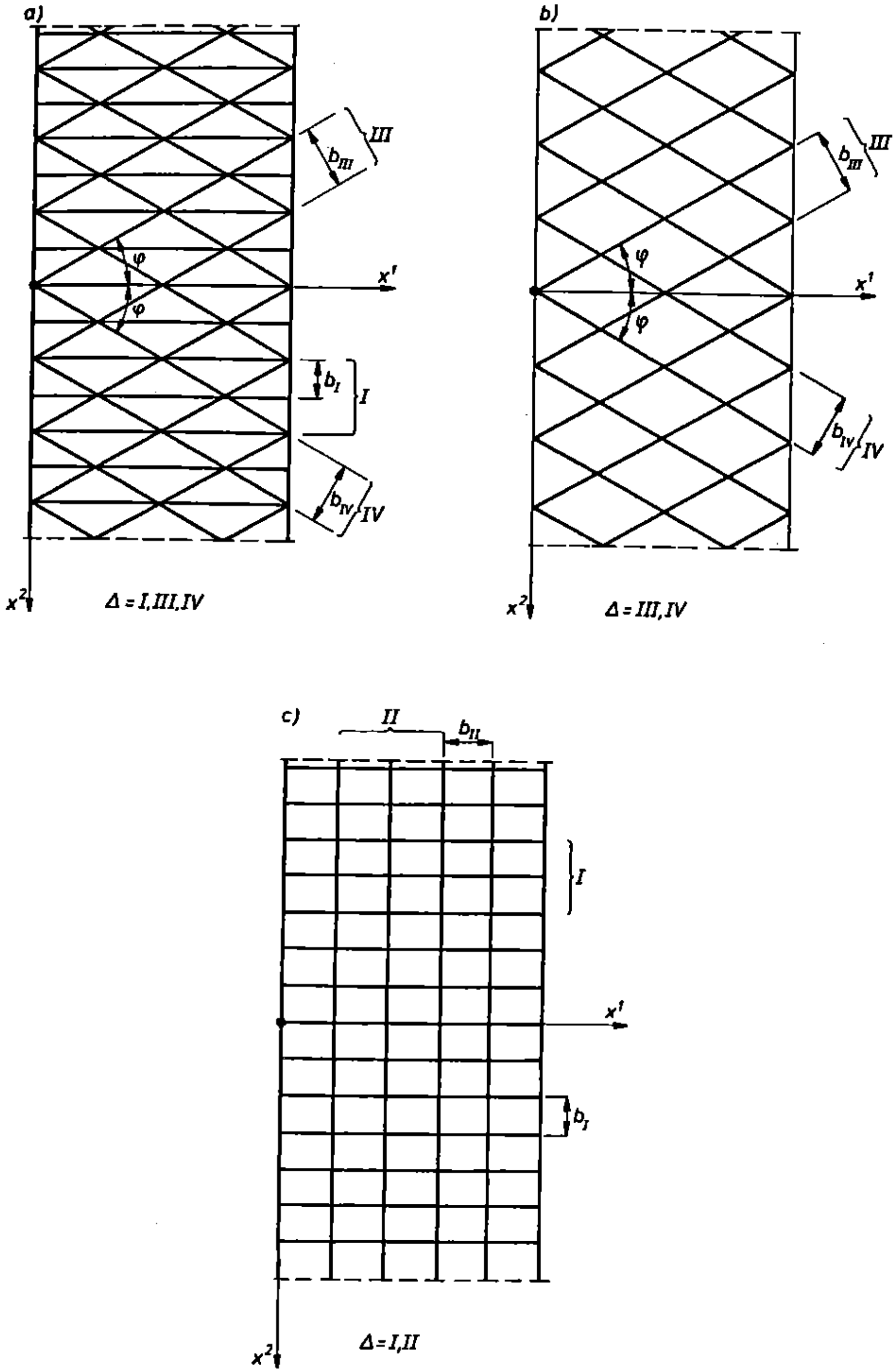


FIG. 3.

model is used to describe this phase. It is assumed, that the meshes are symmetrically distributed within the cross-section, and that the system of individual fibre families (Δ) in the r-th mesh ensures orthotropy in directions $x^\alpha = \text{const}$. An example of $\Delta = \text{I, II, III, IV}$ fibre families distribution in the r-th mesh is shown in Fig. 4 (with some notation conventions adopted from papers [28, 29]). It is assumed, that the fibres are very thin, they transmit longitudinal forces only and that they are made of a linearly elastic material [9]. The forces in individual fibres are replaced by the force that is distributed in a continuous manner along a line in the plane of fibres. The force $\overset{r}{S}_{(\Delta)}$ in



a)
III
b_{III}
φ
φ
x^1
b_I
I
b_{IV}
IV
x^2
$\Delta = I, III, IV$
b)
III
b_{III}
φ
φ
x^1
b_{IV}
IV
x^2
$\Delta = III, IV$
c)
II
b_{II}
I
x^1
b_I
x^2
$\Delta = I, II$

FIG. 4.

fibres of (Δ) family and its components $\overset{r}{S}{}^{\alpha\beta}_{(\Delta)}$ in x^α coordinate system are as follows:

$$\overset{r}{S}_{(\Delta)} = \frac{\overset{r}{E}_{(\Delta)}\overset{r}{A}_{(\Delta)}}{\overset{r}{b}_{(\Delta)}}\left(\overset{r}{e}_{(\Delta)} - \overset{0r}{\varepsilon}_{(\Delta)}\right), \tag{2.6}$$

$$\overset{r}{S}{}^{\alpha\beta}_{(\Delta)} = \frac{\overset{r}{E}_{(\Delta)}\overset{r}{A}_{(\Delta)}}{\overset{r}{b}_{(\Delta)}}\left(\overset{r}{e}_{(\Delta)} - \overset{0r}{\varepsilon}_{(\Delta)}\right)\overset{r}{t}{}^{\alpha}_{(\Delta)}\overset{r}{t}{}^{\beta}_{(\Delta)}, \tag{2.7$_1$}$$

or

$$\overset{r}{S}{}^{\alpha\beta}_{(\Delta)} = \frac{\overset{r}{E}_{(\Delta)}\overset{r}{A}_{(\Delta)}}{\overset{r}{b}_{(\Delta)}}\left(\overset{r\delta}{t}_{(\Delta)}\overset{r\lambda}{t}_{(\Delta)}\overset{r}{e}_{(\Delta)\delta\lambda} - \overset{0r}{\varepsilon}_{(\Delta)}\right)\overset{r}{t}{}^{\alpha}_{(\Delta)}\overset{r\beta}{t}_{(\Delta)}, \tag{2.7$_2$}$$

where $\overset{0r}{\varepsilon}_{(\Delta)}$ denotes an initial distortion in fibres. For all (Δ) fibre families, which belong to the r-th mesh, we will obtain

$$\overset{r}{S}{}^{\alpha\beta} = \sum_{\Delta}\overset{r}{S}{}^{\alpha\beta}_{(\Delta)}. \tag{2.8}$$

It is assumed, that there is a full adhesion between the matrix and the fibres, so that the cross-sectional internal forces of the fibrous composite are equal to the sum of its all individual components.

To define the bifurcational, critical load of the plate, the considerations are limited to the linear theory of stability. Within this theory, the subcritical (pre-buckling) state is bending-free, while the plate buckling phenomenon means that there are equilibrium locations, which are arbitrarily close to the subcritical equilibrium configuration [30, 31, 32]. The basic differential equations were assumed in the form:

$$\begin{gathered} N^{\alpha\beta}_{,\alpha} = 0, \qquad M^{\alpha\beta}_{,\alpha} - Q^{\beta} = 0, \\ Q^{\alpha}_{,\alpha} + \overset{0}{N}{}^{\alpha\beta}\left(w + \overset{*}{w}\right)_{,\alpha\beta} = 0, \end{gathered} \tag{2.9}$$

where subcritical forces $\overset{0}{N}{}^{\alpha\beta}$ are calculated from equations of the membrane (disk) state in which appropriate boundary conditions are satisfied, $\overset{*}{w}$ represents small initial deflection, while w is the post-buckling deflection. In formulae $(2.7)_2$ and (2.9) the traditional summation convention holds for Greek indices which are located on different levels.

The equilibrium equations (2.9) result from the principle of virtual work with utilization of the Hencky – Bolle's hypothesis. On the other hand, the corresponding equations derived by a consistent use of the Vlasov's hypothesis are of much more complex form, and they contain quantities which are difficult to be physically interpreted. The theory based on Vlasov's hypothesis (2.9) is energetically (variationally) inconsistent (exactly such a theory was presented by Vlasov), while the theory of Hencky–Bolle is variationally consistent.

The internal forces are defined in the following way:

$$
\begin{aligned}
N^{\alpha\beta} &= \int_{-h/2}^{h/2} \sigma^{\alpha\beta}\, dz + \sum_r \overset{r}{S}{}^{\alpha\beta}, \\
M^{\alpha\beta} &= \int_{-h/2}^{h/2} \sigma^{\alpha\beta} z\, dz + \sum_r \overset{r}{S}{}^{\alpha\beta} z^r, \qquad Q^{\beta} = \int_{-h/2}^{h/2} \sigma^{\beta 3}\, dz
\end{aligned}
\tag{2.10}
$$

which can be presented as below after substitution of Eqs. (2.2), (2.3) and (2.4):

$$
\begin{aligned}
N^{\alpha\alpha} &= B(\varepsilon_{\alpha\alpha} + \nu\varepsilon_{\beta\beta}) + \sum_r \overset{r}{S}{}^{\alpha\alpha} \qquad \text{for} \quad \alpha \neq \beta, \\
N^{21} &= N^{12} = \frac{1-\nu}{2} B\varepsilon_{12} - h \int_0^t R^{1212}(t-\tau)\varepsilon_{12}(\tau)\, d\tau + \sum_r \overset{r}{S}{}^{12}, \\
M^{\alpha\alpha} &= \overset{0}{D} \left(\overset{0}{\kappa}_{\alpha\alpha} + \nu \overset{0}{\kappa}_{\beta\beta} \right) + C \overset{2}{D} \left(\overset{2}{\kappa}_{\alpha\alpha} + \nu \overset{2}{\kappa}_{\beta\beta} \right) \\
&\qquad + \sum_r \overset{r}{S}{}^{\alpha\alpha} z^r \qquad \text{for} \quad \alpha \neq \beta, \\
M^{12} &= M^{21} = \frac{1-\nu}{2} \overset{0}{D}\overset{0}{\kappa}_{12} + \frac{1-\nu}{2} C \overset{2}{D}\overset{2}{\kappa}_{12} \\
&\qquad - \frac{h^3}{12} \int_0^t R^{1212}(t-\tau) \overset{0}{\kappa}_{12}(\tau)\, d\tau \\
&\qquad - C\frac{h^5}{80} \int_0^t R^{1212}(t-\tau) \overset{2}{\kappa}_{12}(\tau)\, d\tau + \sum_r \overset{r}{S}{}^{12} z^r, \\
Q^1 &= G'h \left(k - \frac{1}{3}C \right) (\psi_1 + w_{,1}) \\
&\qquad - h \left(k - \frac{1}{3}C \right) \int_0^t R^{1313}(t-\tau) [\psi_1(\tau) + w_{,1}(\tau)]\, d\tau,
\end{aligned}
\tag{2.11}
$$

where

$$B = \frac{Eh}{1-\nu^2}, \qquad \overset{0}{D} = \frac{Eh^3}{12(1-\nu^2)}, \qquad \overset{2}{D} = \frac{Eh^5}{80(1-\nu^2)} \tag{2.12}$$

and the k factor, which appears in expressions for the shearing forces, has the following values: $k = 1$ for Vlasov's theory (where simultaneously $C = 1$) and $k = 5/6$ for Hencky–Bolle's theory (where additionally $C = 0$).

3. Determination of critical loads

The considerations concern the time-dependent buckling of a fibrous composite plate in a shape of a strip, which is subjected to uni-axial compression by the force p being uniformly distributed along the supported edges $x^1 = 0$, $x^1 = a$ (Fig. 1). The plate has support conditions, which do not vary along the support line, and it has orthotropic physical properties in directions parallel to parametric lines $x^a = \text{const}$. Under the assumed conditions, the plate experiences cylindrical bending, and the unknown functions are functions of one variable, namely x^1. The aim of considerations includes the determination of bifurcational loads and a study of the effects of appropriate parameters.

Under the assumption of the bending-free (disk) pre-buckling state, the subcritical forces are:

$$\overset{0}{N}{}^{11} = -p, \qquad \overset{0}{N}{}^{22} = 0, \qquad \overset{0}{N}{}^{12} = \overset{0}{N}{}^{21} = 0, \tag{3.1}$$

while basic equations and relations take the suitably simplified form. For example, physical equations for the fibrous phase consisting of m pairs of identical meshes located symmetrically in the cross-section at distances $z^r = \pm e_1, \pm e_2, \ldots, \pm e_m$; $e_m \in (0, h/2)$, take the form:

$$\begin{aligned}
N^{11} &= B\varepsilon_{11} + 2m\sum_{\Delta}\frac{\overset{r}{E}_{(\Delta)}\overset{r}{A}_{(\Delta)}}{\overset{r}{b}_{(\Delta)}}\left[\left(\overset{r}{t}{}^1_{(\Delta)}\right)^2 \overset{0}{u}_{1,1} - \overset{0r}{\varepsilon}_{(\Delta)}\right]\left(\overset{r}{t}{}^1_{(\Delta)}\right)^2,\\
N^{22} &= \nu B\varepsilon_{11} + 2m\sum_{\Delta}\frac{\overset{r}{E}_{(\Delta)}\overset{r}{A}_{(\Delta)}}{\overset{r}{b}_{(\Delta)}}\left[\left(\overset{r}{t}{}^1_{(\Delta)}\right)^2 \overset{0}{u}_{1,1} - \overset{0r}{\varepsilon}_{(\Delta)}\right]\left(\overset{r}{t}{}^2_{(\Delta)}\right)^2,\\
M^{11} &= \overset{0}{D}\overset{0}{\kappa}_{11} + C\,\overset{2}{D}\overset{2}{\kappa}_{11} + 2\sum_{m}\sum_{\Delta}\frac{\overset{r}{E}_{(\Delta)}\overset{r}{A}_{(\Delta)}}{\overset{r}{b}_{(\Delta)}}\\
&\quad\cdot\left(\overset{r}{t}{}^1_{(\Delta)}\right)^4 e_m^2\left[\psi_{1,1} - C\frac{4}{3}\left(\frac{e_m}{h}\right)^2(\psi_{1,1} + w_{,11})\right],
\end{aligned} \tag{3.2}$$

(3.2) [cont.]

$$M^{22} = \nu \overset{0\,0}{D}\kappa_{11} + \nu C \overset{2\,2}{D}\kappa_{11} + 2\sum_{m}\sum_{\Delta} \frac{\overset{r}{E}_{(\Delta)}\overset{r}{A}_{(\Delta)}}{\overset{r}{b}_{(\Delta)}} \cdot \left(\overset{r}{t}{}^{1}_{(\Delta)}\right)^2 \left(\overset{r}{t}{}^{2}_{(\Delta)}\right)^2 e_m^2 \left[\psi_{1,1} - C\frac{4}{3}\left(\frac{e_m}{h}\right)^2 (\psi_{1,1} + w_{,11})\right],$$

$$Q^1 = G'h\left(k - \frac{1}{3}C\right)(\psi_1 + w_{,1}) - h\left(k - \frac{1}{3}C\right)\int_0^t R^{1313}(t-\tau)\left[\psi_1(\tau) + w_{,1}(\tau)\right] d\tau.$$

Assume the relaxation kernel R^{1313} in an exponential form, which is typical for linear viscoelasticity, as

$$R^{1313} = \frac{G' - H'}{n_{13}} \exp\left(-\frac{t-\tau}{n_{13}}\right); \tag{3.3}$$

it corresponds to the model of the body for which the constitutive relationship is $\sigma^{13} + n_{13}\dot{\sigma}^{13} = H'e_{13} + n_{13}G'\dot{e}_{13}$, where n_{13}-relaxation time; G', H' – immediate and sustained Kirchhoff's moduli; $(\dot{\ }) = \frac{d}{dt}(\ )$; then, after substitution of (2.3), (3.1) and (3.2) into equilibrium equations (2.9), the system of integro-differential equations is obtained with $w(x^1,t)$, $\overset{0}{u}_1(x^1,t)$ and $\psi_1(x^1,t)$ displacement functions as unknowns. Now, we express the unknown functions as the products of functions with separated arguments and which satisfy the boundary conditions of the problem. Then, having eliminated the appropriate integral expressions, the system of governing equations can be brought to the form of a differential equation for the deflection amplitude $f(t)$ being a function of time. This equation can be presented as:

$$\overline{n}_{13}\left(p^N_{\rm cr} - p\right)\dot{f}(t) + \left(p^D_{\rm cr} - p\right)f(t) = pf_0, \tag{3.4}$$

where $p^N_{\rm cr}$, $p^D_{\rm cr}$ are immediate and sustained critical loads for a fibrous composite plate strip, respectively.

Let us now consider, in a detailed manner, the case of buckling of a fibrous composite plate strip, which is simply supported along its edges $x^1 = 0$, $x^1 = a$, with the cross-section including two fibrous meshes only ($m = 1$), which are spaced symmetrically at distances $z^r = \pm e$. Here we have:

$$\begin{aligned} N^{11} &= \left(B + 2\overset{r}{B}{}^{11}\right)u_{,1}, \\ N^{22} &= \left(\nu B + 2\overset{r}{B}{}^{22}\right)u_{,1}, \end{aligned} \tag{3.5}$$

(3.5) [cont.]

$$M^{11} = \left\{ \left(1 - \frac{1}{5}C\right) \overset{0}{D} + 2 \overset{r}{B}{}^{11} e^2 \left[1 - C\frac{4}{3}\left(\frac{e}{h}\right)^2\right] \right\} \psi_{1,1} - C\left[\frac{1}{5}\overset{0}{D} + \frac{8}{3}\overset{r}{B}{}^{11} e^2 \left(\frac{e}{h}\right)^2\right] w_{,11},$$

$$M^{22} = \left\{ \left(1 - \frac{1}{5}C\right) \nu \overset{0}{D} + 2 \overset{r}{B}{}^{22} e^2 \left[1 - C\frac{4}{3}\left(\frac{e}{h}\right)^2\right] \right\} \psi_{1,1} - C\left[\frac{1}{5}\nu\overset{0}{D} + \frac{8}{3}\overset{r}{B}{}^{22} e^2 \left(\frac{e}{h}\right)^2\right] w_{,11},$$

$$Q^1 = G'h\left(k - \frac{1}{3}C\right)(\psi_1 + w_{,1}) - h\left(k - \frac{1}{3}C\right)\frac{(G' - H')}{n_{13}}\int_0^t [\psi_1(\tau) + w_{,1}(\tau)] \exp\left(-\frac{t-\tau}{n_{13}}\right) d\tau,$$

where

$$\overset{r}{B}{}^{11} = \sum_{\Delta} \frac{\overset{r}{E}_{(\Delta)}\overset{r}{A}_{(\Delta)}}{\overset{r}{b}_{(\Delta)}} \left(\overset{r}{t}{}^1_{(\Delta)}\right)^4, \qquad \overset{r}{B}{}^{22} = \sum_{\Delta} \frac{\overset{r}{E}_{(\Delta)}\overset{r}{A}_{(\Delta)}}{\overset{r}{b}_{(\Delta)}} \left(\overset{r}{t}{}^1_{(\Delta)}\right)^2 \left(\overset{r}{t}{}^2_{(\Delta)}\right)^2 .$$

The following boundary conditions and equilibrium equations are assumed:

$$\text{(3.6)} \quad \begin{array}{lll} \text{for} \quad x^1 = 0: & \overset{0}{u}_1 = 0, \quad w = 0, & M^{11} = 0, \\ \text{for} \quad x^1 = a: & N^{11} = -p, \quad w = 0, & M^{11} = 0, \end{array}$$

(3.7)

$$N^{11}_{,1} = 0,$$

$$\overline{D}{}^{11}_1 \psi_{1,11} - \overline{D}{}^{11}_2 w_{,111} = G'h\left(k - \frac{1}{3}C\right)(\psi_1 + w_{,1}) - h\left(k - \frac{1}{3}C\right)\frac{(G' - H')}{n_{13}}\int_0^t (\psi_1 + w_{,1}) \exp\left(-\frac{t-\tau}{n_{13}}\right) d\tau,$$

$$\overline{D}{}^{11}_1 \psi_{1,111} - \overline{D}{}^{11}_2 w_{,1111} = p(w_{,11} + \overset{*}{w}_{,11}).$$

The following notations in equations (3.7) have been introduced:

(3.8)

$$\overline{D}{}^{11}_1 = \left(1 - \frac{1}{5}C\right)\overset{0}{D} + 2\overset{r}{B}{}^{11} e^2 \left[1 - \frac{4}{3}C\left(\frac{e}{h}\right)^2\right],$$

$$\overline{D}{}^{11}_2 = C\left(\frac{1}{5}\overset{0}{D} + \frac{8}{3}\overset{r}{B}{}^{11}\frac{e^4}{h^2}\right),$$

$$\overline{D}{}^{11} = \overline{D}{}^{11}_1 + \overline{D}{}^{11}_2 = \overset{0}{D} + 2\overset{r}{B}{}^{11} e^2 .$$

The quantity $\overline{D}^{11} = \overline{D}_1^{11} + \overline{D}_2^{11}$ is the bending stiffness of the fibrous composite plate. From $(3.7)_1$ we obtain $N^{11} = \text{const}$, and from the boundary conditions (3.6) $N^{11} = -p$. The remaining two equations (3.7) form the system of integro-differential equations, which can be reduced to the following differential equations:

$$
\begin{gathered}
n_{13}\overline{D}_1^{11}\dot{\psi}_{1,11} + \overline{D}_1^{11}\psi_{1,11} - n_{13}\overline{D}_2^{11}\dot{w}_{,111} - \overline{D}_2^{11}w_{,111} \\
= n_{13}G'h\left(k - \frac{1}{3}C\right)\left(\dot{\psi}_1 + \dot{w}_{,1}\right) + H'h\left(k - \frac{1}{3}C\right)(\psi_1 + w_{,1}), \\
\overline{D}_1^{11}\psi_{1,111} - \overline{D}_2^{11}w_{,1111} = p(w_{,11} + \overset{*}{w}_{,11}).
\end{gathered} \tag{3.9}
$$

Searching then for solutions of the system of equations (3.9) in the form:

$$
w(x^1,t) = f(t)\sin\frac{\pi x^1}{a}, \qquad \psi_1(x^1,t) = g(t)\cos\frac{\pi x^1}{a}, \tag{3.10}
$$

and by approximating the initial deflection by $\overset{*}{w}(x^1) = f_0\sin(\pi x^1/a)$, we will finally have:

$$
\begin{gathered}
g(t) = \frac{\left(\frac{\pi}{a}\right)^2\overline{D}_2^{11} - p}{\frac{\pi}{a}\overline{D}_1^{11}}f(t) - \frac{p}{\frac{\pi}{a}\overline{D}_1^{11}}f_0, \\
\overline{n}_{13}\left(p_{cr}^N - p\right)\dot{f}(t) + \left(p_{cr}^D - p\right)f(t) = pf_0,
\end{gathered} \tag{3.11}
$$

where

$$
\overline{n}_{13} = n_{13}\frac{1 + \dfrac{G'h\left(k - \frac{1}{3}C\right)}{\overline{D}_1^{11}}\left(\dfrac{a}{\pi}\right)^2}{1 + \dfrac{H'h\left(k - \frac{1}{3}C\right)}{\overline{D}_1^{11}}\left(\dfrac{a}{\pi}\right)^2}, \tag{3.12}
$$

$$
p_{cr}^N = \frac{\overline{D}_1^{11}\left(\dfrac{\pi}{a}\right)^2}{1 + \dfrac{\overline{D}_1^{11}\left(\frac{\pi}{a}\right)^2}{G'h\left(k - \frac{1}{3}C\right)}} = \overline{D}^{11}\left(\frac{\pi}{a}\right)^2\frac{1}{1 + \overline{\varepsilon}_N\left(\dfrac{\pi}{a}\right)^2}, \tag{3.13}
$$

$$
p_{cr}^D = \frac{\overline{D}^{11}\left(\dfrac{\pi}{a}\right)^2}{1 + \dfrac{\overline{D}_1^{11}\left(\frac{\pi}{a}\right)^2}{H'h\left(k - \frac{1}{3}C\right)}} = \overline{D}^{11}\left(\frac{\pi}{a}\right)^2\frac{1}{1 + \overline{\varepsilon}_D\left(\dfrac{\pi}{a}\right)^2}, \tag{3.14}
$$

The following notation

$$(3.15)\qquad \bar{\varepsilon}_N = \frac{\overline{D}_1^{11}}{G'h\left(k - \frac{1}{3}C\right)}, \qquad \bar{\varepsilon}_D = \frac{\overline{D}_1^{11}}{H'h\left(k - \frac{1}{3}C\right)}$$

was used to denote the ratios of the strip bending stiffness in x^1 direction to the immediate and the sustained lateral stiffnesses.

The solution of Eq. $(3.11)_2$ satisfying the initial condition

$$f(t=0) = \frac{p}{(p_{cr}^N - p)} f_0$$

which corresponds to the elastic solution, is the function:

$$(3.16)\qquad f(t) = \frac{pf_0}{p_{cr}^D - p} - \left(\frac{pf_0}{p_{cr}^D - p} - \frac{pf_0}{p_{cr}^N - p}\right)\exp\left(-\frac{p_{cr}^D - p}{p_{cr}^N - p}\frac{t}{\bar{n}_{13}}\right),$$

which for $t = \infty$ has the value:

$$f(\infty) = \frac{p}{p_{cr}^D - p} f_0 .$$

So, the time-dependence of the displacements represented by the buckling mode shape defined by (3.10), are described by the expressions:

$$(3.17)\qquad \begin{aligned} w(x^1, t) &= \left\{\frac{pf_0}{p_{cr}^D - p} - \left(\frac{pf_0}{p_{cr}^D - p} - \frac{pf_0}{p_{cr}^N - p}\right) \cdot \exp\left(-\frac{p_{cr}^D - p}{p_{cr}^N - p}\frac{t}{\bar{n}_{13}}\right)\right\}\sin\frac{\pi x^1}{a}, \\ \psi_1(x^1, t) &= \frac{a}{\pi \overline{D}_1^{11}}\left\{\left(\frac{\pi^2}{a^2}\overline{D}_2^{11} - p_{cr}^D\right)\frac{pf_0}{p_{cr}^D - p} \right. \\ &\quad \left. - \left(\frac{\pi^2}{a^2}\overline{D}_2^{11} - p\right)\left(\frac{pf_0}{p_{cr}^D - p} - \frac{pf_0}{p_{cr}^N - p}\right)\exp\left(-\frac{p_{cr}^D - p}{p_{cr}^N - p}\frac{t}{\bar{n}_{13}}\right)\right\}\cos\frac{\pi x^1}{a}. \end{aligned}$$

It follows from the analysis of expression (3.16) and $(3.17)_1$ that at the moment when the compressive force $p < p_{cr}^D$ is applied, the plate will deflect elastically and this deflection will grow with decreasing velocity to the asymptotic value of $w(x^1, \infty) = \frac{pf_0}{p_{cr}^D - p}\sin\frac{\pi x^1}{a}$. For $p = p_{cr}^D$ the deflection growth velocity will be constant and equal to $\dot{w}(x^1, t) = \frac{1}{\bar{n}_{13}}\frac{pf_0}{(p_{cr}^N - p)}\sin\frac{\pi x^1}{a}$. However, for the force $p_{cr}^D < p < p_{cr}^N$ the deflection will grow at an increasing velocity, and when $p = p_{cr}^N$, the immediate loss of stability will appear. Thus, the stable state of the plate is the state, in which deflection growth rate is diminishing, while at the unstable state the velocity of the deflection is increasing. According to the above statement, $p = p_{cr}^D$ is the limit of the sustained stability, since at lower loads ($p < p_{cr}^D$) the deflections growth decreases with time to the upper bound equal to the asymptotic limiting value of $w = \dfrac{p f_0}{p_{cr}^D - p} \sin \dfrac{\pi x^1}{a}$.

Before we list the formulae for critical loads corresponding to individual plate theories, it is necessary to specify the way of reaching the solution, which corresponds to the Kirchhoff's classical theory. As a matter of fact, assumption of the rotation angles of normals in the form of $\psi_a = -w_{,\alpha}$ leads to the elimination of the components of shear deformation $e_{\alpha 3} = \varepsilon_{\alpha 3} = 0$, see (2.2), (2.3), but simultaneously it makes the transversal forces Q^α, $(2.11)_5$, $(3.2)_5$, $(3.5)_5$ vanish, though, in fact, they must have a nonzero value. However, if the constitutive equation for transverse shearing is presented in the inverted form, that is:

$$e_{\alpha 3} = \varepsilon_{\alpha 3} = \frac{1}{G'h\left(k - \frac{1}{3}C\right)} \left\{ Q^\alpha + \left(1 - \frac{H'}{G'}\right) \frac{1}{n_{13}} \cdot \int_0^t Q^\alpha(\tau) \exp\left[-\frac{H'(t-\tau)}{n_{13}G'}\right] d\tau \right\}, \tag{3.18}$$

we can see, that simultaneous satisfaction of the conditions $e_{\alpha 3} = \varepsilon_{\alpha 3} = 0$ and $Q^\alpha \neq 0$ is possible when the transverse shear stiffness of the plate becomes infinite.

So, the critical loads for a viscoelastic plate strip, considered in this paper, classified with respect to the plate theory type, are expressed by the following formulae:

- for Vlasov's theory ($C = 1$, $k = 1$)

$$p_{cr} = p_{cr}^D = \left(\overset{0}{D} + 2\overset{r}{B}{}^{11} e^2\right) \left(\frac{\pi}{a}\right)^2 \cdot \frac{1}{1 + \dfrac{3}{2}\dfrac{1}{H'h}\left[\dfrac{4}{5}\overset{0}{D} + 2\overset{r}{B}{}^{11} e^2 \left(1 - \dfrac{4}{3}\dfrac{e^2}{h^2}\right)\right]\left(\dfrac{\pi}{a}\right)^2}, \tag{3.19}$$

- for the Hencky–Bolle theory ($C = 0$, $k = 5/6$)

$$p_{cr} = p_{cr}^D = \left(\overset{0}{D} + 2\overset{r}{B}{}^{11} e^2\right) \left(\frac{\pi}{a}\right)^2 \frac{1}{1 + \dfrac{6}{5}\dfrac{1}{H'h}\left(\overset{0}{D} + 2\overset{r}{B}{}^{11} e^2\right)\left(\dfrac{\pi}{a}\right)^2}, \tag{3.20}$$

• for Kirchhoff's theory

$$p_{\mathrm{cr}(K)} = p^N_{\mathrm{cr}(K)} = \overline{D}^{11}\left(\frac{\pi}{a}\right)^2 = \left(\overset{0}{D} + 2\overset{r}{B}{}^{11}e^2\right)\left(\frac{\pi}{a}\right)^2. \tag{3.21}$$

As we have already found, the elastic solution of the fibrous composite plate strip for the buckling problem corresponds to the immediate solution, which imposes the requirement, that in formulae (3.19) and (3.20) G' is substituted instead of the sustained Kirchhoff's modulus, H'. Of course, the formula (3.21) remains unchanged.

The formula for critical loads for a homogeneous plate strip can be obtained after elimination of the phase II (fibrous phase) from the plate, and by assuming that $\overset{r}{A}_{(\Delta)} = 0$, which results in the following:

$$\overset{r}{B}{}^{11} = \overset{r}{B}_{22} = 0, \qquad \overline{D}^{11}_1 = \left(1 - \frac{1}{5}C\right)\overset{0}{D},$$

$$\overline{D}^{11}_2 = \frac{1}{5}C\,\overset{0}{D}, \qquad \overline{D}^{11} = \overset{0}{D} = \frac{Eh^3}{12(1-\nu^2)}$$

and from (3.13), (3.14):

$$p^N_{\mathrm{cr}} = \overset{0}{D}\left(\frac{\pi}{a}\right)^2 \frac{1}{1 + \dfrac{\left(1 - \frac{1}{5}C\right)\overset{0}{D}}{G'h\left(k - \frac{1}{3}C\right)}\left(\dfrac{\pi}{a}\right)^2},$$

$$p^D_{\mathrm{cr}} = \overset{0}{D}\left(\frac{\pi}{a}\right)^2 \frac{1}{1 + \dfrac{\left(1 - \frac{1}{5}C\right)\overset{0}{D}}{H'h\left(k - \frac{1}{3}C\right)}\left(\dfrac{\pi}{a}\right)^2}. \tag{3.22}$$

From (3.22) it follows, that the critical load for Vlasov's theory ($C = 1$, $k = 1$) and for Hencky–Bolle theory ($C = 0$, $k = 5/6$) are identical and equal to

$$p_{\mathrm{cr}} = p^D_{\mathrm{cr}} = \overset{0}{D}\left(\frac{\pi}{a}\right)^2 \frac{1}{1 + \dfrac{6}{5}\dfrac{\overset{0}{D}}{H'h}\left(\dfrac{\pi}{a}\right)^2}, \tag{3.23}$$

while for Kirchhoff's theory the critical load is

$$p_{\mathrm{cr}} = p_{\mathrm{Eul}} = \overset{0}{D}\left(\frac{\pi}{a}\right)^2. \tag{3.24}$$

The solution (3.23) is identical with a solution given by TETERS [23] (page 283, where the plate with thickness $2h$ was considered). However, the solution (3.24), which corresponds to Kirchhoff's model, is a well known classical Euler's solution for a homogeneous plate strip [30, 31].

4. Parametric analysis

Let us perform a parametric analysis to illustrate the closed-form analytical solutions for critical loads obtained in Sec. 3 for a simply supported fibrous composite plate strip, which possesses some viscoelastic properties. Let us consider the plate made of plastic and of the cross-section reinforced symmetrically ($z^r = \pm e$) by means of two identical meshes according to variants a), b), c), which are presented in Fig. 4. Moreover, we will assume that the tension (compression) stiffnesses $(\overset{r}{E}_{(\Delta)}\overset{r}{A}_{(\Delta)})/\overset{r}{b}_{(\Delta)}$ are identical for all families of fibres in a given mesh.

It is more convenient to perform the analysis after the expressions (3.13), (3.14) are transformed to the form:

$$p^N_{\text{cr}} = \overset{0}{D}\left(\frac{\pi}{a}\right)^2 \frac{\gamma}{1+\gamma_N}, \qquad p^D_{\text{cr}} = \overset{0}{D}\left(\frac{\pi}{a}\right)^2 \frac{\gamma}{1+\gamma_D}, \tag{4.1}$$

where

$$\gamma = 1 + 24(1-\nu^2)n\mu_{(\Delta)}\left(\frac{e}{h}\right)^2 \overset{r}{d},$$

$$\gamma_N = \frac{E}{G'}\left(\frac{h}{a}\right)^2 \frac{\pi^2\left[\left(1-\frac{1}{5}C\right) + 24(1-\nu^2)n\mu_{(\Delta)}\left(\frac{e}{h}\right)^2\left(1-\frac{4}{3}C\frac{e^2}{h^2}\right)\overset{r}{d}\right]}{12(1-\nu^2)\left(k-\frac{1}{3}C\right)},$$

$$\gamma_D = \frac{E}{H'}\left(\frac{h}{a}\right)^2 \frac{\pi^2\left[\left(1-\frac{1}{5}C\right) + 24(1-\nu^2)n\mu_{(\Delta)}\left(\frac{e}{h}\right)^2\left(1-\frac{4}{3}C\frac{e^2}{h^2}\right)\overset{r}{d}\right]}{12(1-\nu^2)\left(k-\frac{1}{3}C\right)}, \tag{4.2}$$

$$n = \frac{\overset{r}{E}_{(\Delta)}}{E}, \qquad \mu_{(\Delta)} = \frac{\overset{r}{A}_{(\Delta)}}{\overset{r}{b}_{(\Delta)}\,h}, \qquad \overset{r}{d} = \sum_{\Delta}\left(\overset{r}{t}{}^1_{(\Delta)}\right)^4.$$

Factor $\overset{r}{d}$, which appears in formulae (4.2) is the quantity, which depends on the selection of the mesh variant and on the fibre inclination angle φ.

Because $\vec{t}^{\,1}_{\mathrm{I}} = 1$, $\vec{t}^{\,2}_{\mathrm{I}} = 0$, $\vec{t}^{\,1}_{\mathrm{II}} = 0$, $\vec{t}^{\,2}_{\mathrm{II}} = 1$, $\vec{t}^{\,1}_{\mathrm{III}} = \cos\varphi$, $\vec{t}^{\,2}_{\mathrm{III}} = -\sin\varphi$, $\vec{t}^{\,1}_{\mathrm{IV}} = \cos\varphi$, $\vec{t}^{\,2}_{\mathrm{IV}} = \sin\varphi$, (Fig. 5), then $\vec{d}$ for the particular variant of the mesh is: $\vec{d}_a = 1 + 2\cos^4\varphi$, $\vec{d}_b = 2\cos^4\varphi$ and $\vec{d}_c = 1$, respectively.

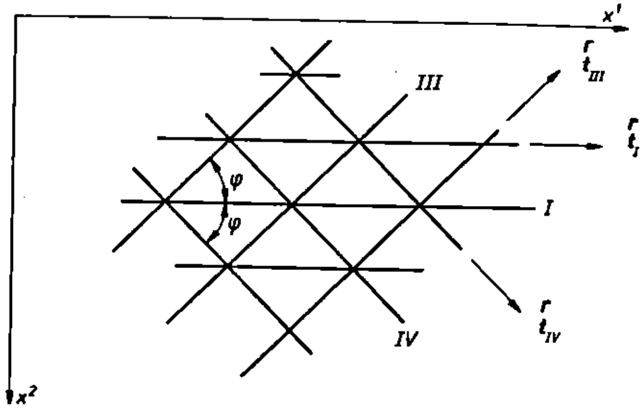


FIG. 5.

The first term in the formula (4.1) is equal to the Euler's critical force for a homogeneous plate strip (3.24), so it is constant for a given strip. According to the above observation, the influence of the following factors: slenderness of the plate h/a, shear ratios E/G', E/H', $\vec{E}_{(\Delta)}/E$, the type of the plate theory used, the mesh variant, the mesh density and its location within the cross-section and the fibre inclination angles φ, are totally included in γ, γ_N, γ_D.

The parametric analysis to illustrate the influence of some important parameters among those mentioned above, which are believed to have the decisive influence on the limit of applicability of the Kirchhoff's hypothesis in the theory of fibrous composite plates, was done in a non-dimensional form. The analysis was performed for the following numerical data:

• physical data: $\nu = 0.2$; $n = \dfrac{\vec{E}_{(\Delta)}}{E} = 10$; $\mu_{(\Delta)} = 0.01(1\%)$; $E/H' = 5$, 10, 25, 50, 60, 80, 100, 120,

• geometrical data: $e/h = 0.45$; $a/h = 5, 8, 10, 15, 20, 25, 30, 60$; $\varphi = 45°$.

The error produced by neglecting the shear effects on values of the critical load is equal to

$$|\varepsilon| = \frac{p^D_{\mathrm{cr}} - p_{\mathrm{cr}(K)}}{p^D_{\mathrm{cr}}}\cdot 100\% = \gamma_D \cdot 100\%\,. \tag{4.3}$$

Numerical values of the error, ε, calculated for two theory types and for three variants of the reinforcement mesh, are given in Table 1.

Table 1.

$\lvert\varepsilon\rvert$%		variant of mesh	E/H'											
			5		10		25		50		60		80	
			theory Vlasov	theory H–B	theory Vlasov	theory H–B	theory Vlasov	theory H–B	theory Vlasov	theory H–B	theory Vlasov	theory H–B	theory Vlasov	theory H–B
$\frac{a}{h}$	5	*a*	33.7	35.0	67.4	69.9	168.5	174.8	336.9	349.5	404.3	419.4	539.0	559.2
		b	24.9	25.4	49.9	50.7	124.7	126.8	249.4	253.6	299.3	304.3	399.0	405.8
		c	29.3	30.2	58.6	60.3	146.6	150.8	293.2	301.5	351.8	361.9	469.0	482.5
	8	*a*	13.2	13.6	26.3	27.3	65.8	68.3	131.6	136.5	157.9	163.8	210.6	218.4
		b	9.7	9.9	19.5	19.8	48.7	49.5	97.4	99.1	116.9	118.9	155.9	158.5
		c	11.5	11.8	22.9	23.6	57.3	58.9	114.5	117.8	137.4	141.6	183.2	188.5
	10	*a*	8.4	8.7	16.8	17.5	42.1	43.7	84.2	87.4	101.1	104.9	134.8	139.8
		b	6.2	6.3	12.5	12.7	31.2	31.7	62.3	63.4	74.8	76.1	99.8	101.4
		c	7.3	7.5	14.7	15.1	36.6	37.7	73.3	75.4	87.9	90.5	117.3	120.6
	15	*a*	3.7	3.9	7.5	7.8	18.7	19.4	37.4	38.8	44.9	46.6	59.9	62.1
		b	2.8	2.82	5.5	5.6	13.9	14.1	27.7	28.2	33.3	33.8	44.3	45.1
		c	3.3	3.4	6.5	6.7	16.3	16.8	32.6	33.5	39.1	40.2	52.1	53.6
	20	*a*	2.11	2.18	4.21	4.37	10.5	10.9	21.1	21.8	25.3	26.2	33.7	35.0
		b	1.56	1.59	3.12	3.17	7.8	7.9	15.6	15.9	18.7	19.0	24.9	25.4
		c	1.83	1.9	3.66	3.77	9.2	9.4	18.3	18.8	22.0	22.6	29.3	30.2
	25	*a*	1.35	1.40	2.70	2.80	6.7	7.0	13.5	14.0	16.2	16.8	21.6	22.4
		b	1.00	1.01	2.00	2.03	4.5	5.1	10.0	10.2	12.0	12.2	16.0	16.2
		c	1.17	1.21	2.35	2.41	5.9	6.0	11.7	12.1	14.1	14.5	18.8	19.3
	30	*a*	0.94	0.97	1.87	1.94	4.7	4.9	9.4	9.7	11.2	11.7	15.0	15.5
		b	0.69	0.70	1.39	1.41	3.5	3.6	6.9	7.0	8.3	8.5	11.1	11.3
		c	0.82	0.84	1.63	1.68	4.1	4.2	8.1	8.4	9.8	10.1	13.0	13.4

Figure 6 presents variation of the critical loads: $p^N_{\rm cr}$ – immediate (continuous line) and $p^D_{\rm cr}$ – sustained (dashed line), both referred to the Euler's force, $p_{{\rm cr}(K)}$ (3.21) – pertaining to the fibrous composite strip, of $G'/H' = 2$. In turn, Fig. 7 presents variation of $p^D_{\rm cr}/p_{{\rm cr}(K)}$ ratio on the plane of variables $\left(\dfrac{h}{a}, \dfrac{E}{H'}\right)$. The graphs presented in Figs. 6 and 7 were obtained for the Vlasov's theory and for the variant of reinforcement as shown in Fig. 4a. Appropriate numerical values are given in Table 2.

From the above analysis it follows that, if the transverse shear deformations in fibrous composite plates are taken into account, the values of critical loads are visibly reduced and they depend mostly on the following parameters: a/h, E/G', E/H'. However, the selection of the theory type,

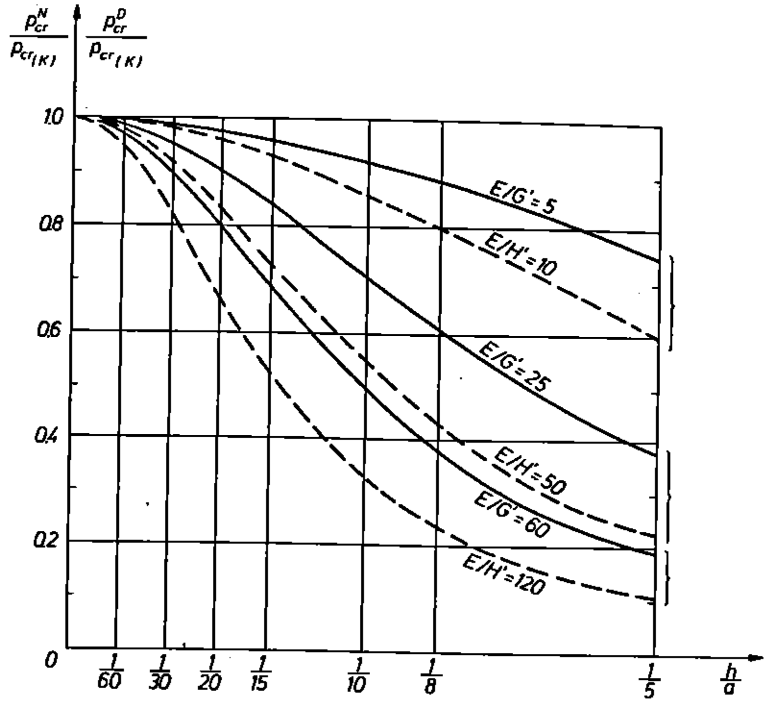


FIG. 6.

Table 2.

$\frac{p_{cr}^D}{p_{cr(K)}}$		h/a								
		0	1/60	1/30	1/25	1/20	1/15	1/10	1/8	1/5
E/H'	0	1	1	1	1	1	1	1	1	1
	5	1	0.998	0.991	0.989	0.979	0.964	0.923	0.883	0.748
	10	1	0.995	0.982	0.974	0.960	0.930	0.856	0.792	0.597
	25	1	0.988	0.955	0.937	0.905	0.842	0.704	0.603	0.372
	50	1	0.977	0.914	0.881	0.826	0.728	0.543	0.432	0.229
	60	1	0.973	0.899	0.861	0.798	0.690	0.497	0.388	0.198
	80	1	0.964	0.870	0.822	0.748	0.625	0.426	0.322	0.156
	100	1	0.955	0.842	0.788	0.704	0.572	0.372	0.275	0.129
	120	1	0.947	0.817	0.756	0.664	0.527	0.331	0.240	0.110

that is, the Vlasov's or Hencky-Bolle theory, is of lesser importance, though, in every case the Hencky-Bolle theory produces results which are by some 1.5 – 4.5% higher than those obtained by employing the Vlasov's hypothesis (see Table 1).

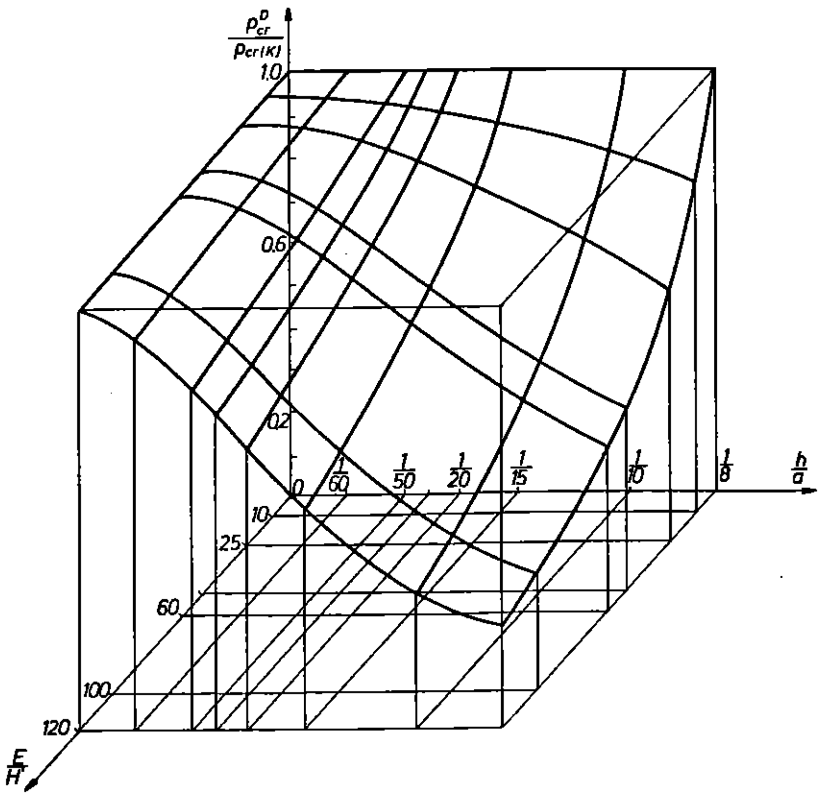


FIG. 7.

It can be observed that Kirchhoff's hypothesis can be used only for slender, viscoelastic fibrous composite plates. The decisive contribution to the plate performance is due to the $(h/a)^2$ and E/H' parameters. If some allowable error is assumed, for example $\varepsilon = 3\%$, then by neglecting the transverse shear, as it follows from Table 1, the Kirchhoff's hypothesis can be used in cases of very slender plates only, that is for $(E/H') > 10$ at the slendernesses $(a/h) > 30$, while for $(E/H') = 10$ at $(a/h) > 20$, and for $(E/H') = 5$ at $(a/h) > 15$.

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