ENGINEERING TRANSACTIONS • *Engng. Trans.* • **43**, 4, 571–590, 1995
Polish Academy of Sciences • Institute of Fundamental Technological Research

INTERACTIVE ELASTIC BUCKLING OF THIN-WALLED CLOSED ORTHOTROPIC BEAM-COLUMNS

Z. KOŁAKOWSKI and M. KRÓLAK (ŁÓDŹ)

The present paper deals with an analysis of global and local stabilities and with the investigation of equilibrium paths in the initial post-buckling behaviour of elastic thin-walled beam-columns consisting of orthotropic rectangular walls. Closed or open cross-section beam-columns, simply supported at the ends, are subject to axial compression, eccentric compression or pure bending. The problem is solved by means of a variational method using the asymptotic nonlinear theory of stability of the conservative systems [7, 8]. Numerical calculations employ Byskov's and Hutchinson's asymptotic expansion [3] in the form of the transition matrix method. The calculation is restricted to the first order nonlinear approximation. The analysis is carried out in order to study the influence of the wall orthotropy factor of closed cross-section beam-columns (square or trapezoid-shaped) subject to axial and eccentric compression influences the critical state and the initial post-buckling behaviour.

NOTATIONS

a_{ijJ} three-index coefficients in the nonlinear equilibrium equations by Eq. (A.3.1) [3],
b_i width of the i-th wall of column,
E_{xi}, E_{yi} Young's moduli of i-th wall along x and y axes, respectively,
G_i modulus of non-dilatational strain of i-th wall,
h_i thickness of the i-th wall of the column,
l length of the column,
m number of axial half-waves of n-th mode,
$M_{xi}^{(n)}, M_{yi}^{(n)}, M_{xyi}^{(n)}$ bending moment resultants for the i-th wall referring to the n-th mode in the first approximation,
n number of mode,
N number of interacting modes,
$\overline{\mathrm{N}}$ force field,
N_{xi}, N_{yi}, N_{xyi} in-plane stress resultants for the i-th wall,
N_{xi}^{o} pre-buckling in-plane stress for the i-th wall,
$N_{xi}^{(n)}, N_{yi}^{(n)}, N_{xyi}^{(n)}$ in-plane stress resultants for the i-th wall referring to the n-th mode in the first approximation,
N_{yi}^{*} Eq. (A.1.4),
N_{xyi}^{*} Eq. (A.1.4),
Q_{yi}^{*} Eq. (A.1.4),
$\overline{\mathrm{U}}$ displacement field,

u_i, v_i, w_i displacement components of middle surface of the i-th wall,
u_i^o, v_i^o, w_i^o pre-buckling displacement fields,
$u_i^{(n)}, v_i^{(n)}, w_i^{(n)}$ buckling displacement fields referring to the n-th buckling mode,
$\beta_i = E_{xi}/E_{yi}$ orthotropy factor of the i-th wall,
Δ_i measure of the applied pressure,
$\varepsilon_{xi}, \varepsilon_{yi}, \varepsilon_{xyi}$ strain tensor components for the middle surface of the i-th wall,
$\kappa_{xi}, \kappa_{yi}, \kappa_{xyi}$ curvature modifications and torsions of the middle surface of the i-th wall,
λ scalar load parameter,
λ_n value of λ at bifurcation mode number n,
λ_s maximum value of λ for imperfect column,
ν_{xyi}, ν_{yxi} Poisson's ratio of the i-th wall; the first index indicates transverse direction and the second shows the direction of load,
ξ_n amplitude of n-th buckling mode,
$\overline{\xi}_n$ imperfection amplitude corresponding to ξ_n,
$\sigma_n^* = \sigma_n 10^3/E_{x1}$ dimensionless stress of the n-th mode,
σ_m^* $= \min(\sigma_1^*, \sigma_2^*, \sigma_3^*)$,
σ_s^* limit dimensionless stress for imperfect column (load carrying capacity),
ζ_i $= x_i/b_i$,
η_i $= y_i/b_i$.

1. Introduction

Orthotropic materials, including fibrous composites, are more and more frequently used as carrying elements of thin-walled structures. Therefore the designers seek information concerning the behaviour of thin-walled structures built of orthotropic materials under various kinds of loads. Carrying elements are particularly endangered by loads causing a loss of stability. A full analysis of the behaviour of thin-walled structures very often includes the following states: pre-buckling, buckling and post-buckling.

It should be remembered, however, that the buckling occurring in thin-walled structures, especially in flat-walled columns, may have many forms, both qualitatively and quantitatively different from one another – for instance global flexural or flexural-torsional buckling, lateral and local buckling.

In cases when the critical load values for different and separately analyzed buckling modes are close to each other, a so-called coupled (or interactive) buckling may take place.

The load value at which an interactive buckling occurs is always lower than the critical loads for each buckling mode considered separately, that is with no regard to the mutual interaction of those uncoupled buckling modes.

It is known that the concept of interactive buckling results from Koiter's general asymptotic theory of stability [7, 8] which is one of the most effective methods for solving nonlinear stability problems. A practical application of Koiter's theory to the analysis of structure stability was done after its modification by BYSKOV and HUTCHINSON [3] when it was converted into a form more convenient for numerical analysis.

While the interactive buckling of thin-walled structures built of the isotropic material is dealt with by many authors [1, 4, 9, 13, 14, 17–29], not very much attention has been paid to the same process in thin-walled orthotropic structures.

For the first order approximation KOITER and VAN DER NEUT [9] have proposed a technique in which the interaction of an overall mode with two local modes (three-modes approach) having the same wavelength has been considered. The fundamental mode is henceforth called "primary", and the nontrivial higher local mode (having the same wavelength as the "primary") corresponding to the mode triggered by overall longwave mode is called a "secondary" one.

A comprehensive review of literature concerning interactive buckling, published before 1990, has been done by KRÓLAK [16, Chapter 5].

In the present paper, basic equations are derived for the stability of thin-walled beam-columns with rectangular flat walls made of orthotropic material. An assumption is made that the principal axes of orthotropy are parallel to the wall edges. Beam-columns with closed or open cross-sections, simply supported at the ends, are subjected to axial compression, eccentric compression or pure bending.

The problem is solved by variational method using Koiter's asymptotic theory of stability of conservative systems.

The following techniques are employed in the solution and in the computer programme prepared: Byskov's and Hutchinson's asymptotic expansion, and the numerical method of transition matrices [6, 30] using Godunov's orthogonalization [2]. Such an approach enables us to find all global and local buckling modes of the structures analysed; moreover, it includes interaction of several buckling modes, the shear-lag effect and the cross-sectional distortions.

Plots are made illustrating the influence of wall orthotropy factor on the critical state and on the initial post-buckling behaviour of the closed cross-section (square, trapezoid) columns subjected to axial and eccentric compression.

2. Solution of the problem

The considerations concern long prismatic beam-column, of the length l, whose flat walls are treated as thin orthotropic rectangular plates. These rectangular plates, of principal axes of orthotropy parallel to their edges, are connected along their longitudinal edges and form a beam-column.

The cross-section of a structure composed of a few plates is presented in Fig. 1 along with local Cartesian systems of coordinates. Columns of closed or open cross-sections are simply supported at their ends.

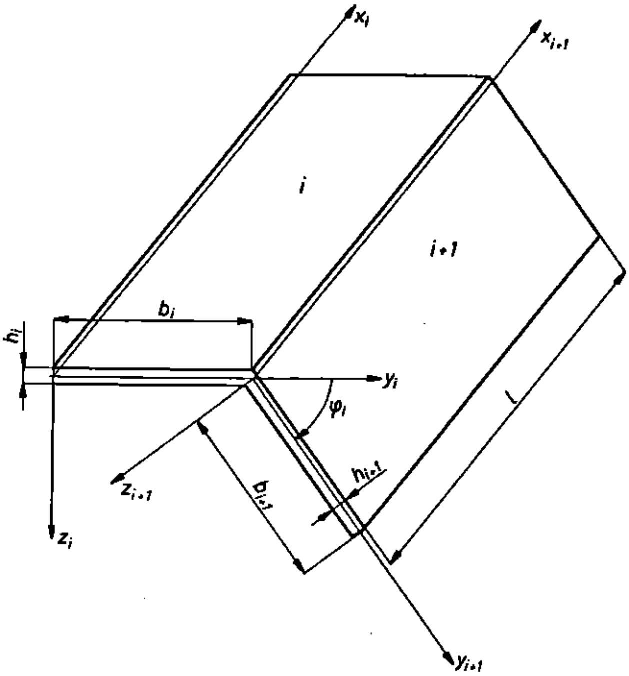

FIG. 1. Prismatic plate structure and the local coordinate system.

The analysis of stability of thin-walled columns is carried out using the plate model. For the i-th wall, exact geometrical relationships are adopted in order to enable the consideration of both out-of-plane and in-plane bending of each wall:

$$
\begin{aligned}
\varepsilon_{xi} &= u_{i,x} + 0.5(u_{i,x}^2 + v_{i,x}^2 + w_{i,x}^2),\\
\varepsilon_{yi} &= v_{i,y} + 0.5(u_{i,y}^2 + v_{i,y}^2 + w_{i,y}^2),\\
\varepsilon_{xyi} &= 0.5(u_{i,y} + v_{i,x} + u_{i,x}u_{i,y} + v_{i,x}v_{i,y} + w_{i,x}w_{i,y}),\\
\kappa_{xi} &= -w_{i,xx}\,, \qquad \kappa_{yi} = -w_{i,yy}\,, \qquad \kappa_{xyi} = -w_{i,xy}\,.
\end{aligned}
\tag{2.1}
$$

Physical relationships for the i-th wall are formulated in the following way:

$$\begin{aligned} \varepsilon_{xi} &= (N_{xi} - \nu_{xyi}N_{yi})/(E_{xi}h_i), \\ \varepsilon_{yi} &= (N_{yi} - \nu_{yxi}N_{xi})/(E_{yi}h_i), \\ \varepsilon_{xyi} &= N_{xyi}/(2G_ih_i). \end{aligned} \tag{2.2}$$

The dependence between Young's moduli and Poisson's ratios in Eqs. (2.2) is as follows:

$$E_{xi}\nu_{yxi} = E_{yi}\nu_{xyi}\,. \tag{2.3}$$

The differential equilibrium equations resulting from the virtual work principle and corresponding to expressions (2.1) for the i-th wall can be written as follows:

$$\begin{aligned} &N_{xi,x}+N_{xyi,y}+(N_{xi}u_{i,x})_{,x}+(N_{yi}u_{i,y})_{,y}+(N_{xyi}u_{i,x})_{,y}+(N_{xyi}u_{i,y})_{,x}=0, \\ &N_{yi,y}+N_{xyi,x}+(N_{xi}v_{i,x})_{,x}+(N_{yi}v_{i,y})_{,y}+(N_{xyi}v_{i,x})_{,y}+(N_{xyi}v_{i,y})_{,x}=0, \\ &D_i\nabla\nabla w_i-(N_{xi}w_{i,x})_{,x}-(N_{yi}w_{i,y})_{,y}-(N_{xyi}w_{i,x})_{,y}-(N_{xyi}w_{i,y})_{,x}=0. \end{aligned} \tag{2.4}$$

The solution of these equations for each plate should satisfy kinematic and static conditions at the junctions of adjacent plates and boundary conditions at the ends $x = 0$ and $x = l$ (see Appendix 1).

The nonlinear problem is solved by the asymptotic KOITER method [7, 8]. Displacement field, $\overline{\mathbf{U}}$, and sectional force field, $\overline{\mathbf{N}}$, are expanded in power series in the buckling mode amplitudes, ξ_n (ξ_n is the amplitude of n-th buckling mode divided by the thickness of the first component wall, h_1):

$$\begin{aligned} \overline{\mathbf{U}} &= \lambda\overline{\mathbf{U}}_i^{(0)} + \xi_n\overline{\mathbf{U}}_i^{(n)} + \ldots, \\ \overline{\mathbf{N}} &= \lambda\overline{\mathbf{N}}_i^{(0)} + \xi_n\overline{\mathbf{N}}_i^{(n)} + \ldots, \end{aligned} \tag{2.5}$$

where the prebuckling fields are $\overline{\mathbf{U}}_i^{(0)}$, $\overline{\mathbf{N}}_i^{(0)}$, the buckling mode fields are $\overline{\mathbf{U}}_i^{(n)}$, $\overline{\mathbf{N}}_i^{(n)}$. The range of indices is $[1, N]$ where N is the number of interacting modes.

By substituting the expansion (2.5) into equations of equilibrium (2.4), junction conditions (A.1.3) and boundary conditions (A.1.5), the boundary value problems of zero and first order can be obtained. The zero approximation describes the pre-buckling state while the first approximation, that is the linear problem of stability, enables us to determine the critical loads of global and local value and the buckling modes. This question can be reduced to a homogeneous system of differential equilibrium equations.

The plates with a linearly varying stresses along their widths are divided into several strips under uniformly distributed compressive (tensile) stresses (Fig. 2). Instead of the finite strips method, the exact transition matrix method is used in this case.

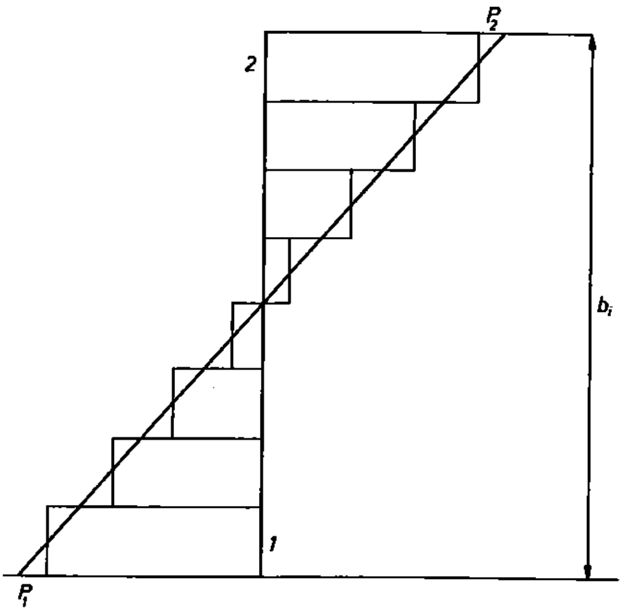

FIG. 2. Discretization of a linear distribution of stresses by means of finite strips.

The pre-buckling solution of an orthotropic wall consisting of homogeneous fields is assumed as:

$$\begin{aligned} u_i^o &= -x_i \Delta_i , \\ v_i^o &= \nu_{xyi} y_i \Delta_i , \\ w_i^o &= 0, \end{aligned} \tag{2.6}$$

where Δ_i is the actual loading. This loading is specified as the product of a unit loading system and a scalar load factor Δ_i.

Numerical aspects of the problem being solved for the first order fields (unlike those of papers [13, 14]), resulted in the introduction of the following orthogonal functions in the sense of boundary conditions for two longitudinal edges (see Appendix 2):

$$\begin{aligned} \bar{a}_i^{(n)} &= N_{yi}^{*(n)} b_i / K_{yi} = (1 + 2\nu_{xyi}\lambda\Delta_i) v_{i,\eta}^{(n)} \\ &\qquad + \nu_{xyi}(1 + \nu_{xyi}\lambda\Delta_i - \lambda\Delta_i) u_{i,\zeta}^{(n)} , \\ \bar{b}_i^{(n)} &= N_{xyi}^{*(n)} b_i / K_{xi} = g_{xi}\Big[(1 - 2\lambda\Delta_i) u_{i,\eta}^{(n)} + (1 + \nu_{xyi}\lambda\Delta_i - \lambda\Delta_i) v_{i,\zeta}^{(n)}\Big], \end{aligned} \tag{2.7}$$

(2.7) [cont.]
$$
\begin{aligned}
\overline{c}_i^{(n)} &= u_i^{(n)}, \\
\overline{d}_i^{(n)} &= v_i^{(n)}, \\
\overline{e}_i^{(n)} &= w_i^{(n)}, \\
\overline{f}_i^{(n)} &= w_{i,\eta}^{(n)}, \\
\overline{g}_i^{(n)} &= -M_{yi}^{(n)} b_i^2/D_{yi} = w_{i,\eta\eta}^{(n)} + \nu_{xyi} w_{i,\zeta\zeta}^{(n)}, \\
\overline{h}_i^{(n)} &= -Q_{yi}^{*(n)} b_i^3/D_{yi} = w_{i,\eta\eta\eta}^{(n)} + (\nu_{xyi} + 4g_{yi}) w_{i,\zeta\zeta\eta}^{(n)},
\end{aligned}
$$

where $(..)_{i,\zeta} = \partial(..)/\partial\zeta_i$, $(..)_{i,\eta} = \partial(..)/\partial\eta_i$.

The differential equilibrium equations for the first order approximation can be written as follows:

(2.8)
$$
\begin{aligned}
\overline{a}_{i,\eta}^{(n)} &= -\left[g_{yi} - (E_{xi}/E_{yi} - \nu_{xyi}^2 - 2\nu_{xyi}g_{yi})\lambda\Delta_i\right] \overline{d}_{i,\zeta\zeta}^{(n)} \\
&\qquad - g_{yi}\left[1 - (1-\nu_{xyi})\lambda\Delta_i\right] \overline{c}_{i,\zeta\eta}^{(n)}, \\
\overline{b}_{i,\eta}^{(n)} &= -\left[1 - (3 - \nu_{xyi}\nu_{yxi})\lambda\Delta_i\right] \overline{c}_{i,\zeta\zeta}^{(n)} - \nu_{yxi}\left[1 - (1-\nu_{xyi})\lambda\Delta_i\right] \overline{d}_{i,\zeta\eta}^{(n)}, \\
\overline{c}_{i,\eta}^{(n)} &= \left[\overline{b}_i^{(n)}/g_{xi} - (1 + \nu_{xyi}\lambda\Delta_i - \lambda\Delta_i)\overline{d}_{i,\zeta}^{(n)}\right] / (1 - 2\lambda\Delta_i), \\
\overline{d}_{i,\eta}^{(n)} &= \left[\overline{a}_i^{(n)} - \nu_{xyi}(1 + \nu_{xyi}\lambda\Delta_i - \lambda\Delta_i)\overline{c}_{i,\zeta}^{(n)}\right] / (1 + 2\nu_{xyi}\lambda\Delta_i), \\
\overline{e}_{i,\eta}^{(n)} &= \overline{f}_i^{(n)}, \\
\overline{f}_{i,\eta}^{(n)} &= \overline{g}_i^{(n)} - \nu_{xyi}\overline{e}_{i,\zeta\zeta}^{(n)}, \\
\overline{g}_{i,\eta}^{(n)} &= \overline{h}_i^{(n)} - 4g_{yi}\overline{f}_{i,\zeta\zeta}^{(n)}, \\
\overline{h}_{i,\eta}^{(n)} &= -E_{xi}/E_{yi}\left[\nu_{yxi}\overline{f}_{i,\zeta\zeta\eta}^{(n)} + \overline{e}_{i,\zeta\zeta\zeta\zeta}^{(n)}\right. \\
&\qquad \left. + 12(1 - \nu_{xyi}\nu_{yxi})(b_i/h_i)\lambda\Delta_i\overline{e}_{i,\zeta\zeta}^{(n)}\right].
\end{aligned}
$$

The first order solutions may be formulated as follows:

(2.9)
$$
\begin{aligned}
\overline{a}_i^{(n)} &= \overline{A}_i^{(n)}(\eta)\sin\frac{m\pi\zeta b_i}{l}, & \overline{b}_i^{(n)} &= \overline{B}_i^{(n)}(\eta)\cos\frac{m\pi\zeta b_i}{l}, \\
\overline{c}_i^{(n)} &= \overline{C}_i^{(n)}(\eta)\cos\frac{m\pi\zeta b_i}{l}, & \overline{d}_i^{(n)} &= \overline{D}_i^{(n)}(\eta)\sin\frac{m\pi\zeta b_i}{l}, \\
\overline{e}_i^{(n)} &= \overline{E}_i^{(n)}(\eta)\sin\frac{m\pi\zeta b_i}{l}, & \overline{f}_i^{(n)} &= \overline{F}_i^{(n)}(\eta)\sin\frac{m\pi\zeta b_i}{l}, \\
\overline{g}_i^{(n)} &= \overline{G}_i^{(n)}(\eta)\sin\frac{m\pi\zeta b_i}{l}, & \overline{h}_i^{(n)} &= \overline{H}_i^{(n)}(\eta)\sin\frac{m\pi\zeta b_i}{l}.
\end{aligned}
$$

Initially the unknown functions $\overline{A}_i^{(n)}$, $\overline{B}_i^{(n)}$, $\overline{C}_i^{(n)}$, $\overline{D}_i^{(n)}$, $\overline{E}_i^{(n)}$, $\overline{F}_i^{(n)}$, $\overline{G}_i^{(n)}$, $\overline{H}_i^{(n)}$ (with the m-th harmonic) will be defined by the numerical method of transition matrix. The system of ordinary differential equations of the first order with appropriate junction conditions for the adjacent plates is solved by the transition matrices method, using numerical integration of the equilibrium equations in transverse direction, in order to obtain a relation between the state vectors on two longitudinal edges by means of the Godunov orthogonalization method [2].

The omission of the displacements of the fundamental state implies that the difference between the configuration of the undeformed state and the fundamental state are neglected; consequently, the displacements defined earlier, u_i^0 and v_i^0, can be considered as additional ones from the fundamental state to the adjacent state.

The assumed fields for the first order of nonlinear approximation ensure compatibility of the corner displacements of the constituent plates. Thus $v = 0$ at the ends, implying that the plates are restrained in their plane at the ends (for a more detailed analysis see [25]).

The global buckling mode occurs at $m = 1$ and the local modes at $m > 1$ (with $b_i \ll l$). Each buckling mode is normalized so that the maximum normal displacement is equal to the thickness of the first constituent wall, h_1.

At the point where load parameter, λ reaches its maximum value, λ_s for the imperfect structure (limit load parameter), the Jacobian of nonlinear system of equations [3]:

$$\left(1 - \frac{\lambda}{\lambda_J}\right)\xi_J + a_{ijJ}\xi_i\xi_j + \ldots = \frac{\lambda}{\lambda_J}\bar{\xi}_J \qquad \text{at} \quad J = 1, 2, \ldots N \tag{2.10}$$

is equal to zero.

Expression for a_{ijJ} is given in Appendix 3. The formulae for the post-buckling coefficients a_{ijJ} involve only the buckling modes. The result of integration along x indicates that post-buckling coefficients a_{ijJ} are zero when the sum of the wave numbers associated with the three modes $(m_i+m_j+m_J)$ is even.

Since for the first order approximation the limit load is always lower than the minimum value of a bifurcational load obtained in the linear analysis, this approximation can be used as a lower bound estimation of a load carrying capacity.

3. Capabilities of the prepared computer programme

On the basis of the formulae and equations obtained, a computer programme is prepared. Due to a suitable formulation of equilibrium equations, stress-strain continuity conditions and boundary conditions, it has a very broad range of application. The programme enables a division of each girder wall into a few plate strips of different material properties, thicknesses, widths and external loads (compressive or tensile) applied at the ends of these strips (Fig. 2).

The program enables the analysis of the stability (critical state) and the initial post-buckling equilibrium paths of thin-walled beam-columns with different cross-section shapes (closed and open profiles) subjected to axial compression, eccentric compression or pure bending. In circumferential direction of the column, the division of the walls into plate strips enables abrupt changes (at the strip boundary) of material properties, wall thickness and beam-column loads. Moreover, the programme allows to calculate columns with longitudinal stiffeners (especially profiled walls – intermediate stiffeners) and with initial deflections of column walls and axes, like in paper [15] where isotropic columns were discussed. The correctness of the solution obtained and of the programme worked out was tested by a comparison of the calculated results with those obtained by other authors, e.g. CHANDRA and RAJU [5] and with those obtained for isotropic beam-column [10–12, 15]. The programme can be easily applied in a computer-aided system, CAD/CAM.

4. Calculation results

Detailed numerical calculations are carried out for a thin-walled square-section column subjected to uniform and eccentric compression and for a thin-walled column of isosceles trapezoidal section subjected to uniform compression. Eccentric compression in a square column is accomplished by means of a triangular load (stress) distribution at two opposite column walls (Fig. 3).

The following column dimensions are assumed:

- square-section column:

$$b_1 = b_2 = b_3 = 100 \text{ [mm]}, \qquad h_1 = h_2 = h_3 = 1 \text{ [mm]}, \qquad l = 2750 \text{ [mm]},$$

- trapezoidal-section column:

$$b_1 = 50 \text{ [mm]}, \qquad b_2 = 95.476 \text{ [mm]}, \qquad b_3 = 100 \text{ [mm]},$$

$$h_1 = 0.5 \text{ [mm]}, \qquad h_2 = 1.075 \text{ [mm]}, \qquad h_3 = 1.7 \text{ [mm]}, \qquad l = 2750 \text{ [mm]}.$$

It should be noted that geometrical dimensions of the cross-section were selected so that the second moment of area of the section relative to the central axes of inertia could be identical [10, 12].

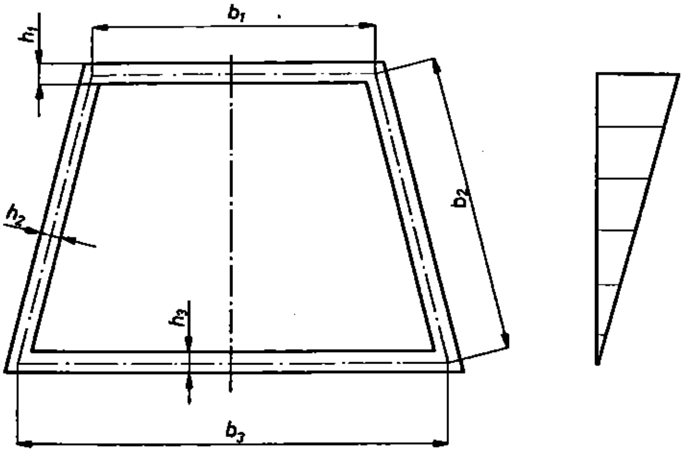

FIG. 3. Type of closed cross-section considered.

The main purpose of calculations is to analyse the influence of the orthotropy factor, $\beta_i = E_{xi}/E_{yi}$ of column walls upon the global and local critical load values and upon the interactive (coupled) buckling. Calculations are carried out for columns where all walls are made of the same orthotropic or isotropic material. Material constants for the orthotropic walls are taken from paper [5]. Calculations are made for columns with 13 different material constants as listed in Table 1.

Table 1. Elastic constants for various cases of composite beam-columns.

Spec. no.	$\beta = E_x/E_y$	ν_{xy}	ν_{yx}	G/E_x
1	0.0728	0.02184	0.3	0.4065
2	0.1315	0.03945	0.3	0.4091
3	0.3031	0.09093	0.3	0.4002
4	0.5064	0.15192	0.3	0.3937
5	0.7041	0.21123	0.3	0.4009
6	0.8358	0.25074	0.3	0.3882
7	1.0000	0.3	0.3	0.3846
8	1.1964	0.3	0.25074	0.3245
9	1.4202	0.3	0.21123	0.2823
10	1.9747	0.3	0.15192	0.1994
11	3.2992	0.3	0.09093	0.1213
12	7.6045	0.3	0.03945	0.0538
13	13.7362	0.3	0.02184	0.0296

Results are presented in a graphic form. Figure 4 shows plots of the global (flexural) critical stress and the lowest local critical stresses of the analysed columns as functions of the wall orthotropy factor, $\beta_i = \beta = E_{ix}/E_{iy} = E_x/E_y$.

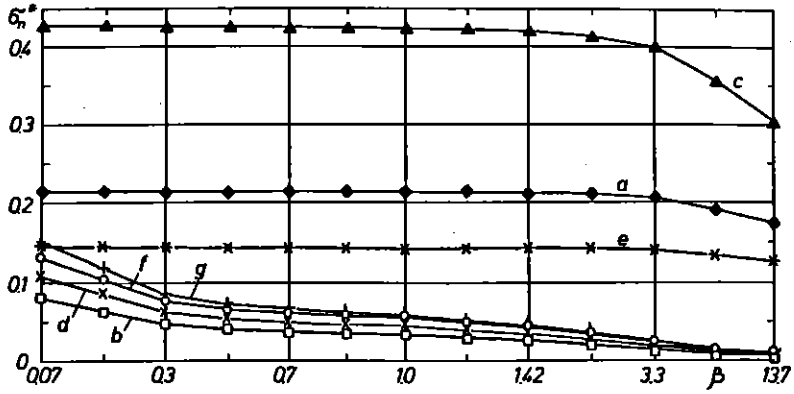

FIG. 4. Dimensionless stress σ_n^* carried by the orthotropy factor β. Curves: a – global buckling for uniformly compressed square column, b – local buckling for uniformly compressed square column, c – global buckling for eccentrically compressed square beam-column, d – local buckling for eccentrically compressed square beam-column, e – global buckling for uniformly compressed trapezoid column, f – local buckling for uniformly compressed trapezoid column at the first minimum, g – local buckling for uniformly compressed trapezoid column at the second minimum.

The plots indicate that for $\beta \leq 1$, the overall critical stresses are practically independent of the wall orthotropy factor, or, to be more accurate, of the moduli E_y when dimensionless critical loads measured along the Y-axes, $\sigma_n^* = \sigma_n 10^3/E_x$, are dependent on $E_x(= E_{xi})$.

If the wall orthotropy factor β varies within the limits $1 \leq \beta \leq 13.7362$, the dimensionless global critical stresses decrease, E_x being constant, by the maximum of:

19% – the square column under axial compression;

13.4% – the trapezoidal column under axial compression;

29% – the square column under eccentric compression.

It should be remembered, however, that dimensional critical stresses change proportionally to the moduli E_x, because $\sigma_n = \sigma_n^* E_x/1000$.

The adoption of the plate model for the column allowed us to take into account the effect of factor β (moduli E_y) on the critical values of global buckling stresses which would not be possible if a bar model was applied. Numerical calculations proved that in the column under analysis, the wall orthotropy factor β influences the lowest critical stresses of local buckling much more significantly than those of global buckling. In the considered range of β factor values, the dimensionless local critical stresses in columns with orthotropic walls, as compared to those with isotropic walls, are changing in the following way:

at $\beta = 0.0728$ they increased by approx. $2.2 - 2.3$ times;

at $\beta = 13.7362$ they decreased by approx. $5.9 - 6.2$ times.

In case of a trapezoid-section column, two minimum values are found of the local buckling critical stresses; one of those is an absolute minimum while the other is a local one.

These minimum values differ in the critical stress values and in the number of half-waves, m, hence also in the eigenvalue modes. Therefore, in a compressed trapezoid-shaped column made of an isotropic material ($\beta = 1$), the absolute minimum occurs at $m = 33$ while the local one at $m = 79$. The critical stress for the second minimum of local buckling mode is by about 16% higher than the critical stress related to the absolute minimum. The number of half-waves being formed over the length of the column analysed, changes along with the factor β. The change in the number of half-waves for the first and the second minimum of local buckling modes of the trapezoidal column under compression is presented in Fig. 5.

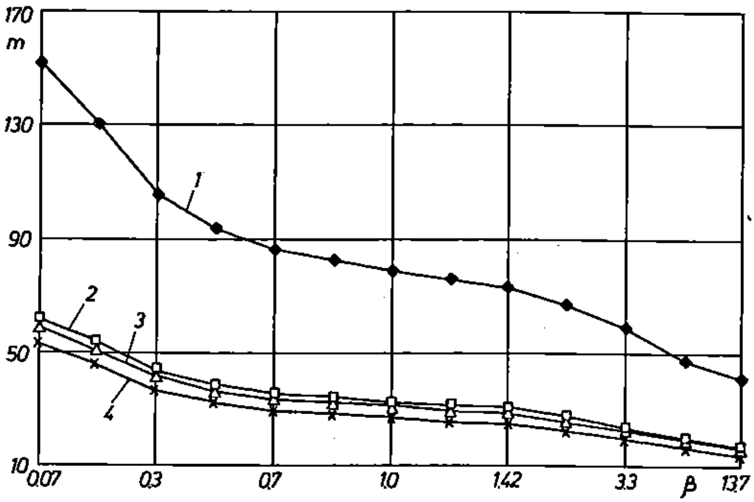

FIG. 5. The number of half-waves m carried by the orthotropy factor β. Curves: *1* – local buckling for uniformly compressed square column, *2* – local buckling for eccentrically compressed square beam-column, *3* – local buckling for uniformly compressed trapezoid column at the first minimum, *4* – local buckling for uniformly compressed trapezoid column at the second minimum.

The plots in Fig. 5 show that the number of half-waves, m, in a column of $\beta = 13.7362$ decreases nearly twice as much in relation to the column made of the isotropic material; for $\beta = 0.0728$ it increases almost by a factor of two. The ratio of half-waves numbers of local bucklings referring to the second minimum (greater number of half-waves) and the first one (smaller number of m) buckling modes is about 2.4, and practically does not depend on β in the analysed range of variation.

Figure 6a presents the curves of wall deflection in a section running through the middle of a half-wave of the local buckling mode corresponding to the first minimum; Figure 6b illustrates the same at the second minimum relating to three trapezoid-section column, their β factor being 0.0728, 1.0 and 13.7362.

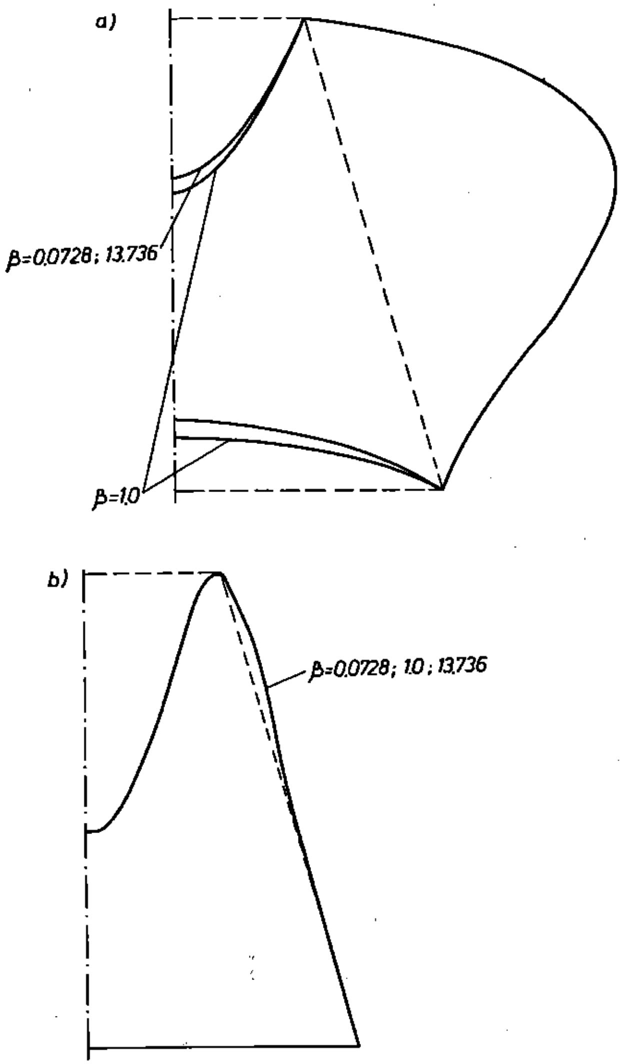

FIG. 6. Local buckling modes for the trapezoid column corresponding to the first (a) and the second (b) minimum.

Figures 6 illustrate that local buckling modes, being completely different for the first and the second minimum, depend only slightly on β (except the half-wave length in the direction of compression).

Figures 7 present plots of the bending moments, $M_{iy}^{(n)}$, also for the first and the second minimum of the trapezoid column; the section is identical to that in Figs. 6.

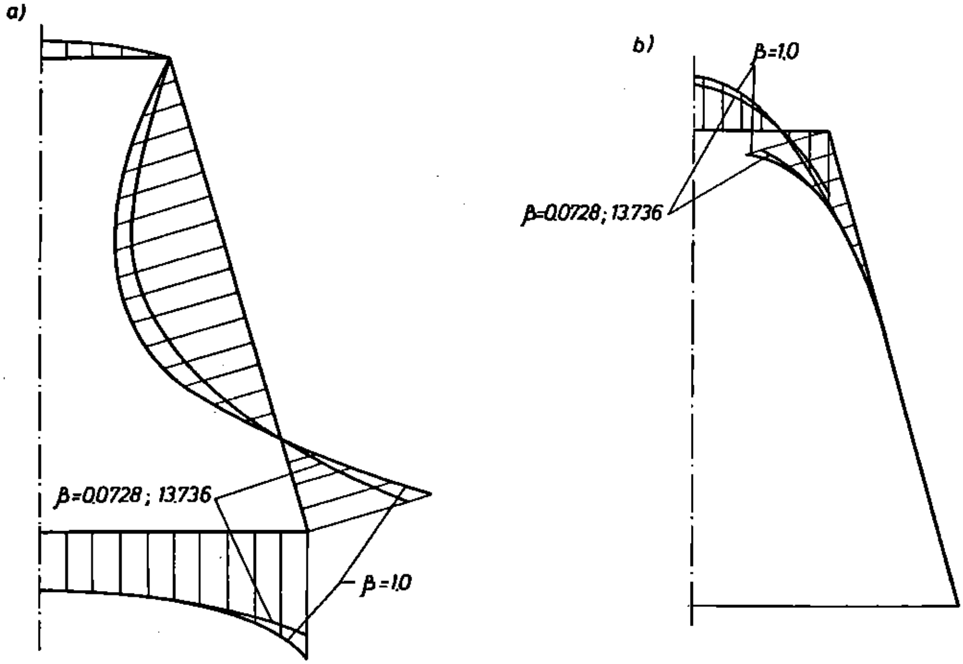

FIG. 7. Bending moments $M_{iy}^{(n)}$ for the first (a) and the second minimum (b) of the trapezoid column.

An analysis of plots representing deflections and bending moments, $M_{iy}^{(n)}$, leads to a conclusion that in the trapezoidal column under analysis the weakest wall, regarding its local stability, is the narrowest wall of the column (in our calculations, wall No. 1). As h_1/b_1 for the first wall has the lowest value 0.01, such a result could be predicted. In the present paper, unlike in papers [10–14], the influence of the number of half-waves with given imperfections, on the load carrying capacity is neglected.

One of the more important aims of this paper is to determine the load carrying capacity, $\sigma_s^* = \sigma_s 1000/E_x = \lambda_s \Delta_i 1000$ of columns calculated for different values of the wall orthotropy factor β. Figure 8 shows plots illustrating the ratio of the limit load σ_s^* to the minimum critical load σ_m^* as a function of β, the imperfections being $|\overline{\xi}_1| = 1.0$, $|\overline{\xi}_2| = 0.2$, $\overline{\xi}_3 = 0.0$.

In each case the sign of the imperfections has been chosen in the most unfavourable way, i.e. so that σ_s^* would have its minimum value (see [10–12] for a more detailed discussion).

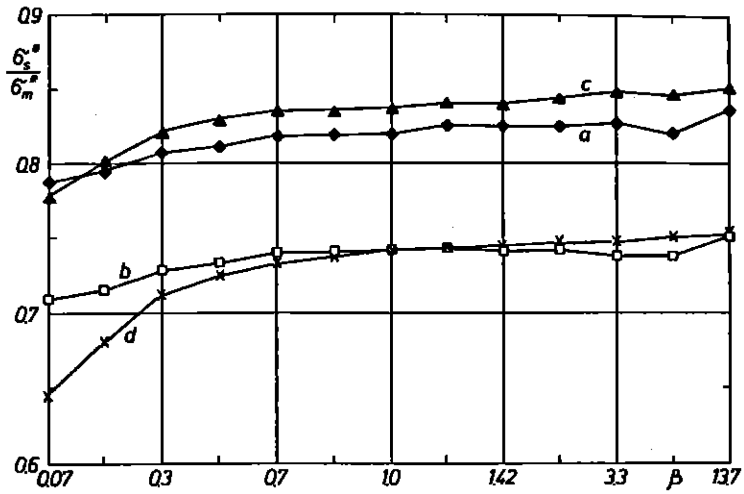

FIG. 8. Load carrying capacity σ_s^*/σ_m^* carried by orthotropy factor β. Curves: a – uniformly compressed square column, b – eccentrically compressed square beam-column, c – uniformly compressed trapezoid column at the first minimum, d – uniformly compressed trapezoid column at the second minimum.

An interaction of the global buckling mode with two local modes, primary and secondary, having the same wavelength are considered.

As can be seen in Fig. 8, the orthotropy factor β has an insignificant effect upon the ratio of the load carrying capacity to the minimum critical stress, σ_s^*/σ_m^*.

In case of eccentric compression of the square-section column, the value of the dimensionless local critical load (Fig. 4) is higher than under uniform compression, due to a greater stability of disk-bent webs and unloaded bottom flange (Fig. 3). Along with the growing eccentricity, the ratio σ_s^*/σ_m^* (compare curves a and b in Fig. 8) decreases; therefore the imperfection sensitivity increases together with the eccentricity of compressive load.

In case of the trapezoid-section columns, the interaction is considered between global buckling mode and two local modes, having the same numbers of half-waves, referring to the two different minimum values. It turned out that the interaction of global mode with two local modes referring to the second local minimum causes a greater relative decrease in the limit load, σ_s^*/σ_m^*, than does the interaction of the global mode with the local ones corresponding to the first minimum.

The dimensionless limit load σ_s^* for the interaction between global buckling and local modes referring to the second minimum is by 2 – 4% higher at $\beta = 0.0728 - 3.2992$ and by less than 3% lower at $\beta = 7.6045 - 13.736$

than σ_s^* corresponding to the interaction between global buckling and local modes referring to the first minimum. Attention should be paid to the proper selection of local buckling modes. This can be accomplished only by means of nonlinear analysis.

5. Conclusions

The paper presents a solution to the stability problem of thin-walled beam-columns built of orthotropic rectangular plates. A computer programme with a broad range of application has been worked out and tested. Detailed calculations are made for thin-walled columns with closed cross-sections (square, trapezoid) subjected to axial and eccentric compression. Analysis is carried out of how the wall orthotropy factor, $\beta = E_x/E_y$ influences the following:

- global and local critical stress values;
- limit load values for columns with imperfections.

Global and local buckling modes are determined for the columns under analysis. The plate model of the column adopted in the analysis enabled us to study the influence of β factor upon the global (flexural and flexural-torsional) stability of the column. An interaction is analysed between global and two local buckling modes, having the same numbers of half-waves referring to the two lowest characteristic values.

The applied method describing the buckling of thin-walled structures from the global to the local loss of stability can be easily adopted in the computer-aided system, CAD/CAM.

The present analysis has to be completed by including the second approximation, in order to investigate post-buckling in case when the first order interaction is weak.

Acknowledgments

The paper has been carried out in the range of research project supported by National Research Committee (KBN No. PB 0923/P4/93/04).

APPENDIX 1

The sectional internal forces appearing in the paper by following relationships:

$$
\begin{aligned}
N_{xi} &= K_{xi}\varepsilon_{xi} + K_{xyi}\varepsilon_{yi}\,,\\
N_{yi} &= K_{xyi}\varepsilon_{xi} + K_{yi}\varepsilon_{yi}\,,\\
N_{xyi} &= 2K_{si}\varepsilon_{xyi}\,,\\
M_{xi} &= D_{xi}\kappa_{xi} + D_{xyi}\kappa_{yi}\,,\\
M_{yi} &= D_{xyi}\kappa_{xi} + D_{yi}\kappa_{yi}\,,\\
M_{xyi} &= 2D_{si}\kappa_{xyi}\,,
\end{aligned}
\tag{A.1.1}
$$

where

$$
\begin{aligned}
K_{xi} &= E_{xi}h_i/(1-\nu_{xyi}\nu_{yxi}),\\
K_{yi} &= E_{yi}h_i/(1-\nu_{xyi}\nu_{yxi}),\\
K_{xyi} &= \nu_{yxi}K_{xi} = \nu_{xyi}K_{yi}\,,\\
K_{si} &= G_i h_i\,,\\
D_{xi} &= E_{xi}h_i^3/(12(1-\nu_{xyi}\nu_{yxi})),\\
D_{yi} &= E_{yi}h_i^3/(12(1-\nu_{xyi}\nu_{yxi})),\\
D_{xyi} &= \nu_{yxi}D_{xi} = \nu_{xyi}D_{yi}\,,\\
D_{si} &= G_i h_i^3/12.
\end{aligned}
\tag{A.1.2}
$$

The kinematical and statical continuity conditions at the junctions of adjacent plates may be written in the form:

$$
\begin{aligned}
u_{i+1}|^0 &= u_i|^+,\\
w_{i+1}|^0 &= w_i|^+\cos(\varphi_i) - v_i|^+\sin(\varphi_i),\\
v_{i+1}|^0 &= w_i|^+\sin(\varphi_i) + v_i|^+\cos(\varphi_i),\\
w_{i+1,y}|^0 &= w_{i,y}|^+,\\
M_{y(i+1)}|^0 - M_{yi}|^+ &= 0,\\
N^*_{y(i+1)}|^0 - N^*_{yi}|^+\cos(\varphi_i) - Q^*_{yi}|^+\sin(\varphi_i) &= 0,\\
Q^*_{y(i+1)}|^0 + N^*_{yi}|^+\sin(\varphi_i) - Q^*_{yi}|^+\cos(\varphi_i) &= 0,\\
N^*_{xy(i+1)} - N^*_{xyi}|^+ &= 0,
\end{aligned}
\tag{A.1.3}
$$

where

$$
\begin{aligned}
N^*_{yi} &= N_{yi} + N_{yi}v_{i,y} + N_{xyi}v_{i,x}\,,\\
Q^*_{yi} &= N_{yi}w_{i,y} + N_{xyi}w_{i,x} - D_{yi}[w_{i,yyy} + (\nu_{xyi} + 4g_{yi})w_{i,xxy}],\\
N^*_{xyi} &= N_{xyi} + N_{xyi}u_{i,x} + N_{yi}u_{i,y}\,,\\
g_{xi} &= K_{si}/K_{xi}\,, \qquad g_{yi} = K_{si}/K_{yi}\,.
\end{aligned}
\tag{A.1.4}
$$

The boundary conditions referring to the simple supports of the beam-columns at both ends are assumed to be:

$$\text{(A.1.5)}\qquad \begin{gathered} \sum_i \frac{1}{b_i} \int N_{xi}(x_i = 0,\, y_i)\, dy_i = \sum_i \frac{1}{b_i} \int N_{xi}(x_i = l,\, y_i)\, dy_i = \sum_i N^0_{xi}\,, \\ v_i(x_i = 0,\, y_i) = v_i(x_i = l,\, y_i) = 0, \\ w_i(x_i = 0,\, y_i) = w_i(x_i = l,\, y_i) = 0, \\ M_{yi}(x_i = 0,\, y_i) = M_{yi}(x_i = l,\, y_i) = 0. \end{gathered}$$

Appendix 2

The conditions resulting from the variational principle for two longitudinal edges on which a relation between the state vectors is derived using the modified transition matrices method, may be written in the form

$$\text{(A.2.1)}\qquad \begin{gathered} \int_0^l N^*_{iy}\delta v\, dx_i = 0, \qquad \int_0^l N^*_{ixy}\delta u\, dx_i = 0, \\ \int_0^l M_{iy}\delta w_{i,y}\, dx_i = 0, \qquad \int_0^l Q^*_{iy}\delta w\, dx_i = 0. \end{gathered}$$

Appendix 3

The coefficients in the nonlinear equilibrium equations (2.10) a_{ijJ} are given by the following expressions (see Byskov and Hutchinson [3] for a more detailed analysis)

$$\text{(A.3.1)}\qquad a_{ijJ} = \frac{\left[\sigma^{(J)}\cdot l_{11}(U^{(i)}, U^{(j)}) + 2\sigma^{(i)}\cdot l_{11}(U^{(j)}, U^{(J)})\right]}{2\sigma^{(J)}\cdot\varepsilon^{(J)})}\,.$$

References

1. M.A. Ali and S. Sridharan, *A versatile model for interactive buckling of columns and beam-columns*, Int. J. Solids Struct., **24**, 5, pp. 481–486, 1988.

2. B.L. Biderman, *Mechanics of thin-walled structures. Statics* [in Russian], Mashinostroenie, pp. 488, Moscow 1977.

3. E. Byskov and J.W. Hutchinson, *Mode interaction in axially stiffened cylindrical shells*, AIAA J., **15**, 7, pp. 941–948, 1977.

4. E. BYSKOV, *An asymptotic expansion applied to van der Neut's column. Collapse: The buckling of structures in theory and practice,* J.M.T. THOMPSON and G.W. HUNT [Ed.], Cambridge University Press, pp. 269–282, Cambridge 1983.

5. R. CHANDRA, and B.B. RAJU, *Postbuckling analysis of rectangular orthotropic plates,* Int. J. Mech. Sci., **16**, pp. 81–97, 1973.

6. K. KLÖPPEL and W. BILSTEIN, *Ein Verfahren zur Ermittlung der Beullasten beliebiger rechtwinkling abgekanteter offener und geschlossener Profile nach der linearen Beultheorie unter Verwendung eines abgewandelten Reduktionsverfahrens,* Veröffentlichungen des Institutes für Statik und Stahlbau der Technischen Hochschule, Darmstadt, Heft 16, 1971.

7. W.T. KOITER, *Elastic stability and post-buckling behaviour,* [in:] Proceedings of the Symposium on Nonlinear Problems, Univ. of Wisconsin Press, Wisconsin, 1963, pp. 257–275.

8. W.T. KOITER, *General theory of mode interaction in stiffened plate and shell structures,* WTHD Report 590, Delft, p. 41, 1976.

9. W.T. KOITER and A. VAN DER NEUT, *Interaction between local and overall buckling of stiffened compression panels,* [in:] Thin-Walled Structures, J. RHODES and A.G. WALKER [Eds.], Granada, St.Albans, part I, pp. 51–56, part II, pp. 66–86, 1980.

10. Z. KOŁAKOWSKI, *Mode interaction in thin-walled trapezoidal column under uniform compression,* Thin-Walled Structures, **5**, pp. 329–342, 1987.

11. Z. KOŁAKOWSKI, *Interactive buckling of thin-walled beams with open and closed cross-section,* Engng. Trans., **37**, 2, pp. 375–397, 1989.

12. Z. KOŁAKOWSKI, *Some thougths on mode interaction in thin-walled columns under uniform compression,* Thin-Walled Structures, **7**, pp. 23–35, 1989.

13. Z. KOŁAKOWSKI, *Influence of modification of boundary conditions on load carrying capacity in thin-walled columns in the second order approximation,* Int. J. Solids Struct., **30**, 19, pp. 2597–2609, 1993.

14. Z. KOŁAKOWSKI, *Interactive buckling of thin-walled beams with open and closed cross-sections,* Thin-Walled Struct., **15**, pp. 159–183, 1993.

15. Z. KOŁAKOWSKI and A. TETER, *Interactive buckling of thin-walled closed elastic beam-columns with intermediate stiffeners,* Int. J. Solids Struct., **32**, 11, 1501–1516, 1995.

16. M. KRÓLAK [Ed.], *Post-buckling behaviour and load carrying capacity of thin-walled plate girders* [in Polish], PWN, Warszawa-Łódź, 1990, pp. 553.

17. A.I. MANEVICH, *Interaction of buckling modes of stiffened plate under compression* [in Russian], Stroitel'naya Mekhanika i Raschet Sooruzhenii, 5, pp. 24–29, 1981.

18. A.I. MANEVICH, *Theory of interactive buckling of stiffened thin-walled structures* [in Russian], Prikl'adnaya Matematika i Mekhanika, **46**, 2, pp. 337–345, 1982.

19. A.I. MANEVICH, *Stability of shells and plates with T-section stiffeners* [in Russian], Stroitel'naya Mekhanika i Raschet Sooruzhenii, 2, pp. 34–38, 1985.

20. A.I. MANEVICH, *Interactive buckling of stiffened plate under compression* [in Russian], Mekhanika Tverdogo Tela, 5, pp. 152–159, 1988.

21. H. MOELLMANN and P. GOLTERMANN, *Interactive buckling in thin-walled beams. Part 1. Theory; Part 2. Applications,* Int. J. Solids Struct., **25**, 7, pp. 715–728 and pp. 729–749, 1989.

22. M. PIGNATARO and A. LUONGO, *Asymmetric interactive buckling of thin-walled columns with initial imperfection*, Thin-Walled Struct., **3**, pp. 365–386, 1987.

23. M. PIGNATARO and A. LUONGO, *Multiple interactive buckling of thin-walled members in compression*, Proc. Int. Colloq. on Stability of Plate and Shell Structures, Ghent, University Ghent, pp. 235–240, 1987.

24. J. ROORDA, *Buckling behaviour of thin-walled columns*, Can. J. Civ. Engng., **15**, pp. 107–116, 1988.

25. S. SRIDHARAN and T.R. GRAVES-SMITH, *Postbuckling analyses with finite strips*, J. Engng. Mech. Div. ASCE., **107**, EM5, pp. 869–888, 1981.

26. S. SRIDHARAN and M.A. ALI, *Interactive buckling in thin-walled beam-columns*, J. Engng. Mech. ASCE, **111**, 12, pp. 1470–1486, 1985.

27. S. SRIDHARAN and M.A. ALI, *An improved interactive buckling analysis of thin-walled columns having doubly symmetric sections*, Int. J. Solids Struct., **22**, 4, pp. 429–443, 1986.

28. S. SRIDHARAN and M.H. PENG, *Performance of axially compressed stiffened panels*, Int. J. Solids and Struct., **25**, 8, pp. 879–899, 1989.

29. V. TVERGAARD, *Imperfections sensitivity of a wide integrally stiffened panel under compression*, Int. J. Solids and Struct., **9**, 1, pp. 177–192, 1973.

30. B. UNGER, *Elastisches Kippen von beliebig gelagerten und aufgehängten Durchlaufträgern mit einfachsymmetrischen, in Trägerachse veränderlichem Querschnitt und einer Abwandlung des Reduktionsverfahrens als Lösungsmethode*, Dissertation D17, Darmstadt 1969.

DEPARTMENT OF STRENGTH OF MATERIALS AND STRUCTURES
ŁÓDŹ UNIVERSITY OF TECHNOLOGY, ŁÓDŹ.

Received April 11, 1995.