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ON THE CONSISTENCY COEFFICIENT OF A POWER-LAW FLOW OF BLOOD THROUGH THE NARROW VESSEL

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In this paper, we study the behaviour of the relative consistency coefficient of an assumed power-law flow of blood through the narrow vessel. The flow field comprises two layers, e.g. a marginal plasma layer near the wall and a core layer which is suspension of red cells in plasma. The relative consistency coefficient is determined by equating the sum of volume rates of flow in the two layers to the volume rate of flow in case the two fluids are replaced by a single power-law fluid with an appropriate consistency coefficient. The results are displayed graphically and discussed.

1. Introduction

A plausible assumption is made that the rheological properties of blood do not influence very much its flow in large vessels and the blood may be treated as a Newtonian fluid (cf. Quemada [1], Rodkiewicz [2]). Some exception may however, result at low flow rates, near quasi-steady conditions and in the vicinity of changes of the cross-section area of the vessel. As the diameter of the vessel is reduced, blood rheological properties appear more and more important from shear thinning, finally complicated by phase separation in narrow vessels. Experiments on steady blood flow in narrow vessels exhibit some anomalous features, e.g. the blunting of velocity profile, the formation of plasma layer and the Fåhraeus–Lindqvist effect (Sutera [3]). The blunting of the velocity profile occurs near the axis of the vessel. Blood is actually a complex fluid with formed elements (red cells, white cells and platelets) suspended in plasma. The red-blood cells (erythrocytes) outnumber the other cells and play an important role in carrying oxygen to all parts of the body. The percent volume concentration of red blood cells in the whole blood is called the hematocrit. The hematocrit value has a definite effect on the apparent viscosity of blood. When blood flows in a narrow vessel, two important inter-related phenomena occur. One of these is the tendency of erythrocytes to migrate toward the center of the flow-vessel leaving a relatively cell-free, slower moving layer of plasma. The other is the reduction of apparent viscosity, as caused essentially by the cell-free layer (FÅHRAEUS and LINDQVIST [4]). BLOCH'S [5] photographic records of blood flow in narrow vessels of animals supported the existence of a marginal (plasma) layer near the wall, leading to consider the flow as a two-phase one, with a particle-rich axial core surrounded by a particle-depleted wall layer. Many authors, e.g., HAYNES and BURTON [6], WHITMORE [7], CHATURANI and SAMY [8], SUKLA *et al.* [9], MAJHI and USHA [10], TANDON and KUSHWAHA [11] have studied the Fåhraeus – Lindqvist phenomena related problems theoretically treating blood either as Newtonian or non-Newtonian fluid. BUGLIARELLO *et al.* [12] carried out measurements in vitro in glass capillaries with diameters in the range of 40 to 83 μ for a possible estimation of plasma layer thickness. Normal human whole blood samples with acid-citrate-dextrose and varying hematocrits were used for this purpose. The shear stresses corresponding to their measurements were in the range of 10 to 100 dynes/cm^2. They measured the plasma layer thickness from frames of high speed motion pictures. And the thickness of the plasma layer was defined by averaging individual measurements of the distance from the capillary wall to the point of closest approach of a red cell. At a hematocrit of 40%, the plasma layer thickness was found to decrease with the decrease of shear stress in the 40 μ capillary. The ratio of the plasma layer thickness to the radius of the vessel were found of about $0.05 - 0.1$ at normal hematocrit. We propose a two-layer model for blood flow through narrow arterial tube in which both the layers consist of power-law fluids having the same power-law index, but different consistency coefficients. On the basis of a prescribed relation between these coefficients, the total volumetric flow rate is to be calculated. Final aim will be to find the relative consistency coefficient in reference to a single power-law fluid flowing through the same tube.

2. MATHEMATICAL FORMULATION

We consider the flow of an inelastic time-independent non-Newtonian fluid through a tube. The tube is sufficiently narrow and of constant diameter. The flow consists of two layers, namely the peripheral cell-free plasma layer and the core layer which is suspension of red cells in plasma (Fig. 1).

In case of a power-law fluid, the relationship between the shear stress and shear rate (or velocity gradient) is expressed generally as

$$\tau = -K\left(\frac{dv_z}{dr}\right)^n, \tag{2.1}$$

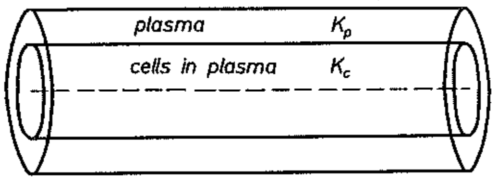

FIG. 1. Two-layer flow.

where K is the consistency coefficient and n the power-law index. v_z is the axial velocity component. K has the dimensions $ML^{-1}\,T^{n-2}$. Equation (2.1) can be rewritten in the form

$$\tau = -K\left(\frac{dv_z}{dr}\right)^n = -\mu_a\left(\frac{dv_z}{dr}\right), \tag{2.2}$$

where

$$\mu_a = K\left(\frac{dv_z}{dr}\right)^{n-1}.$$

μ_a is the apparent viscosity. It is to be mentioned that if $n = 1$, then Eq. (2.1) reduces to the Newtonian form. When $n < 1$, (2.1) describes the pseudo-plastic or shear thinning fluid, and when $n > 1$, it is of the form of dilatant fluid. Pseudo-plastic behaviour is characteristic of many polymers, polymer solutions and suspensions (cf. ANDERSSON and IRGENS [13]). We shall deal in the present case with a value of n which is less than 1. The choice $n < 1$ will be evident from the following discussions. WELL *et al.* [14] have shown that the apparent viscosity of human blood varies with the shear rate. The apparent viscosity increases with decreasing shear rate. The whole blood thus can be considered as non-Newtonian and shear-thinning. They also demonstrated that red cell suspension in saline solution is non-Newtonian and its absolute value of viscosity is lower than that of whole blood. On the other hand, plasma exhibited nearly Newtonian viscosity in their measurements. Some authors, however, consider plasma as non-Newtonian, emphasizing the influence of fibrinogen on its behaviour (OKA, [15]). Plasma is also known to exhibit a unique relaxation phenomenon. Let us now consider a typical blood flow curve (Fig. 2) after RAHN *et al.* [16]. Figure 2 presents shear stress vs. shear rate for the blood sample which had a physiologically normal hematocrit of 46 and contained an anticoagulant acid-citrate-dextrose. This "shear stress-shear rate" curve appears to correspond to such a curve for a commonly observed behaviour often exhibited by solutions of high polymers (MIDDLEMAN [17]). If the data of Fig. 2 are replotted as $\tau^{1/2}$ vs. $\dot{\gamma}^{\,1/2}$, $\dot{\gamma}$ being the shear rate, it can be shown by expanding the low shear rate scale that blood has a finite yield

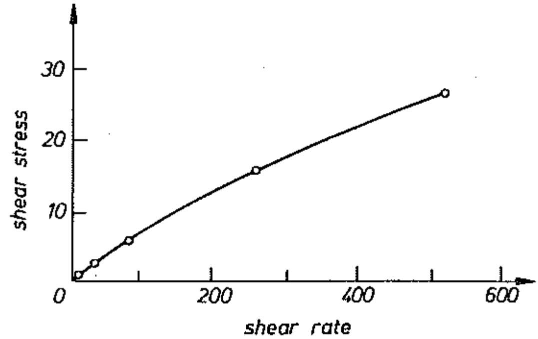

FIG. 2. A typical flow curve for blood (Ref. 16).

stress. But the magnitude of the yield stress (about 0.05 dyne/cm^2 in this sample) is actually quite small. And, we consider the power law fluid with $n < 1$, alternative to casson fluid as representative of blood.

3. VOLUMETRIC FLOW RATE, RELATIVE CONSISTENCY COEFFICIENT

Assuming that peripheral plasma layer is of thickness δ and the radius of the tube R (Fig. 1), we can write the equations governing the motion in the peripheral and core layers, respectively, as

$$\frac{d}{d\eta}\left[\eta K_p\left(-\frac{dv_{zp}}{d\eta}\right)^n\right] = R^{n+1}P\eta, \tag{3.1}$$

$$1 - \delta/R \leq \eta \leq 1$$

and

$$\frac{d}{d\eta}\left[\eta K_c\left(-\frac{dv_{zc}}{d\eta}\right)^n\right] = R^{n+1}P\eta, \tag{3.2}$$

$$0 \leq \eta \leq 1 - \delta/R,$$

where $\eta = r/R$, $P = -(dp/dz)$ the pressure gradient, K_p and K_c are the consistency coefficients, respectively in the peripheral and core layers. v_{zp} and v_{zc} are the axial velocity components. Without loss of generality, we may assume that K_p is constant and K_c is expressible in the form

$$K_c = \frac{K_p}{1 - \beta h(\eta)}, \qquad h(\eta) = h_m\left\{1 - \left(\frac{\eta}{1 - \delta/R}\right)^{n_1}\right\}, \tag{3.3}$$

where β is a constant having the value 2.5, n_1 is the shape parameter and h_m is the maximum hematocrit at the centre of the tube. The boundary conditions are given by

$$\begin{array}{lllll} & \text{i)} & v_{zp} = 0 & \text{at} & \eta = 1, \\ & \text{ii)} & \dfrac{dv_{zc}}{d\eta} = 0 & \text{at} & \eta = 0, \\ (3.4) & \text{iii)} & v_{zp} = v_{zc} & \text{at} & \eta = 1 - \delta/R, \\ & \text{iv)} & K_p \dfrac{dv_{zp}}{d\eta} = K_c \dfrac{dv_{zc}}{d\eta} & \text{at} & \eta = 1 - \delta/R. \end{array}$$

The last relation (iv) is due to equality of shear stress at the interphase of the two layers. To ease the analysis, we assume the value of the power-law index $n = 1/2$. Solution of Eq. (3.1) with $n = 1/2$ is given by

$$v_{zp} = \frac{R^3P^2}{12K_p^2}(1-\eta^3) + A^2\left(\frac{1}{\eta} - 1\right) + \frac{R^{3/2}PA}{K_p}(1-\eta), \tag{3.5}$$

where A is constant of integration. In obtaining (3.5), the condition (i) of (3.4) has been utilized. It remains to determine A.

Solution of Eq. (3.2) for the case $n = 1/2$ is given by

$$v_{zc} = -\frac{R^3P^2}{4K_p^2}\left[\frac{\eta^3}{3}(1-L)^2 + \frac{L^2a^{2n}\eta^{2n+3}}{2n+3} + \frac{2a^n\eta^{n+3}}{n+3}L(1-L)\right] + D, \tag{3.6}$$

where $L = \beta h_m$, $a = (1 - \delta/R)^{-1}$ and D is constant of integration. In obtaining (3.6), the condition (ii) of (3.4) and the relation (3.3) have been used.

Utilizing the condition (iv) of (3.4), i.e.

$$K_p \frac{dv_{zp}}{d\eta} = K_c \frac{dv_{zc}}{d\eta} \qquad \text{at} \qquad \eta = 1 - \delta/R$$

in Eqs. (3.5) and (3.6), we obtain $A = 0$. Thus the solution (3.5) reduces to

$$v_{zp} = \frac{R^3p^2}{12k_p^2}(1-\eta^3). \tag{3.7}$$

The constant D in (3.6) is determined by applying the condition (iii) of (3.4), $v_{zp} = v_{zc}$ at $\eta = 1 - \delta/R$ to Eqs. (3.6) and (3.7), as

$$D = \frac{R^3p^2}{12K_p^2}\left[(1-\delta/R)^3\left\{L^2 - 2L + \frac{6L(1-L)}{n_1+3} + \frac{3L^2}{2n_1+3}\right\} + 1\right]. \tag{3.8}$$

Thus (3.6) and (3.8) constitute the solution for v_{zc}. Let us denote the volumetric flow rate in the peripheral plasma layer and the core layer, respectively, by Q_p and Q_c. We calculate Q_p to obtain

$$Q_p = \int_{R-\delta}^{R} v_{zp} 2\pi r\, dr = 2\pi R^2 \int_{1-\delta/R}^{1} v_{zp}\eta\, d\eta = \int_{1-\delta/R}^{1} \frac{R^3P^2}{12K_p^2}(1-\eta^3)\eta\, d\eta$$
$$= \frac{2\pi R^5 P^2}{12K_p^2}\left[\frac{3}{10} + \frac{(1-\delta/R)^5}{5} - \frac{(1-\delta/R)^2}{2}\right], \tag{3.9}$$

Q_c can be calculated as

$$Q_c = \int_0^{R-\delta} v_{zc} 2\pi r\, dr = 2\pi R^2 \cdot \int_0^{1-\delta/R} v_{zc}\eta\, d\eta$$
$$= 2\pi R^2 \int_0^{1-\delta/R} \left\{ -\frac{R^3P^2}{4K_p^2}\left[\frac{\eta^3}{3}(1-L)^2 + \frac{2a^n}{n_1+3}L(1-L)\eta^{n_1+3} + \frac{a^{2n_1}}{2n_1+3}L^2\eta^{2n_1+3}\right]\eta + D\eta \right\} d\eta$$
$$= 2\pi R^2 \left[-\frac{R^3P^2}{4K_p^2}(1-\delta/R)^5 \left\{ \frac{(1-L)^2}{15} + \frac{2L(1-L)}{(n_1+3)(n_1+5)} + \frac{L^2}{(2n_1+3)(2n_1+5)} \right\} + \frac{D(1-\delta/R)^2}{2} \right]. \tag{3.10}$$

We denote by Q the total flux as

$$Q = Q_p + Q_c\,. \tag{3.11}$$

From (3.9) and (3.10), we obtain

$$Q = \frac{2\pi R^5 P^2}{12K_p^2}\left[(1-\delta/R)^5 \left\{ \frac{1}{5} - \frac{(1-L)^2}{5} - \frac{6L(1-L)}{(n_1+3)(n_1+5)} - \frac{3L^2}{(2n_1+3)(2n_1+5)} + \frac{L^2-2L}{2} + \frac{3L(1-L)}{n_1+3} + \frac{3L^2}{2(2n_1+3)} \right\} + \frac{3}{10} \right]. \tag{3.12}$$

If the tube were filled completely by a single power-law fluid having the consistency coefficient K_e (effective consistency coefficient of the two fluids), the volumetric flow rate would then be given by

$$Q_0 = \frac{\pi R^5 P^2}{20 K_e^2}. \tag{3.13}$$

Assuming that the fluxes Q and Q_0 are the same, we calculate the relative consistency coefficient

$$\begin{aligned} K_r &= K_e/K_p \qquad \text{from (3.12) and (3.13) as} \\ K_r &= \{3/10S\}^{1/2}, \end{aligned} \tag{3.14}$$

where

$$S = (1-\delta/R)^5 \left\{ \frac{3L(L-2)}{10} + \frac{3L(1-L)}{(n_1+5)} + \frac{3L^2}{(4n_1+10)} \right\} + \frac{3}{10}.$$

Variations of K_r with δ/R for the cases $n_1 = 2$, 4 and 6 are shown, respectively, in Figs. 3–5. All these cases have been illustrated for h_m values 0, 0.2, 0.3 and 0.46.

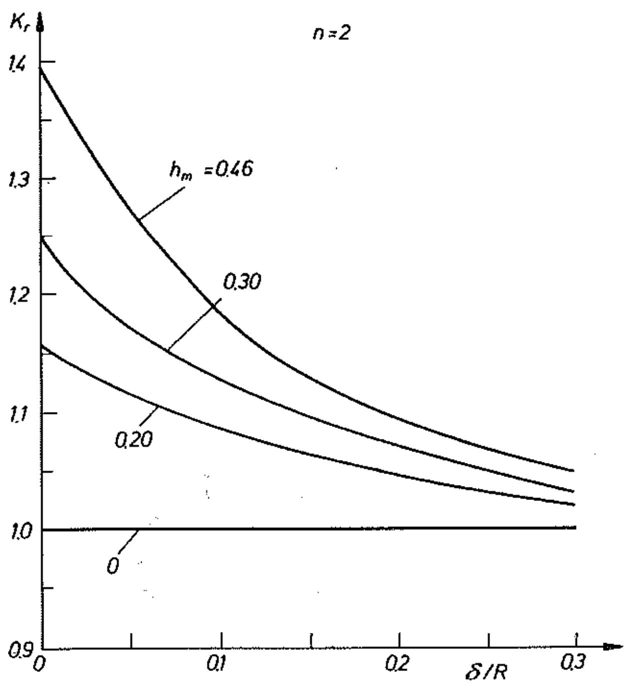

FIG. 3. Variations of relative consistency coefficient for different hematocrit values ($n = 2$).

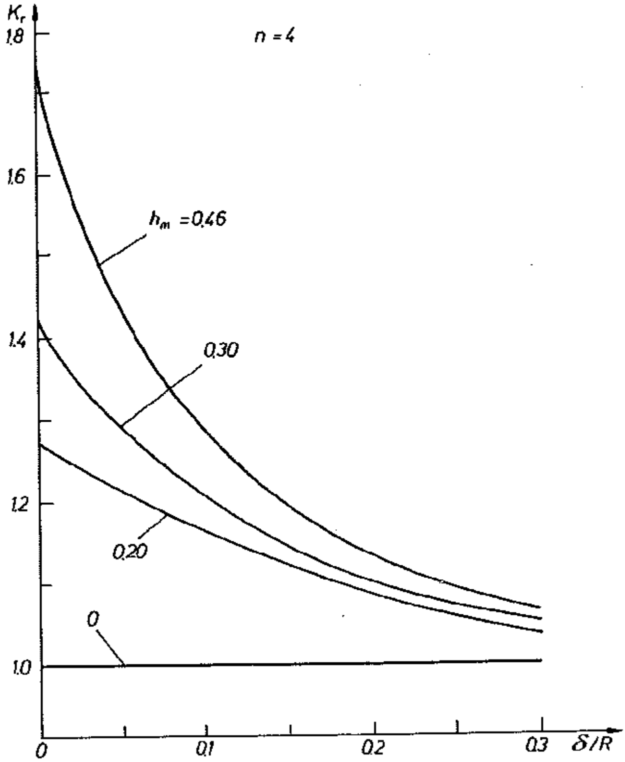

FIG. 4. Variations of relative consistency coefficient for different hematocrit values ($n = 4$).

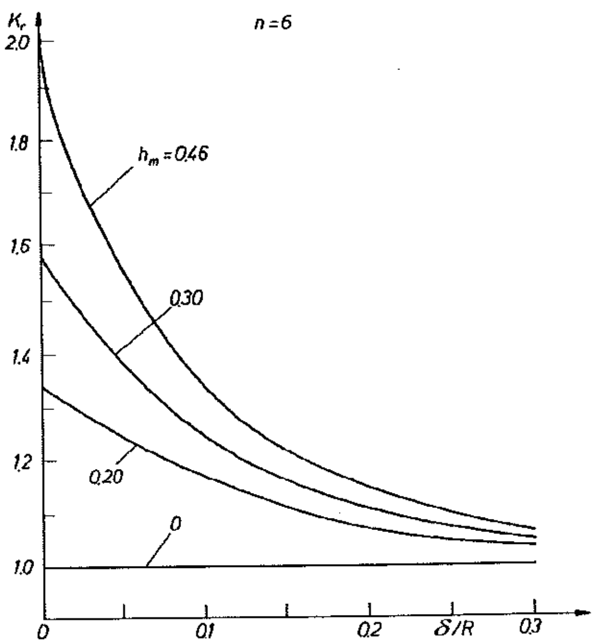

FIG. 5. Variations of relative consistency coefficient for different hematocrit values ($n = 6$).

4. Discussion

Examining Figs. 3 – 5, we arrive at the following conclusions.

In the absence of hematocrit, the relative consistency coefficient takes the constant value 1 in all the cases. For a fixed value of δ/R and fixed n_1, the relative consistency coefficient increases with the increase of hematocrit.

At any δ/R and fixed hematocrit, the relative consistency coefficient increases with the increase of n_1.

For any n_1, the relative consistency coefficient decreases with the increase of δ/R. The rate of decrease becomes faster as the hematocrit increases.

While the exact non-Newtonian behaviour of blood is not known, the simple phenomenological model presented in this paper seems to be useful for better understanding of the blood flow through narrow vessels and in clinical applications.

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References

1. D. Quemada, *Blood rheology and its implication in flow of blood,* [in:] Arteries and Arterial Blood Flow C.M. Rodkiewicz [Ed.], Springer Verlag, Wien, 1–127, 1983.
2. C.M. Rodkiewicz, *Flow in large arteries,* [in:] Arteries and Arterial Blood Flow, C.M. Rodkiewicz [Ed.], Springer Verlag, Wien, 327–411, 1983.
3. S.P. Sutera, *Red cell motion and deformation in the microcirculation,* [in:] Cardiovascular and Pulmonary Dynamics, M.Y. Jaffrin [Ed.] Editions, Inserm Paris, 221–242, 1978.
4. R. Fåhraeus and T. Lindqvist, *The viscosity of the blood in narrow capillary tubes,* Amer. J. Physiol. **96**, 562–568, 1931.
5. E.H. Bloch, *A quantitative study of the hemodynamics in the living microvascular system,* Amer. J. Anatomy, **110**, 125–153, 1962.
6. R.H. Haynes and A.C. Burton, *Role of non-Newtonian behaviour of blood in hemodynamics,* Amer. J. Physiol., **197**, 943, 1959.
7. R.L. Whitmore, *A theory of blood flow in small vessels,* J. Appl. Physiol., **22**, 767–771, 1967.
8. P. Chaturani and R.P. Samy, *A two-layered model for blood flow through stenosed arteries,* Proc. Nat. Conf. Fluid Mech. and Fluid Power, Hyderabad, India, Cf., 16–22, 1982.

9. J.B. SUKLA *et al.*, *Effects of peripheral layer viscosity on blood flow through the artery with mild stenosis*, Bull. Math. Bio., **42**, 797–805, 1980.

10. S.N. MAJHI and L. USHA, *A mathematical note on the Fåhraeus–Lindqvist effect in power law fluid*, Bull. Math. Biology, **47**, 765–769, 1985.

11. P.N. TANDON and K. KUSHWAHA, *A study of Fåhraeus–Lindqvist effect in oscillatory capillary – blood flows*, Proc. Nat. Acad. Sci. India, **62** (A), III, 412–421, 1992.

12. G. BUGLIARELLO *et al.*, *The profile viscosity and other characteristics of blood flow in a non-uniform shear field*, Proc. Four. Int. Congr. on Rheology, A.L. COPLEY [Ed.], **4**, Interscience, N.Y., 351-370, 1965.

13. H.I. ANDERSSON and F. IRGENS, *Numerical solution of the laminar boundary layer equations for power-law fluids*, J. Non-Newtonian Fluid Mech., **32**, 175–195, 1989.

14. R.E. WELL *et al.*, *Shear-rate dependence of viscosity of blood: interaction of red cells and plasma proteins*, Tran. Soc. Rheol., **6**, 19, 1962.

15. S. OKA, *Cardiovascular hemorheology*, Cambridge University Press, Cambridge, London 1981.

16. A.W. RAHN *et al.*, [in:] Chemical Engineering in Medicine and Biology, D. HERSHEY [Ed.], Plenum, New York, 45, 1967.

17. S. MIDDLEMAN, *Transport phenomena in the cardiovascular system*, Wiley - Interscience, New York 1972.

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