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SHAPE OPTIMIZATION OF 2D ELASTIC STRUCTURES USING ADAPTIVE GRIDS (*)

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The paper deals with an effective adaptive method for 2D shape optimization of linear elastic structures. The design objective is to find the shape of the kinematically unconstrained boundary assuring minimum volume of the structural material with constraints imposed on equivalent stresses. The geometrical shape parameters are the design variables. The method is based on a special kind of finite element automatically adaptive grids. The original iterative algorithm for solving the nonlinear system of equations and inequalities arising from the Kuhn-Tucker conditions is presented. The proposed approach is successfully tested on two classical examples.

1. Introduction

Shape optimization is an important topic in structural design research. Due to the iterative nature of shape optimization, the efficiency of the method is very essential in order to decrease the number of iteration steps [2]. The paper deals with further development of the adaptive method for 2D shape optimization of linear elastic structure, presented in the earlier authors' paper [1]. In our study we propose an automatic adaptability: the change of positions of finite element nodes is a consequence of the iterative change of the geometrical shape parameters which are the design variables. This is possible by introduction of a special kind of grids.

In the present paper we are dealing with two different types of grids. The main feature of these grids is that they consist of two families of layers. The numbers of layers are constant in the iteration process. In the paper the polar-type grid is introduced in which the circumferential layers have different sizes in order to prevent the degeneration of finite elements (Fig. 1) what is important [6]. We reduce the number of design variables by introducing the other geometric shape parameter vector $\mathbf{a}$ instead of $\mathbf{h}$.

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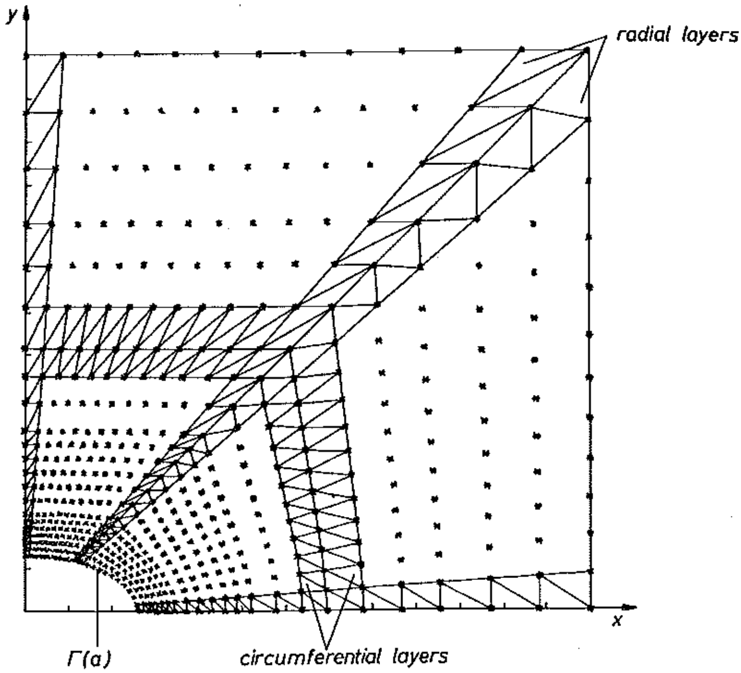


FIG. 1. Polar-type grid.

The grid remeshing should be suitably controlled to assure desirable accuracy of the method [3]. In the paper the grid remeshing is controlled by the Kuhn - Tucker necessary conditions. We calculate analytically the derivatives of the stiffness matrix and those of stresses with respect to design variables, what considerably improves the accuracy of the method.

The efficiency and accuracy of the method are illustrated by two classical numerical examples. The first one (Subsec. 7.1) is calculated using a rectangular type of grid, the second one (Subsec. 7.2) – a polar-type grid.

2. STATEMENT OF THE PROBLEM

The design objective is to find the shape of the kinematically unconstrained boundary of 2D linear elastic structure subjected to static forces, assuring minimum volume under constraint imposed on equivalent stresses.

The manner of division of the structure domain by the rectangular type of grid of triangular finite elements is presented in [1]. The structure is divided into p_0 "horizontal" layers and r_0 vertical layers, the latter having constant sizes.

In the paper we introduce the polar-type grid (Fig. 1) having p_0 circumferential layers and r_0 radial layers. In order to avoid the degeneration of triangles, the sizes of circumferential layers decrease, approaching the pole, by a proper multiplier m_p of the base h_r (Fig. 2), prescribed for each radius r, respectively. However, all radial layers have the same angle size θ_0 (Fig. 2).

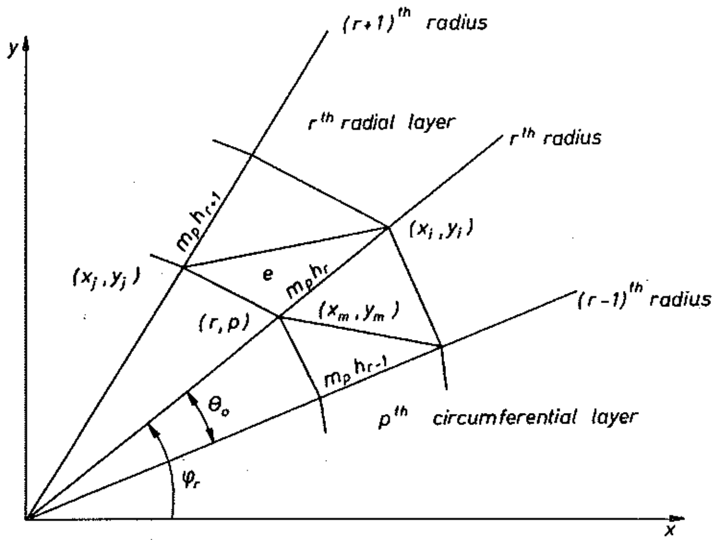


FIG. 2. Four typical triangles of the polar-type grid.

Such a problem of shape optimization is formulated mathematically as follows:

Find the shape of the unconstrained boundary $\Gamma(\mathbf{a})$ of a structure assuring minimum volume f as the objective function

$$f(\mathbf{a}) = t\sum_{i=1}^{N} A_i(\mathbf{a}), \tag{2.1}$$

where t – thickness (t = constant), A_i – area of the i-th finite element, and N – number of finite elements.

The objective function is subjected to the equality constraint in the form of the equilibrium equation

$$\mathbf{K}(\mathbf{a})\mathbf{u} - \mathbf{P} = \mathbf{0}, \tag{2.2}$$

where $\mathbf{K}$ – global stiffness matrix, $\mathbf{u}$ – displacement vector of FE nodes, $\mathbf{P}$ – vector of external forces, and to the inequality constraints in the form of the Huber–Mises yield condition for equivalent stress in each of the FE

$$\sigma^2 - \sigma_0^2 \le 0, \tag{2.3}$$

with

$$\sigma^2 = \sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2,$$

where σ_{xx}, σ_{yy} are normal stresses along the x and y directions, respectively, and τ_{xy} is the shear stress.

$\mathbf{a} = (a_1, \ldots, a_I)$ is the design vector that contains the geometrical parameters which control the shape of the structure, and $\mathbf{h} = (h_1, \ldots, h_{r_0+1})$ presented in Fig. 2 is the function of $\mathbf{a}$.

Lagrangian of the problem has the form

$$L = f + \boldsymbol{\lambda}^{e^T}[\mathbf{K}(\mathbf{a})\mathbf{u} - \mathbf{P}] + \boldsymbol{\lambda}^{s^T}(\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}_0^2), \tag{2.4}$$

where $\boldsymbol{\lambda}^{e^T}$ is the vector of Lagrange multipliers associated with the equilibrium equation, $\boldsymbol{\lambda}^{s^T}$ is the vector of Lagrange multipliers associated with the stress constraints, $\boldsymbol{\sigma}^2$ is the square of equivalent stresses in all finite elements of the body.

3. System of governing equations

According to the Kuhn - Tucker theorem, the necessary conditions for the problem have the following form:

$$\mathbf{K}(\mathbf{a})\mathbf{u} - \mathbf{P} = \mathbf{0}, \tag{3.1}$$

$$\mathbf{K}(\mathbf{a})\boldsymbol{\lambda}^e + \nabla_{\mathbf{u}}(\boldsymbol{\lambda}^{s^T}\boldsymbol{\sigma}^2) = \mathbf{0}, \tag{3.2}$$

$$\frac{\partial L}{\partial a_i} = \sum_{r=1}^{r_0+1} \frac{\partial L}{\partial h_r}\frac{\partial h_r}{\partial a_i} = 0, \qquad i = 1, \ldots, I, \tag{3.3}$$

where

$$G_r = \frac{\partial L}{\partial h_r} = \frac{\partial f}{\partial h_r} + \boldsymbol{\lambda}^{e^T}\frac{\partial \mathbf{K}}{\partial h_r}\mathbf{u} + \boldsymbol{\lambda}^{s^T}\frac{\partial \boldsymbol{\sigma}^2}{\partial h_r},$$

$$\boldsymbol{\lambda}^{s^T}(\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}_0^2) = 0, \qquad \boldsymbol{\lambda}^s \geq \mathbf{0}. \tag{3.4}$$

The above conditions constitute a system of nonlinear equations and inequalities, in which the number of unknowns $\mathbf{a}$, $\mathbf{u}$, $\boldsymbol{\lambda}^s$, $\boldsymbol{\lambda}^e$ is equal to the number of equations. The solution of this system gives us the solution of our optimum problem from the point of view of necessary conditions (i.e., at least the local minimum), whereas the question of sufficient conditions remains open.

4. Derivatives of the stiffness matrix and of the square of equivalent stresses with respect to variable h_r

Our system of equations requires derivatives of the stiffness matrix and of the square of equivalent stresses, with respect to the variable h_r (3.3). In order to avoid the errors connected with the numerical calculation of these derivatives we determine them analytically. The demonstration below is made for one of four typical triangles e in our polar-type triangular discretization (Fig. 2).

The stiffness matrix of one element has the known form

$$\mathbf{K}^e = tA^e \mathbf{B}^{e^T} \mathbf{D} \mathbf{B}^e; \tag{4.1}$$

we may write

$$\mathbf{B}^e = \frac{1}{2A^e} \widetilde{\mathbf{B}}^e,$$

then

$$\mathbf{K}^e = tA^e \frac{1}{2A^e} \widetilde{\mathbf{B}}^{e^T} \mathbf{D} \frac{1}{2A^e} \widetilde{\mathbf{B}}^e = \frac{t}{4A^e} \widetilde{\mathbf{B}}^{e^T} \mathbf{D} \widetilde{\mathbf{B}}^e, \tag{4.2}$$

$$\frac{\partial \mathbf{K}^e}{\partial h_r} = \frac{t}{2} \left[\frac{\partial}{\partial h_r} \left(\frac{1}{2A^e} \right) 4(A^e)^2 \mathbf{B}^{e^T} \mathbf{D} \mathbf{B}^e + \widetilde{\mathbf{P}}^{e^T} \mathbf{D} \mathbf{B}^e + \mathbf{B}^{e^T} \mathbf{D} \widetilde{\mathbf{P}}^e \right], \tag{4.3}$$

where

$$\widetilde{\mathbf{P}}^e = \frac{\partial \widetilde{\mathbf{B}}^e}{\partial h_r},$$

$$\frac{\partial \mathbf{K}^e}{\partial h_r} = \frac{t}{2} \left[-2 \frac{\partial A^e}{\partial h_r} \mathbf{B}^{e^T} \mathbf{D} \mathbf{B}^e + \widetilde{\mathbf{P}}^{e^T} \mathbf{D} \mathbf{B}^e + \mathbf{B}^{e^T} \mathbf{D} \widetilde{\mathbf{P}}^e \right]. \tag{4.4}$$

In our case A^e is the area of the triangle e

$$A^e = \frac{1}{2} [x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)], \tag{4.5}$$

where

$$\begin{aligned}
x_i &= (d_r - (p_s - 1)h_r) \cos \varphi_r\,, \\
y_i &= (d_r - (p_s - 1)h_r) \sin \varphi_r\,, \\
x_j &= (d_{r+1} - p_s h_{r+1}) \cos \varphi_{r+1}\,, \\
y_j &= (d_{r+1} - p_s h_{r+1}) \sin \varphi_{r+1}\,, \\
x_m &= (d_r - p_s h_r) \cos \varphi_r\,, \\
y_m &= (d_r - p_s h_r) \sin \varphi_r\,,
\end{aligned}$$

d_r is the distance from a fixed boundary point of the structure on radius r to the pole, p_s is a multiplier of the base h_r for j and m apex of triangle e, taking in account the sizes of all circumferential layers, $\varphi_r = r \cdot \theta_0$, where θ_0 – base angle.

$$\frac{\partial A^e}{\partial h_r} = \frac{1}{2}[(1-p_s)\cos\varphi_r(y_j - y_m) + x_i p_s \sin\varphi_r - x_j \sin\varphi_r - p_s(y_i - y_j)\cos\varphi_r + x_m(1-p_s)\sin\varphi_r], \tag{4.6}$$

$$\widetilde{\mathbf{P}}^{eT} = \begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix}, \tag{4.7}$$

where

$$\begin{aligned} b_i &= p_s \sin\varphi_r, \\ c_i &= -p_s \cos\varphi_r, \\ b_j &= -\sin\varphi_r, \\ c_j &= -\cos\varphi_r, \\ b_m &= (1-p_s)\sin\varphi_r, \\ c_m &= (p_s - 1)\cos\varphi_r. \end{aligned}$$

For the square of equivalent stresses $\boldsymbol{\sigma}^2$ we may write for one element

$$\frac{\partial \sigma^2}{\partial h_r} = (2\sigma_{xx} - \sigma_{yy})\frac{\partial \sigma_{xx}}{\partial h_r} + (2\sigma_{yy} - \sigma_{xx})\frac{\partial \sigma_{yy}}{\partial h_r} + 6\tau_{xy}\frac{\partial \tau_{xy}}{\partial h_r}, \tag{4.8}$$

where

$$[\sigma_{xx}, \sigma_{yy}, \tau_{xy}]^e = \mathbf{D}\mathbf{B}^e\mathbf{u}^e,$$

$$\frac{\partial}{\partial h_r}[\sigma_{xx}, \sigma_{yy}, \tau_{xy}]^e = \left(-\frac{1}{A^e}\frac{\partial A^e}{\partial h_r}\mathbf{D}\mathbf{B}^e + \frac{1}{2A^e}\mathbf{D}\widetilde{\mathbf{P}}^e\right)\mathbf{u}^e.$$

5. Derivatives of equivalent stresses with respect to displacements of the nodes

The adjoined equation (3.2) contains the derivatives of squares of the equivalent stresses with respect to the components of the displacement vector. We find the analytical expressions for these derivatives. In order to do it, at first we express the square of the equivalent stress in a single element applying the summation convention

$$\sigma^{e^2} = (D_{1k}B^e_{kl}u^e_l)^2 + (D_{2k}B^e_{kl}u^e_l)^2 - (D_{1k}B^e_{kl}u^e_l)(D_{2i}B^e_{ij}u^e_j) + 3(D_{3k}B^e_{kl}u^e_l)^2 \tag{5.1}$$

with $j, l = 1, 2, \ldots, 6,\ i, k = 1, 2, 3$.

The derivative of σ^{e^2} with respect to u_i takes, with above notations, the form

$$\frac{\partial(\sigma^e)^2}{\partial u_i} = 2\sigma^e_{xx}D_{1k}B^e_{ki} + 2\sigma^e_{yy}D_{2k}B^e_{ki} - \sigma^e_{xx}D_{2k}B^e_{ki} - D_{1k}B^e_{ki}\sigma^e_{yy} + 6\tau^e_{xy}D_{3k}B^e_{ki}. \tag{5.2}$$

6. Outline of solution algorithm

In order to solve the system of nonlinear algebraic equations (3.1) – (3.4), an optimizing Fortran program was written according to our iterative algorithm presented below. To solve the equilibrium equations (3.1) and (3.2) in the steps 2(A) and 6(A) – "A" steps defined as belonging to an "Analyser" – we use the standard professional FEM program presented in [5]. By "O" we define steps belonging to an "Optimizer", where adjustable coefficients β_λ and α are used.

Step 1. (O)
Take $n := 0$ (n is iteration counter).
Assume the starting values of $\mathbf{a}(0)$ and find $\mathbf{h}(0) = \mathbf{h}(\mathbf{a}(0))$.

Step 2. (A)
Find from (3.1) the displacement vector $\mathbf{u}(n)$
$\mathbf{K}[\mathbf{h}(n)]\mathbf{u}(n) = \mathbf{P}$,
calculate the square of equivalent stresses $\boldsymbol{\sigma}^2(n)$ for all elements and calculate the volume $f(n)$.

Step 3. (O)
If

$$\max_j \left[\left|\frac{\sigma_j^2(n) - \sigma_0^2}{\sigma_0^2}\right|\right] < \mu,$$

where μ is a given small number and j indicates the elements of the boundary Γ, then STOP.

STEP 4. (O)
Find derivatives of $\mathbf{K}$ and of $\boldsymbol{\sigma}^2$ with respect to h_r from (4.4) and (4.8), and of $\boldsymbol{\sigma}^2$ with respect to u_i from (5.2).

STEP 5. (O)
Assume initial value of $\boldsymbol{\lambda}^s(0)$. In this case, using additional relation 20 from [4], we take

$$\lambda_j^s(0) = \frac{f(n)}{\sigma_0^2} \qquad \text{for all } j;$$

for $n \neq 0$ we take

$$\lambda_j^s(n) = \lambda_j^s(n-1)\left(1 + \beta_\lambda \frac{\sigma_j^2(n) - \sigma_0^2}{[\Sigma_{i=1}^m(\sigma_i^2(n) - \sigma_0^2)^2]^{1/2}}\right),$$

where m is the number of elements of the boundary Γ.

STEP 6. (A)
Find $\boldsymbol{\lambda}^e(n)$ from (3.2)

$$\mathbf{K}[\mathbf{h}(n)]\boldsymbol{\lambda}^e(n) + \nabla_u(\boldsymbol{\lambda}^{s^T}(n)\boldsymbol{\sigma}^2(n)) = 0.$$

STEP 7. (O)
Find $G_r(n)$ from the optimality conditions (3.3).

$$G_r(n) = \frac{\partial f(n)}{\partial h_r(n)} + \boldsymbol{\lambda}^{e^T}(n)\frac{\partial \mathbf{K}(n)}{\partial h_r(n)}\mathbf{u} + \boldsymbol{\lambda}^{s^T}(n)\frac{\partial \boldsymbol{\sigma}^2(n)}{\partial h_r(n)}.$$

STEP 8. (O)
Find derivatives of $\mathbf{h}$ with respect to a_i for the optimized structure. Calculate design variables a_i for the next iteration

$$a_i(n+1) = a_i(n) - \alpha(i)\sum_{r=1}^{r_0+1} G_r(n)F_r(n),$$

where $F_r(n) = \dfrac{\partial h_r(n)}{\partial a_i(n)}$, $\quad i = 1, \ldots, I$, find $\quad \mathbf{h}(n+1) = \mathbf{h}(\mathbf{a}(n+1))$ and $\Gamma(n+1) = \Gamma(\mathbf{h}(\mathbf{a}(n+1)))$.

STEP 9. (O)
$n := n+1$, go to Step 2.

7. Numerical examples

7.1. *Shape design of the minimum volume cantilever beam*

We consider a cantilever beam of length L loaded by a concentrated load P at the free end. The beam is of a rectangular cross-sections with a variable depth H and the constant width t. Our aim is to find the shape of the minimum volume beam with the constraints imposed on equivalent stress which should not exceed σ_0. According to the classical beam theory, the distribution of the depth H of such a beam is:

$$H^2(x) = \frac{6P(L-x)}{t\sigma_0}, \qquad 0 \le x \le L, \tag{7.1}$$

where x denotes the distance of the section from the clamped edge.

The solution of our numerical problem will be referred to the above theoretical distribution of H. In order to verify the method, we assume a parabolic shape function as in the theoretical solution for a beam of a given length L, but with an unknown coefficient a_1 in the transformed formula (7.1):

$$x(H) = a_1 H^2 + L. \tag{7.2}$$

The domain of the beam is divided by the rectangular type of grid [1] having $p_0 = 12$ "horizontal" layers and $r_0 = 48$ vertical layers. It means that the beam is divided into 1152 triangles with 637 nodes that leads to 1274 components of the displacement vector $\mathbf{u}$. Following values of the problem parameters were assigned: Young's modulus E: $0.21 \times 10^5\,\mathrm{kN/cm^2}$, Poisson's ratio: 0.25, $\sigma_0 = 21\,\mathrm{kN/cm^2}$, $t = 1.42\,\mathrm{cm}$, $L = 170\,\mathrm{cm}$, $H_{\max} = 17\,\mathrm{cm}$, $P = 6\,\mathrm{kN/cm}$. Numerical calculations are performed using the following dimensionless quantities: $\overline{E} = E/\sigma_0 = 1000$, $\overline{\sigma}_0 = \sigma_0/\sigma_0 = 1$, $\overline{t} = t/h_0 = 1$, $\overline{L} = L/h_0 = 120$, $\overline{P} = P/(\sigma_0 h_0) = 0.2$, where $h_0 = H_{\max}/p_0 = 1.42\,\mathrm{cm}$.

In order to check the efficiency of the method we start with the initial value a_1 which considerably differs (by about 50%) from the theoretical one. The theoretical shape and the shapes obtained from six consecutive iterations are presented in Fig. 3a. Maximum equivalent stresses at several sections of the beam and material volume vs. the number of iterations are presented in Figs. 3b and 3c, respectively.

7.2. *Shape design of the hole in the minimum volume square plate*

Let us find parameters of an elliptic hole in a square plate under biaxial tension (Fig. 4a). The plate has to be of minimum volume and the equivalent

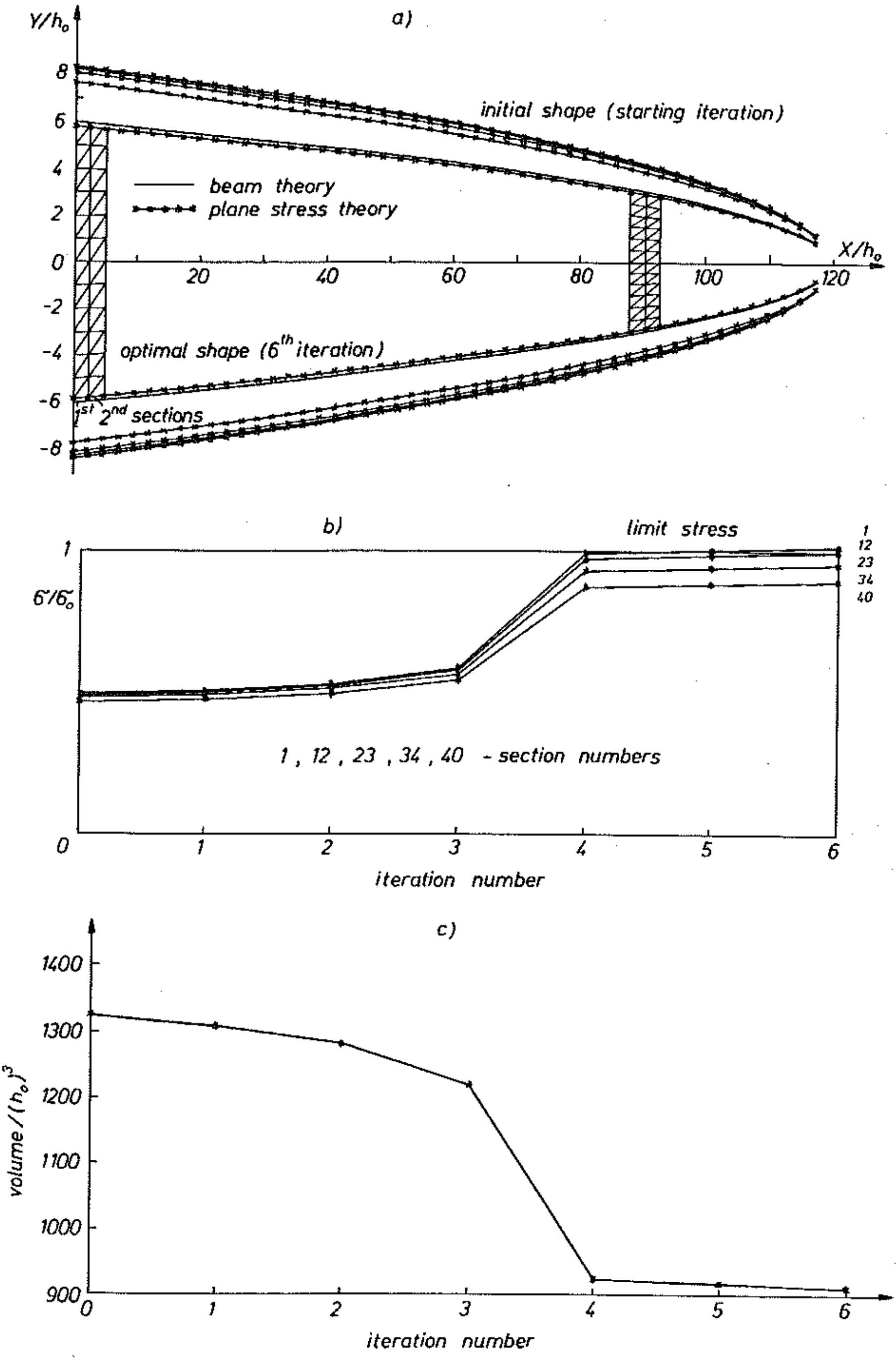
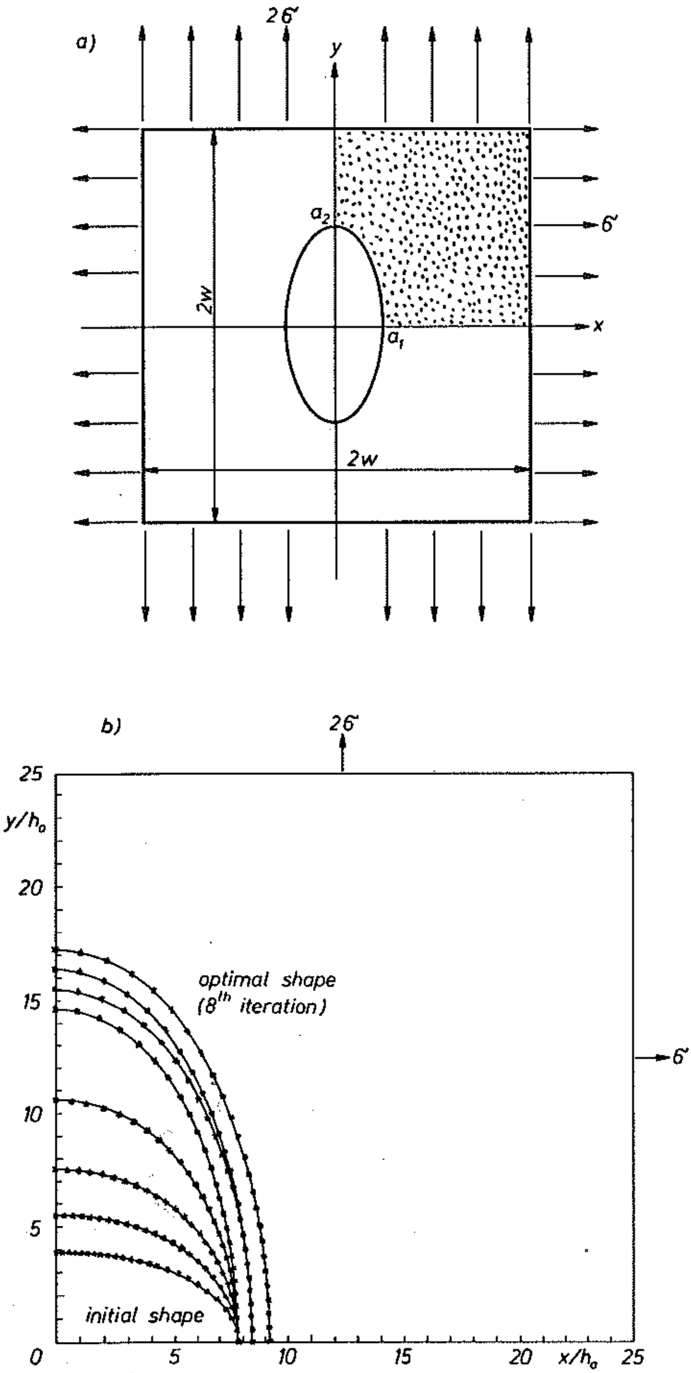


FIG. 3. Shape design for a minimum volume of a cantilever beam. a) Shape iterations, b) equivalent stress in some sections of the beam vs. iteration number, c) volume of the beam vs. iteration number.



a)
2σ
y
a₂
σ
2w
x
a₁
2w
b)
2σ
25
y/h₀
20
15
10
5
0
optimal shape
(8th iteration)
σ
initial shape
5
10
15
20
x/h₀
25

[FIG. 4]

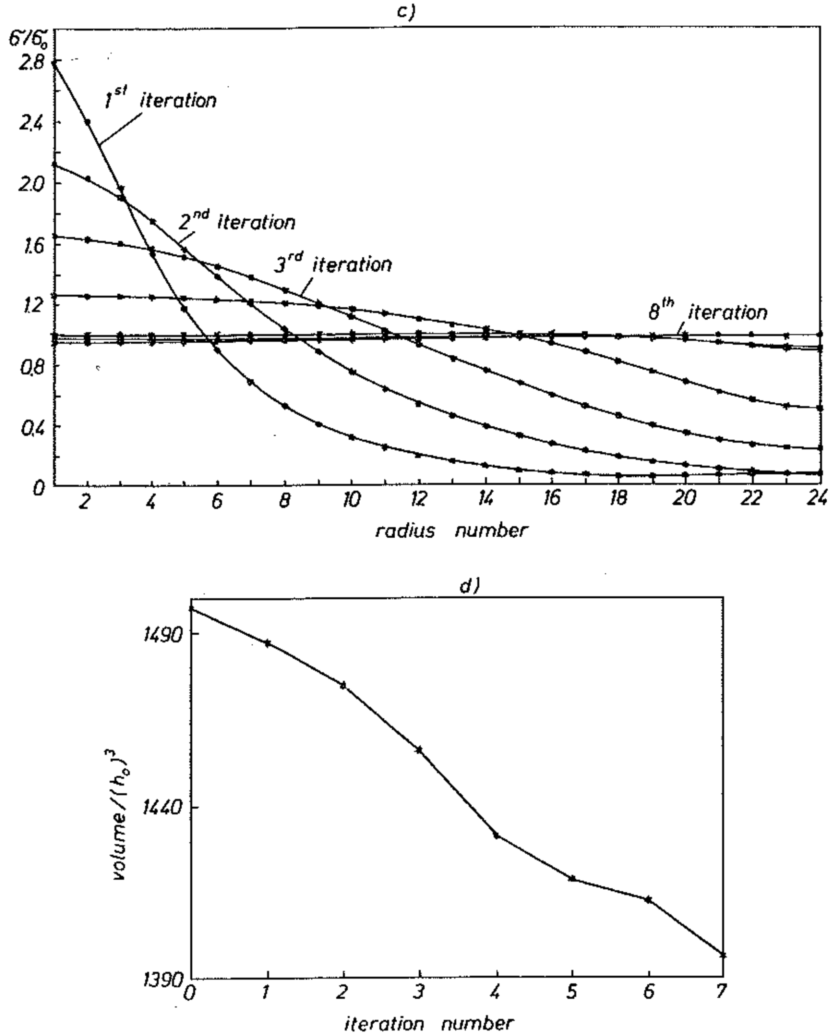


FIG. 4. Shape design for a plate with an elliptic hole under biaxial tension. a) Design model of the plate, b) shape iterations, c) equivalent stress on the free boundary for consecutive iterations, d) volume of the plate vs. iteration number.

stress in all FE should be smaller than a given value σ_0. Due to the double symmetry we are considering one quarter of the plate with polar-type FE grid presented in Fig. 1. The shape of the hole is given parametrically:

$$x = a_1 \cos\alpha,$$
$$y = a_2 \sin\alpha.$$

The semi-axes a_1 and a_2 are our design variables.

According to the Fig. 1, we have $p_0 = 21$ circumferential layers and $r_0 = 24$ radial layers. It means that we have 1008 triangles with 550 nodes that leads to 1100 components of the displacement vector $\mathbf{u}$. Following values of the problem parameters were assigned: Young's modulus E: $0.21 \times 10^5\,\mathrm{kN/cm^2}$, Poisson's ratio: 0.25, $\sigma_0 = 21\,\mathrm{kN/cm^2}$, thickness $t = 4\,\mathrm{cm}$, $W = 156\,\mathrm{cm}$, $\sigma = 6.3\,\mathrm{kN/cm^2}$. Numerical calculations are performed using the following dimensionless quantities: $\overline{E} = E/\sigma_0 = 1000$, $\overline{\sigma}_0 = \sigma_0/\sigma_0 = 1$, $\overline{t} = t/h_0 = 1$, $\overline{W} = W/h_0 = 39$, $\overline{\sigma} = \sigma/\sigma_0 = 0.3$, where $h_0 = W/p_b \cong 4\,\mathrm{cm}$, where $p_b = 36$ is the base number of circumferential layers.

In order to check the efficiency of the method, we start with the initial values of parameters a_1 and a_2, the ratio of which is quite different from the theoretical solution, i.e. we take $a_1/a_2 = 2$. We assume $a_1/W = 0.2$.

In Fig. 4b eight iterations of the hole shape are shown, reaching finally the proper ratio of $a_1/a_2 = 0.5$. In Fig. 4c we see the changes of the equivalent stresses in consecutive iterations in all elements of the boundary. Finally, in Fig. 4d the decrease of material volume with increasing number of iterations is presented.

8. Concluding remarks

The efficiency of the presented method results from its automatic adaptability; the change of dimensions of finite elements due to the change of positions of their nodes is a consequence of the iterative change of the design variable $\mathbf{a}$. It is essential that the design variable $\mathbf{a}$ is representing the natural shape-determining parameter. The FE remeshing and the change of design variables during successive iterations are controlled by optimality conditions arising from the Kuhn – Tucker necessary conditions. At the same time the method allows us to avoid undesirable geometrical distortions of FE. In addition to that, the analytical expressions of all derivatives with respect to independent variables improve the accuracy of the method. Finally, what is important, the standard FEM code is imbedded in the computational algorithm.

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