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# OPTIMIZATION OF THE STRUCTURE OF A MULTILAYER CYLINDRICAL SHELL UNDER STABILITY LOSS CONDITIONS

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**The computer program presented here has been devised for minimizing the thickness of a thin cylindrical shell composed of linearly elastic, macro-homogeneous, orthotropic layers, resting on hinged supports and threatened with stability loss under the action of static compressive forces directed along the axis and uniformly distributed along the curvilinear edges of the shell. The optimization process is based on the method for determining the critical loads ($T_{\mathrm{cr}}$), presented in [1] and on the kinematic broken line theory, and the static distribution theory of lateral shear stresses.**

## 1. INTRODUCTION

The recent years were characterized by a growing intensity of development of research works devoted to the design of safe optimum structures. This requirement, which is a consequence of the growing needs and the fact that there exist methods for producing new materials, give rise to a natural desire to make use of the properties of the latter. The number of works devoted to problems of optimum design of a structure is now more than a few thousand. They are discussed in detail in many monographs, textbooks and surveys [2 – 5, ...]. Among other fundamental structures in the leading domains of technology there are multilayer thin-walled shells [6 – 26].

It is a well known fact that elements made of composites are more complicated than those made of traditional materials, and the methods required for forecasting their behaviour under definite types of load are more sophisticated [6, 7]. An essential feature of composite materials which is decisive for the growth of their popularity is the possibility of selection of optimum composition, structure and form of the object, to obtain the required physical properties, satisfying the loading conditions of the structure.

The subject of investigation in the theory of optimum design are situations in which trends or requirements mutually exclusive or opposite have to be reconciled. The type of application, the working conditions and the production methods of the object considered are represented in its mathematical model. There are several directions in which contemporary research works are conducted as regards the shaping criteria or objective functions or, finally, the solution methods.

The classical way of seeking for optimum solution is based on solutions obtained by the methods of variational calculus and makes it possible, in the case of composite materials, to solve problems of determining appropriate reinforcement of the structures to satisfy the criterion assumed as requirements for uniform strength, or maximum load, value of the natural vibration frequency, uniform reduced stress distribution or coincidence of the directions of the reinforecement with those of the principal stresses. Those criteria do not always lead to optimum solutions, especially in problems in which the geometrical parameters are fixed. The most important group of criteria now being used are those of minimum volume of the structure, minimum weight of the material and minimum cost, geometrical parameters being used as decision variables. The limiting values determining the load carrying capacity depend, among other factors, on the geometrical parameters and the internal structure (the mechanical properties of the materials, the direction angles and the relative contents of the layers). It follows that, to formulate a problem of optimum design we must include into the set of decision variables not only the geometrical parameters, but also those describing the internal structure. This increases, however, the problem to the extent at which no closed-form solution is possible. In addition to the analytical optimization methods in which a few variables can be used at most, numerical methods (such as those of finite elements, mathematical programming, multi-criterial methods etc.) are now of growing popularity, and are particularly useful for greater systems, or more accurate models of behaviour of the structure.

The multitude of practical problems to be solved is a strong stimulus for the development of various minimization methods. The existing commercial software packages offered by computation centres of large aircraft or automobile factories are expensive and require high quality hardware. Their adaptation for particular needs being not always possible, an idea has been conceived to prepare a program for minimizing the thickness of a multilayer shell, which would require not too large a store and not too long a computation time, and could be used to evaluate the critical loads according to [1].

## 2. FORMULATION OF THE PROBLEM

The subject of the present considerations is a thin circular shell constructed of $N$ homogeneous layers of linearly elastic, orthotropic composite materials (Fig. 1). Our task is to find its optimum initial structure and the load $T^{\text{load}}$ to be carried. Some aspects of the method of determining the critical load $T_{\text{cr}}$ were discussed in (among other works) [1], and in the doctoral thesis of the present author. It is only the result concerning the optimization of the internal structure of the shell that will be presented below.

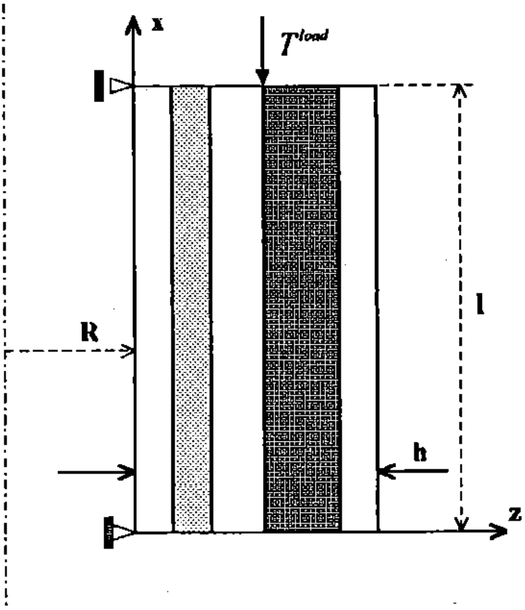


FIG. 1.

The decision variables $\mathbf{x} = (h_1, \varphi_1, \ldots, h_N, \varphi_N)$ are thicknesses $h_i$ and the direction angles $\varphi_i$ of particular layers constituting the shell. They can assume values belonging to a domain $C \subset R^{2N}$ determined mainly by constraint $\psi(\mathbf{x}) = T^{\text{load}} - T_{\text{cr}} \leq 0$. The optimum solution $\mathbf{x}^*$ is sought for, in $C$, satisfying the condition of minimum shell thickness

$$F(\mathbf{x}^*) = \min_{\mathbf{x} \in C} F(\mathbf{x}) = \min_{\mathbf{x} \in C} h.$$

The constraint for $\psi(\mathbf{x})$ being nonlinear, the optimization problem cannot be actually solved by analytical methods. In view of the linearity of the object function $F(\mathbf{x})$ and, therefore, the constancy of its gradient, the presence of $\mathbf{x}^*$ at the edge $C$ and the fact that the constraints are of the inequality type, the optimization procedure is based on the Rosen gradient projection method [27 – 29], which yields solutions of high reliability, approaching global solutions, even with non-convex constraints. This method consists essentially in projecting the gradient $\nabla F(\mathbf{x})$ on the surface tangent to the constraints, then seeking for an extremum of the object function in direction **d** thus determined, which is corrected at each subsequent $k$-th iteration step.

To determine the direction **d**, we construct a matrix

$$P(\mathbf{x}^k) = I - A(A^TA)^{-1}A^T$$

of the projection on the sub-space by the set of active constraints:

$$W(\mathbf{x}^k) = \left[g_{i_1}(\mathbf{x}^k), g_{i_2}(\mathbf{x}^k), g_{i_{\mathrm{NAC}}}(\mathbf{x}^k)\right].$$

The $i$-th constraint is an active constraint for $\mathbf{x} \in C$ if $g_i(\mathbf{x}) = 0$ (in the numerical practice we have $|g_i(\mathbf{x})| \le \delta$, where $\delta$ is the assumed accuracy). Symbol $i_j$ is $j$-th element of sequence of numbers of active constraints. The quantity NAC is a number of active constraints in $\mathbf{x}^k$.

$I$ is an NAC-by-NAC identity matrix, $A$ is a matrix of active constraint gradients

$$A = \left[\nabla g_{i_j}(\mathbf{x}^k)\colon\ j = 1, \ldots, \mathrm{NAC}\right].$$

The direction **d** tangent to the constraints is determined by the formula:

$$\mathbf{d} = -P(\mathbf{x}^k)\nabla F(\mathbf{x}^k)\Big/\|P(\mathbf{x}^k)\nabla F(\mathbf{x}^k)\|.$$

Then, knowing the direction just determined, we seek for $\tau$:

$$\begin{gathered}
\mathbf{x}(\tau) = \mathbf{x}^k + \tau\mathbf{d}(\mathbf{x}^k), \\
\mathbf{x}^{k+1}(\tau) = \mathbf{x}(\tau) - \left[A(A^TA)^{-1}\right](\mathbf{x}^k)\left[g_{i_1}(\mathbf{x}(\tau)), \ldots, g_{i_{\mathrm{NAC}}}(\mathbf{x}(\tau))\right], \\
\|W(\mathbf{x}^{k+1}(\tau))\| \le \delta .
\end{gathered}$$

The algorithm is finished, when the criteria connected with the assumed accuracy of computation are satisfied.

On the basis of the algorithm of the Rosen method (the Appendix) and the procedure of determining the critical load $T_{\mathrm{cr}}$ [1], a program for optimization of a multilayer shell has been written. It will be further referred to as ProgContShell. Because for the determination of the value of $T_{\mathrm{cr}}$, determinants of symmetric matrices of the order of $4N+5$ and $4N+4$ are computed for each combination of buckling modes, the method for computing those determinants proves to be decisive for the efficiency of the program. The method selected for this purpose is the Householder three-diagonalization method [30]. For inverting a symmetric matrix the *syminversion* procedure is used [30]. In addition, a procedure for finding and arranging the set of active constraints was worked out as well as some procedures for iterative return to the boundary of the domain $C$, selection of the step length $\tau$, etc.

In its realized version, the ProgContShell program enables us to select the optimum structure (by determining the thickness $h_i$ and direction angles $\varphi_i$ of $i$-th layer) in the case of an axially symmetric load $T^{\mathrm{load}}$ acting on a cylindrical shell, the length $l$ of which is fixed, as well as radius $R$ of the internal (or middle) surface, the number of layers is $N$ and their properties are given (Fig. 2).

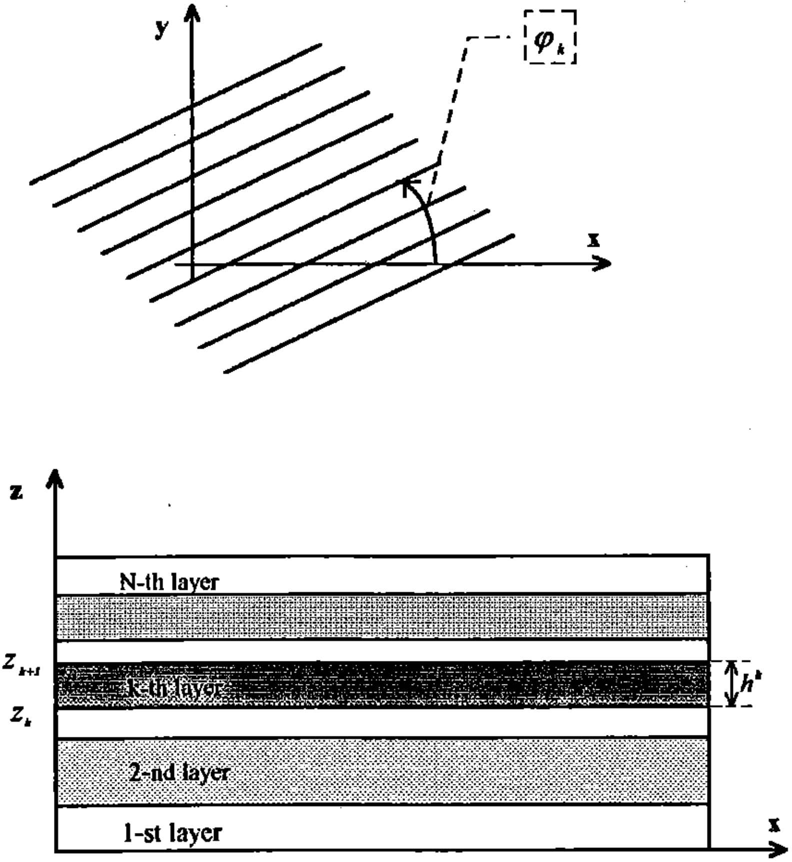


FIG. 2.

For complete option of the program the decision variables are the direction angles and the thicknesses of the layers. In the case of a simplified option, they are only the thicknesses of the layers. In reality the number of layers $N$ is also a decision variable. If the algorithm is used with floating-point arithmetic, the thicknesses of certain layers are negligible, therefore they are rejected to remove the singularities, which have been generated in the structure.

## 3. THE RESULTS AND DISCUSSION

Initially, the shell is composed of layers of the same thickness $h_i^0 = h^0/N$, made of No. 1 material, the elastic constants of which are $E_1 = E_2 = 13.9\,\mathrm{GPa}$, $E_3 = 6.56\,\mathrm{GPa}$, $G_{12} = 1.134\,\mathrm{GPa}$, $G_{23} = G_{13} = 1.869\,\mathrm{GPa}$, $\nu_{12} = \nu_{21} = 0.1628$, $\nu_{32} = \nu_{31} = 0.163$ [31] or a material, the angle $\varphi_i$ of which is marked by a subscript 2, the elastic constants being one-tenth of those for the material No. 1; $R = 19.75\,\mathrm{mm}$, $l = 0.26\,\mathrm{m}$ and $h^0 = 1.5\,\mathrm{mm}$. The direction angles of the principal anisotropy axes in the layers are listed from the inner to the outer layer. The shell is to carry a compressive load, the intensity of which is $T^{\mathrm{load}} = 10/(\pi R)\,\mathrm{kNm}^{-1}$.

**Table 1.**

| $\mathbf{x}^0$ | | | | $\mathbf{x}^1$ | $\mathbf{x}^*$ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $N$ | $\varphi_1$ | $\varphi_2$ | $\varphi_3$ | $h$ [mm] | $N$ | $\varphi_1$ | $\varphi_2$ | $h_1$ [mm] | $h$ [mm] |
| 1 | 0° | | | 0.78 | 1 | 9.85° | | | 0.77 |
| 1 | 45° | | | 0.97 | | 79.55° | | | |
| 1 | 90° | | | 0.78 | | 99.91° | | | |
| 2 | 0° | $0^\circ_2$ | | 1.27 | | 9.83° | | | |
| 2 | $0^\circ_2$ | 0° | | 1.29 | | 9.86° | | | |
| 2 | 0° | $45^\circ_2$ | | 1.27 | | 9.91° | | | |
| 2 | $0^\circ_2$ | 45° | | 1.38 | | 79.52° | | | |
| 2 | 45° | $0^\circ_2$ | | 1.55 | | 79.56° | | | |
| 2 | $45^\circ_2$ | 0° | | 1.27 | | 9.87° | | | |
| 2 | 0° | 0° | | 0.78 | 2 | −.45° | 44° | 0.41 | 0.73 |
| 2 | 45° | 45° | | 0.73 | | −.50° | 44° | | |
| 2 | 45° | 0° | | 0.87 | | −.42° | −46° | | |
| 3 | 0° | 45° | | 0.98 | | 89.5° | 44° | | |
| 3 | 0° | 0° | 0° | 0.79 | | −.48° | 44° | | |
| 3 | 45° | 45° | 0° | 0.75 | | −.44° | 44° | | |
| 3 | 45° | 0° | 45° | 0.83 | | −.44° | 44° | | |
| 3 | 0° | 45° | 45° | 0.98 | | −89.55° | 44° | | |
| 3 | $0^\circ_2$ | 0° | 0° | 1.09 | | −.44° | −46° | | |
| 3 | 45° | $45^\circ_2$ | 0° | 0.87 | | −.48° | 44° | | |
| 3 | 45° | 0° | $45^\circ_2$ | 1.18 | | −.52° | −46° | | |
| 3 | $45^\circ_2$ | 45° | 45° | 1.35 | | 89.5° | 44° | | |

The results of the complete option of the ProgContShell program as used for shells of prescribed original structures are presented in Table 1. The $\mathbf{x}^1$ column contains shell thicknesses on attaining in an iterative manner the boundary of the domain of admissible solutions $C$, that is after the first step of the algorithm has been made. Depending on the type of the structure of $\mathbf{x}^0$ considered, the use of the ProgContShell program gives the following more or less apparent results:

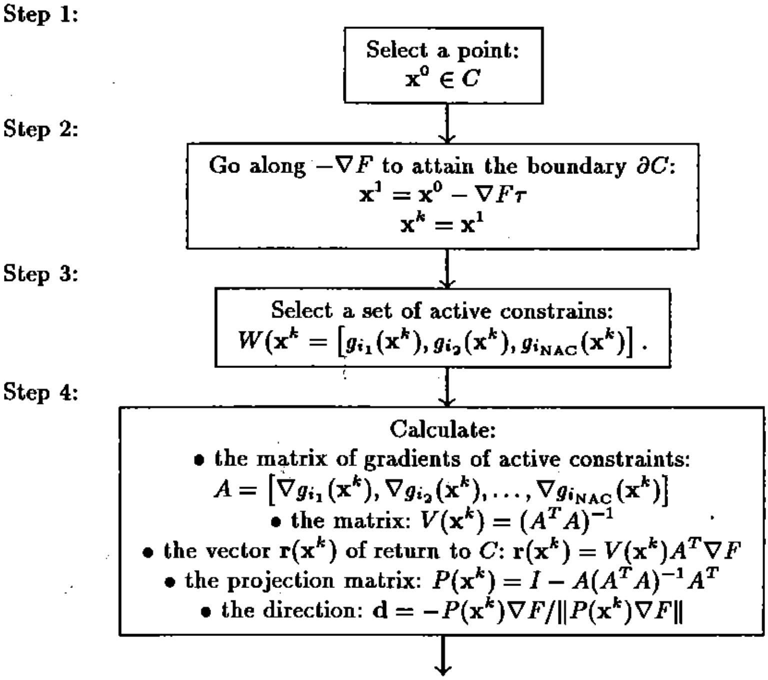
- For a two-layer shell $[0^\circ, 45^\circ]$, that is for a shell approaching the optimum structure, the optimization process reduces the thickness by scarcely 0.24%.
- For a three-layer shell $[45^\circ_2, 45^\circ, 45^\circ]$ the thickness of the shell can be reduced, as a result of an optimization process, by 46%.
- For single-layer structures $[0^\circ]$, $[90^\circ]$, change in the direction angle enables us to reduce the thickness by hardly 1.5%.
- For a structure $[45^\circ_2, 0^\circ]$ the result is improved by about 50.2%.

Apparent disagreement between the optimum direction angles may be explained by the fact that the directions 1 and 2 are identical in the materials considered.

To conclude our considerations it may be said, on the basis of the results obtained, that for various initial structures $\mathbf{x}^0$ as determined by $[h_1, \varphi_1, \ldots, h_N, \varphi_N]$, the individual solutions obtained are located in a sufficiently small neighbourhood of the optimum solution $\mathbf{x}^*$.

# APPENDIX

The algorithm of the gradient projection method devised by J.B. Rosen



**Step 1:**

Select a point:
$\mathbf{x}^0 \in C$

**Step 2:**

Go along $-\nabla F$ to attain the boundary $\partial C$:
$\mathbf{x}^1 = \mathbf{x}^0 - \nabla F \tau$
$\mathbf{x}^k = \mathbf{x}^1$

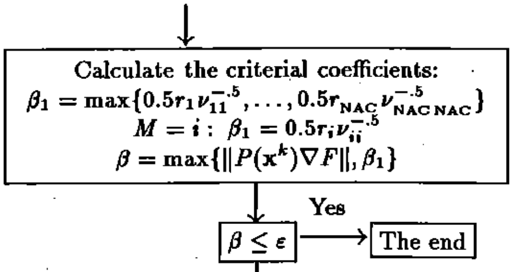
**Step 3:**

Select a set of active constrains:
$W(\mathbf{x}^k = \left[g_{i_1}(\mathbf{x}^k), g_{i_2}(\mathbf{x}^k), g_{i_{\mathrm{NAC}}}(\mathbf{x}^k)\right].$

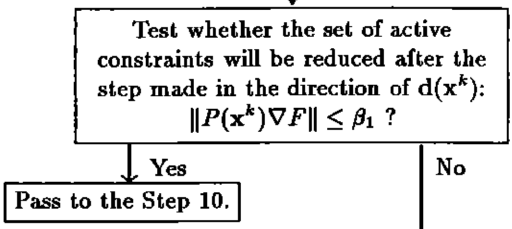
**Step 4:**

Calculate:
- the matrix of gradients of active constraints:
$A = \left[\nabla g_{i_1}(\mathbf{x}^k), \nabla g_{i_2}(\mathbf{x}^k), \ldots, \nabla g_{i_{\mathrm{NAC}}}(\mathbf{x}^k)\right]$
- the matrix: $V(\mathbf{x}^k) = (A^T A)^{-1}$
- the vector $\mathbf{r}(\mathbf{x}^k)$ of return to $C$: $\mathbf{r}(\mathbf{x}^k) = V(\mathbf{x}^k) A^T \nabla F$
- the projection matrix: $P(\mathbf{x}^k) = I - A(A^T A)^{-1} A^T$
- the direction: $\mathbf{d} = -P(\mathbf{x}^k)\nabla F / \| P(\mathbf{x}^k) \nabla F \|$

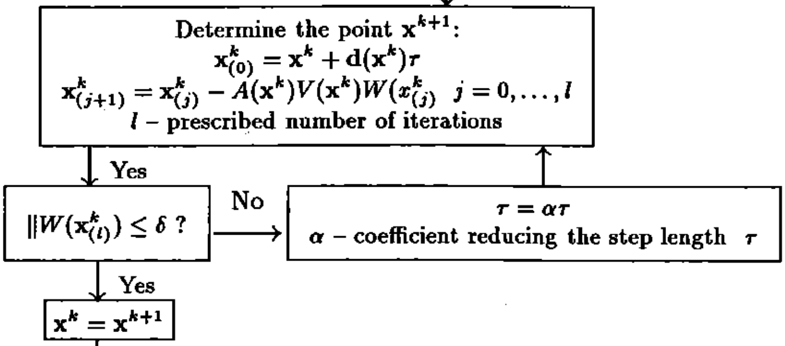
Step 5:



Step 6:



Step 7:



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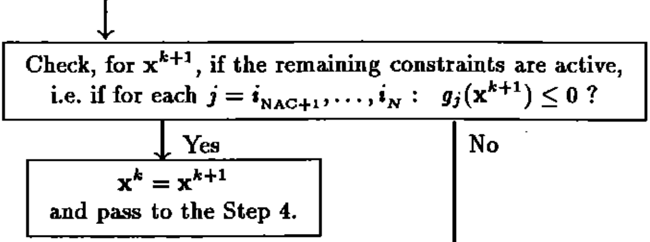
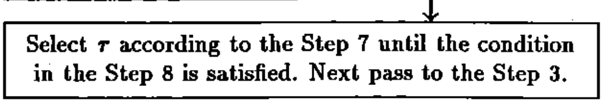
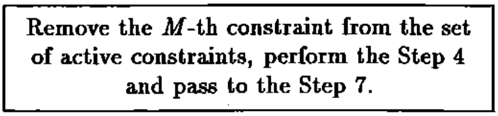


Step 9:



Step 10:



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