*ENGINEERING TRANSACTIONS* • *Engng. Trans.* • **43**, 1 – 2, 83–99, 1995
Polish Academy of Sciences • Institute of Fundamental Technological Research

# ON THE STRESS DISTRIBUTION IN BENDING OF STRONGLY ANISOTROPIC BEAMS (*)

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The displacement function method for the plane problems of linearly elastic orthotropic bodies has been proposed. The method has been used for the estimation of the rate of decay of the end effects in beams and slabs subjected to bending; the results obtained turned out to be in agreement with the earlier estimates obtained by Choi and Horgan for the influence of the end effects in the case of laboratory test specimens. Some possibilities of obtaining the rigorous solutions modeling special cases of bending have been shown. A set of explicit formulae has been proposed for the approximate solutions of bending problems taking into account the end effects, normal stress nonlinear distribution at the cross-sections and the contribution of the shear deformation to the beam deflection.

## 1. Introduction

Rapidly growing application field of laminated and fibre-reinforced composites gives rise to the two entirely different groups of problems which are actually being solved or have to be solved by continuum and structural mechanics. The first class of problems – predicting of the effective parameters of the composites, considered as simple or non-simple materials, becomes a subject of research interest of many scientific teams. Particularly, dozens of researchers and research groups employing homogenization techniques should be pointed out. Results of such considerations can be, and really are, very important for material design and manufacturing.

As it has already been mentioned, there exists however a second class of problems, which have to be solved to meet the needs of engineering structural design. The mechanical sciences have to point out the most effective ways of application of the specific mechanical properties of strongly anisotropic materials. Despite the fact that the fibre-reinforced composites are known

(*) Paper presented at 30th Polish Solid Mechanics Conference, Zakopane, September 5 – 9, 1994.

since at least five decades, still no simple engineering methods exist making possible the semi-qualitative considerations at the preliminary stages of the engineering structural design (1). We shall focus our attention here on the problems of strongly anisotropic uniform composite beams (and – what is equivalent – cylindricaly bent plates).

Three main differences (as compared with the classical beam theory) should be pointed out here:

- Nonlinear normal stress distribution ("stress channeling").
- Large (within the linear theory) deflections due to shear deformation.
- Long range influence of the stress and displacement profiles at the end cross-sections.

None of these items is new, all of them have a long history in the literature of last decades. The first two of them were discussed e.g. in [10, 11, 14, 17]. The problem of the contribution of the shear strain to the beam (plate) deflection has a particularly long history beginning with the works of TIMOSHENKO and REISSNER [1, 2, 4] (2). Among the numerous papers on the Saint-Venant's principle and its re-formulation concerning the strongly anisotropic structures (cf. [5–9, 15]), the work by I. Choi and C.O. Horgan, who were able to estimate effectively the decay rate of the influence of the end effects in plane problems has to be pointed out especially (3). Until now, however, no single approach simultaneously taking into account all these peculiarities has been proposed. In the following sections the authors will try to fill, at least partially, this gap.

## 2. FORMULATION OF THE PROBLEM

Let a linearly elastic orthotropic material be given. Let the directions of the coordinate axes coincide with the axes of orthotropy. We shall consider, in the framework of the small strain approach, a class of plane strain or plane stress problems in the $\{x_1, x_2\}$ plane. We shall assume that the body under considerations occupies the rectangle of the length $l$ and the height $2h$ (Fig. 1), where $l \gg h$. For convenience we place the coordinate origin

(1) An illustrative example of the question which cannot be analyzed on the basis of "isotropic" intuition is the behaviour of the composite material subjected to compression along the fibres with the stress of absolute value exceeding the shear modulus [14, 13]; for any real isotropic material such a question is meaningless.

(2) It has been considered subsequently almost in every paper dealing with bending of strongly anisotropic beams or plates.

(3) When the present authors obtained the estimates for the range of influence of the end loads in plane problems, they were not aware of the results reported in [12].

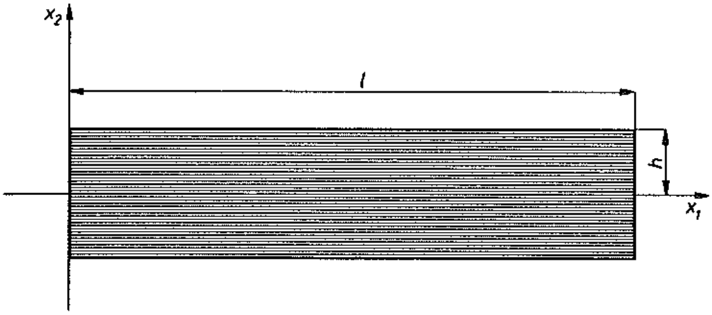


FIG. 1.

in such a way, that for all points of the body we have $-h \leq x_2 \leq h$. We shall confine our interests to the bending mode of loading, i.e. we assume that the stress components $\sigma_{11}$ and $\sigma_{22}$ are odd functions of $x_2$, while $\sigma_{12}$ is even with respect to this variable; this imposes certain symmetries on the load scheme. We shall assume at last that there are no tangent loads at the longer sides, $\sigma_{12}|_{x_2=\pm h} = 0$.

Plane elasticity constitutive relations can be expressed as follows:

$$
\begin{aligned}
\varepsilon_{11} &= \frac{1}{E_1}\left(\sigma_{11} + \frac{\gamma_1^2+\gamma_2^2-2\gamma_3^2}{2}\sigma_{22}\right),\\
\varepsilon_{22} &= \frac{1}{E_1}\left(\gamma_1^2\gamma_2^2\sigma_{22} + \frac{\gamma_1^2+\gamma_2^2-2\gamma_3^2}{2}\sigma_{11}\right),\\
\varepsilon_{12} &= \frac{\gamma_3^2}{E_1}\sigma_{12},
\end{aligned}
\tag{2.1}
$$

or, after inversion,

$$
\begin{aligned}
\sigma_{11} &= \frac{4E_1}{4\gamma_1^2\gamma_2^2-(\gamma_1^2+\gamma_2^2-2\gamma_3^2)^2}\left(\gamma_1^2\gamma_2^2\varepsilon_{11} - \frac{\gamma_1^2+\gamma_2^2-2\gamma_3^2}{2}\varepsilon_{22}\right),\\
\sigma_{22} &= \frac{4E_1}{4\gamma_1^2\gamma_2^2-(\gamma_1^2+\gamma_2^2-2\gamma_3^2)^2}\left(\varepsilon_{22} - \frac{\gamma_1^2+\gamma_2^2-2\gamma_3^2}{2}\varepsilon_{11}\right),\\
\sigma_{12} &= \frac{E_1}{\gamma_3^2}\varepsilon_{12},
\end{aligned}
\tag{2.1'}
$$

where dimensionless constants $\gamma_1, \gamma_2, \gamma_3$ can be expressed using two-dimensional elastic moduli

$$
\gamma_1^2\gamma_2^2 = \frac{E_1}{E_2}, \qquad \gamma_1^2+\gamma_2^2 = 2\left(\frac{E_1}{2\mu} - \nu_{21}\right), \qquad \gamma_3^2 = \frac{E_1}{2\mu}.
$$

Our present considerations are aimed towards the description of the fibre-reinforced structures, thus we shall bear in mind all the time that we assume the following inequalities to be fulfilled:

$$\frac{E_1}{\mu} \gg 1, \qquad \frac{E_1}{E_2} \gg 1. \tag{2.2}$$

By $E_1$, $E_2$, $\mu$, $\nu_{12}$ and $\nu_{21}$ ($\nu_{12}E_1 = \nu_{21}E_2$) we denote two-dimensional moduli and Poisson's coefficients; in the case of plane stress they coincide with the corresponding three-dimensional values (4).

In the course of forthcoming consideration we shall use, in fact, the well-known method of the Airy stress function. Since we shall have to deal not only with the static boundary conditions in terms of tractions, but also with "kinematic" conditions, expressed by displacements, we shall modify this technique to some extent. We shall introduce to this aim such a displacement function $G(x_1, x_2)$ that the displacement fields $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ can be expressed as follows:

$$\begin{aligned} u_1 &= \frac{1}{E_1}\left(G_{,22} + \frac{\gamma_1^2 + \gamma_2^2 - 2\gamma_3^2}{2} G_{,11}\right)_{,2} = \frac{1}{E_1}(G_{,22} - \nu_{21}G_{,11})_{,2}, \\ u_2 &= \frac{1}{E_1}\left(\gamma_1^2\gamma_2^2 G_{,11} + \frac{\gamma_1^2 + \gamma_2^2 - 2\gamma_3^2}{2} G_{,22}\right)_{,1} = \frac{1}{E_2}(G_{,11} - \nu_{12}G_{,22})_{,1}, \end{aligned} \tag{2.3}$$

where comma denotes the partial derivative (5). Using this representation and the inverted constitutive relations (2.1′) one can rewrite the equilibrium conditions

$$\begin{aligned} \sigma_{11,1} + \sigma_{12,2} &= 0, \\ \sigma_{12,1} + \sigma_{22,2} &= 0 \end{aligned} \tag{2.4}$$

in the following form:

$$\begin{aligned} \left[G_{,2222} + (\gamma_1^2 + \gamma_2^2)G_{,1122} + \gamma_1^2\gamma_2^2 G_{,1111}\right]_{,2} &= 0, \\ \left[G_{,2222} + (\gamma_1^2 + \gamma_2^2)G_{,1122} + \gamma_1^2\gamma_2^2 G_{,1111}\right]_{,1} &= 0. \end{aligned} \tag{2.5}$$

This set of two equations is equivalent to the following single equation:

$$G_{,2222} + (\gamma_1^2 + \gamma_2^2)G_{,1122} + \gamma_1^2\gamma_2^2 G_{,1111} = C, \tag{2.6}$$

where $C$ is an arbitrary constant.

(4) For general case, as well as concerning the expression of coefficients $\gamma_i$ in terms of the Kelvin moduli see [16, 19].

(5) Representation (2.3) imposes some differential constraints on the displacement fields. These constraints, however, can be independently obtained from Lamé equations, thus they do not reduce the generality of the representation.

One can prove easily that the following relations hold true:

$$\begin{aligned} \sigma_{11} &= G_{,1222}\,, \\ \sigma_{22} &= G_{,1211}\,, \\ \sigma_{12} &= \frac{C}{2\gamma_3^2} - G_{,1212} \end{aligned} \tag{2.7}$$

thus, assuming $C = 0$ one gets $\Phi = G_{,12}$, where $\Phi$ denotes the Airy stress function; in general case

$$\Phi = G_{,12} - \frac{C}{2\gamma_3^2} x_1 x_2\,. \tag{2.8}$$

Thus $\Phi$ satisfies the following differential equation (6):

$$\Phi_{,2222} + (\gamma_1^2 + \gamma_2^2)\Phi_{,1122} + \gamma_1^2\gamma_2^2\Phi_{,1111} = 0. \tag{2.9}$$

It is a proper place here to quote an instructive example. It is well known that for the strongly anisotropic cantilever beam loaded at the free end one obtains significantly nonlinear normal stress distribution at the cross-sections (cf. [11]) (7). It is not difficult, however, to point out another plane solution, fulfilling the same integral load conditions at the ends which yields *linear* distibution of normal stress at the cross-sections. Indeed, let us take

$$G(x_1, x_2) = \frac{\sigma}{8h^2}\left[\frac{x_1^2 x_2^4}{2} - \frac{x_1^6}{30\gamma_1^2\gamma_2^2} - \frac{(\gamma_1^2+\gamma_2^2)x_2^6}{30} - 3h^2 x_1^2 x_2^2 + Ax_1^4 + \left(\frac{\gamma_1^2+\gamma_2^2}{2} - \gamma_1^2\gamma_2^2 A\right)x_2^4 + Bx_1^3\right], \tag{2.10}$$

where $A$ and $B$ are arbitrary constants. For such a function $G(x_1, x_2)$ one obtains $C = 0$ i.e. $\Phi = G_{,12}$, thus

$$\begin{aligned} \sigma_{11} &= G_{,1222} = \frac{3\sigma x_1 x_2}{h^2}\,, \\ \sigma_{22} &= G_{,1112} = 0, \\ \sigma_{12} &= -G_{,1122} = -\frac{3\sigma}{2h^2}(x_2^2 - h^2). \end{aligned} \tag{2.11}$$

(6) In fact, expressions (2.3) have been originally obtained by integration of Eq. (2.1), stress components being expressed using Airy function derivatives.

(7) Very similar results were obtained by one of the present authors for bending of the strongly anisotropic plates [18].

It is not difficult to see that the stress field (2.11) corresponds to the cantilever beam loaded at the side $x_1 = 0$ with the tangent force $F = 2h\sigma$, i.e $\sigma$ is the mean tangent stress at the cross-sections. By the proper choice of the constants $A$ and $B$:

$$(2.12)\qquad \begin{aligned} A &= \frac{h^2}{2\gamma_1^2\gamma_2^2}\left(\frac{\gamma_1^2+\gamma_2^2+3\gamma_3^2}{3}+\frac{l^2}{h^2}\right), \\ B &= \frac{h^2}{3\gamma_1^2\gamma_2^2}\left(\gamma_1^2+\gamma_2^2-10\gamma_3^2-\frac{4l^2}{h^2}\right), \end{aligned}$$

one can set the horizontal displacement of the corners at some cross-section $x_1 = l$ to be zero and the vertical displacement to vanish in $(l, 0)$. In this case, the horizontal displacement profile at $x_1 = l$ can be described by the following expression:

$$(2.13)\qquad u_1(l, x_2) = \frac{\sigma}{4h^2E_1}\left[(\gamma_1^2+\gamma_2^2)+2\gamma_3^2\right](h^2-x_2^2)x_2\,,$$

while vertical displacement at this cross-section can be expressed as

$$(2.14)\qquad u_2(l, x_2) = \frac{3\sigma}{4h^2E_1}\left[(\gamma_1^2+\gamma_2^2)+2\gamma_3^2\right]lx_2^2\,.$$

The horizontal displacements are thus pinned in three points only ($x_2 = 0$, $x_2 = h$, $x_2 = -h$) instead of vanishing across the whole section. It seems that this constitutes the main difference with respect to the case considered in [11]. This example demonstrates evidently, that the mode of the end support, (even within the same "beam scheme" – in both cases the mean rotations of the cross-section vanish) can have significant impact on the stress distribution in all cross-sections.

## 3. Estimation of the reach of end effects

We shall consider a class of the stress fields defined on the elongated rectangular domain modeling a linearly elastic beam supported at the ends. Assuming the following form of the displacement function:

$$(3.1)\qquad G(x_1, x_2) = Ue^{-kx_1}\cos(\alpha k x_2),$$

where $U$ denotes a constant multiplier, and choosing zero value for the constant $C$ in the equation (2.6), we arrive at the following algebraic equation for $\alpha$:

$$(3.2)\qquad \alpha^4 - (\gamma_1^2+\gamma_2^2)\alpha^2 + \gamma_1^2\gamma_2^2 = 0.$$

Taking $\gamma_1 > \gamma_2 > 0$ one obtains readily the following (real) ascending sequence of roots of Eq. (3.2):

$$\alpha_1 = -\gamma_1, \qquad \alpha_2 = -\gamma_2, \qquad \alpha_3 = \gamma_2, \qquad \alpha_4 = \gamma_1. \tag{3.3}$$

The discussion about the reach of the influence of the differences between the modes of the stress distribution at the ends is meaningful as long as we consider the equivalent total loadings which differ from each other only by the self-equilibrated stress distributions with vanishing resultant force and total bending moment. Thus, in fact, in virtue of the principle of superposition it is enough to consider only the reach of the stress fields generated by such loads. Let us then impose on the stress fields, generated by the displacement functions of the form (3.1), the following conditions at the end, (e.g. for $x_1 = 0$):

$$\begin{aligned} \int_{-h}^{h} \sigma_{12}\, dx_2 &= 0, \\ \int_{-h}^{h} \sigma_{11}\, dx_2 &= 0, \\ \int_{-h}^{h} \sigma_{11} x_2\, dx_2 &= 0. \end{aligned} \tag{3.4}$$

A simple reasoning concerning the global equilibrium of each finite rectangle $\{-h \le x_2 \le h,\ a \le x_1 \le b\}$, where $a$ and $b$ are arbitrary constants, shows that in the case of separated variables: $G(x_1, x_2) = \Xi(x_1)\Psi(x_2)$, where $\Xi(x_1)$ is a monotonic function of $x_1$ and $\Psi(x_2)$ is an even function of $x_2$, conditions (3.4) are equivalent to the following boundary conditions at the longer sides (i.e. for $x_2 = \pm h$):

$$\begin{aligned} \sigma_{12}(x_1, \pm h) &= 0, \\ \sigma_{22}(x_1, \pm h) &= 0. \end{aligned} \tag{3.5}$$

In general, conditions (3.5) can not be fulfilled by the single product such as in (3.1), thus we have to consider linear combinations of two solutions having the same exponential term $\exp(-kx_1)$,

$$G(x_1, x_2) = e^{-kx_1}(a\cos(\gamma_1 k x_2) + b\cos(\gamma_2 k x_2)). \tag{3.6}$$

Taking

$$\begin{aligned} G(x_1, x_2) = Ae^{-kx_1}\Big(&\gamma_2^2\cos(\gamma_2 kh)\cos(\gamma_1 k x_2) \\ &-\gamma_1^2\cos(\gamma_1 kh)\cos(\gamma_2 k x_2)\Big), \end{aligned} \tag{3.7}$$

one satisfies identically the first condition $(3.5)_1$, while the second condition $(3.5)_2$ can not be fulfilled for arbitrary value of $k$; to this end the following relation should be satisfied:

$$\gamma_2 \cos(\gamma_2 kh) \sin(\gamma_1 kh) = \gamma_1 \cos(\gamma_1 kh) \sin(\gamma_2 kh). \tag{3.8}$$

Denoting

$$\beta \equiv k\gamma_2 h, \qquad \gamma \equiv \frac{\gamma_1}{\gamma_2}, \tag{3.9}$$

we can rewrite condition (3.8) as the following transcendental equation for a dimensionless unknown $\beta$:

$$\gamma \operatorname{tg}(\beta) = \operatorname{tg}(\gamma\beta). \tag{3.10}$$

Equation (3.10) has multiple roots. Figure 2 shows the first and second one plotted versus $\gamma$. Eventually one obtains the following expression for the normal stress at the cross-sections produced by any mode of self-equilibrated end load generated by the displacement function (3.7):

$$\sigma_{11} = -A \exp\left(-\frac{\beta x_1}{\gamma_2 h}\right) \gamma^2 \beta^4 \times \left(\gamma_1 \cos(\beta) \sin\left(\beta\gamma \frac{x_2}{h}\right) - \gamma_2 \cos(\gamma\beta) \sin\left(\beta \frac{x_2}{h}\right)\right). \tag{3.11}$$

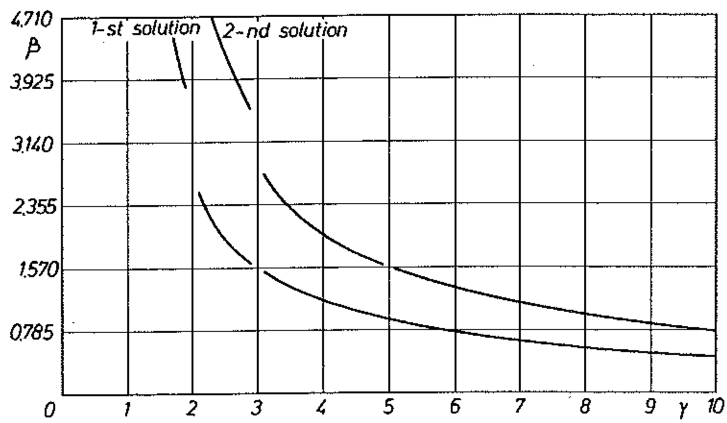


FIG. 2.

Similarly one can consider *odd* displacement function giving rise the *even* normal stress distribution at the cross-section. To this end it is enough to interchange sin and cos symbols in the formulae (3.6)–(3.8). Denoting this time $k\gamma_2 h$ by $\delta$ we obtain, instead of (3.10), another transcendent equation:

$$\operatorname{tg}(\delta) = \gamma \operatorname{tg}(\gamma\delta). \tag{3.12}$$

The first two values of $\delta$ for different $\gamma$ are shown at Fig. 3. For this case the normal stress distribution has the following form:

$$\sigma_{11} = \sigma_0 \exp\left(-\frac{\delta x_1}{\gamma_2 h}\right)\left(\gamma_1 \sin(\delta)\cos\left(\delta\gamma\frac{x_2}{h}\right) - \gamma_2 \sin(\gamma\delta)\cos\left(\delta\frac{x_2}{h}\right)\right). \tag{3.13}$$

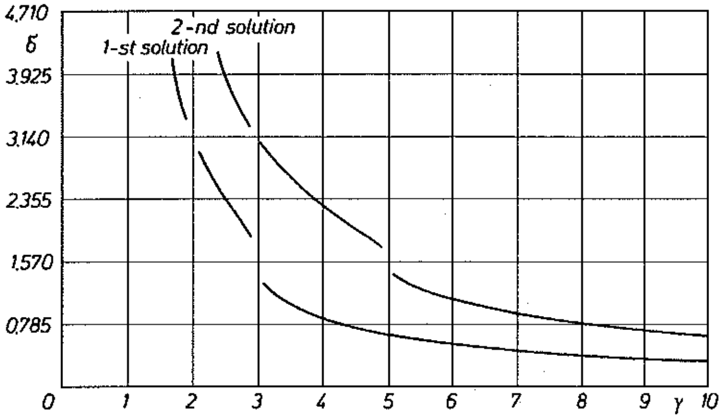


FIG. 3.

Thus, using a different approach and considering a different engineering problem, we have arrived at the same estimates as the one obtained by CHOI and HORGAN [12] who were concerned in the decay rate of the influence of conditions at the clamped ends of laboratory specimens rather than in the analysis of the stress distribution in engineering structures. We shall postpone for the time being the detailed discussion of the results (3.11) and (3.13); we shall mention only that, for certain values of elastic constants, they stay far away from the expectations of the Saint Venant principle in its usual formulation. We shall return to this question in the concluding part of the present paper.

Concluding this section we would like to show one possible way of using the modified representation of the form (3.1) for direct derivation of the solutions of some selected problems. Assuming imaginary value of $k$ in (3.1), taking linear combination of solutions and changing notation, one can obtain the following family of solutions describing bending under different modes of normal loads and different end conditions,

$$G(x_1, x_2) = \cosh(\alpha k x_2)\,(A_1 \sin(kx_1) + A_2 \cos(kx_1)), \tag{3.14}$$

where $\alpha$ is the same as in (3.1), i.e. the solution of (3.2).

Taking for instance

$$G(x_1, x_2) = \frac{P_0 l^4}{2\pi^4 \gamma_1 \gamma_2} \cos\left(\frac{\pi x_1}{l}\right) \times \frac{\gamma_2^2 \cosh\left(\frac{\gamma_2 \pi h}{l}\right) \cosh\left(\frac{\gamma_1 \pi x_2}{l}\right) - \gamma_1^2 \cosh\left(\frac{\gamma_1 \pi h}{l}\right) \cosh\left(\frac{\gamma_2 \pi x_2}{l}\right)}{\gamma_2 \cosh\left(\frac{\gamma_2 \pi h}{l}\right) \sinh\left(\frac{\gamma_1 \pi h}{l}\right) - \gamma_1 \cosh\left(\frac{\gamma_1 \pi h}{l}\right) \sinh\left(\frac{\gamma_2 \pi h}{l}\right)}, \tag{3.15}$$

one obtains a scheme of a beam, simply supported at $x_1 = 0$ and $x_1 = l$, and loaded by a normal load

$$P(x_1) = P_0 \sin\left(\frac{\pi x_1}{l}\right), \tag{3.16}$$

equally distributed between the lower an upper surfaces ($x_2 = \pm h$). For the normal stress distribution at the half-span cross-section we have

$$\sigma_{11}\left(\frac{l}{2}, x_2\right) = \frac{P_0 \gamma_1 \gamma_2}{2} \times \frac{\gamma_1 \cosh\left(\frac{\gamma_2 \pi h}{l}\right) \sinh\left(\frac{\gamma_1 \pi x_2}{l}\right) - \gamma_2 \cosh\left(\frac{\gamma_1 \pi h}{l}\right) \sinh\left(\frac{\gamma_2 \pi x_2}{l}\right)}{\gamma_2 \cosh\left(\frac{\gamma_2 \pi h}{l}\right) \sinh\left(\frac{\gamma_1 \pi h}{l}\right) - \gamma_1 \cosh\left(\frac{\gamma_1 \pi h}{l}\right) \sinh\left(\frac{\gamma_2 \pi h}{l}\right)}, \tag{3.17}$$

while at the end cross-sections we obtain the following stress and normal displacement distributions:

$$\begin{aligned} &\sigma_{11} = 0, \\ &\sigma_{12} = \pm\frac{P_0 \gamma_1 \gamma_2}{2} \times \frac{\cosh\left(\frac{\gamma_2 \pi h}{l}\right) \cosh\left(\frac{\gamma_1 \pi x_2}{l}\right) - \cosh\left(\frac{\gamma_1 \pi h}{l}\right) \cosh\left(\frac{\gamma_2 \pi x_2}{l}\right)}{\gamma_2 \cosh\left(\frac{\gamma_2 \pi h}{l}\right) \sinh\left(\frac{\gamma_1 \pi h}{l}\right) - \gamma_1 \cosh\left(\frac{\gamma_1 \pi h}{l}\right) \sinh\left(\frac{\gamma_2 \pi h}{l}\right)}, \\ &u_1 = 0. \end{aligned} \tag{3.18}$$

We do not see any special reasons for a detailed discussion of the above solution, because we can hardly point out any reasonable conditions at the supported ends, which can be described by the tangent tractions given by $(3.18)_2$ and vanishing normal displacement. One can see that in the case of $\gamma_1$ large enough, the normal stress distribution given by (3.17) reveals strong nonlinearity, similar to that described in [18] for cylindrical bending of strongly anisotropic plates. We are not able however to tell, at this stage of our consideration, if this is an innate property of all such solutions (modeling, in integral sense, transversely loaded beams with simply supported ends) or, possibly, an effect of the specific distribution, either of tangent tractions at the ends and/or of normal transverse load. Building up the solutions of real engineering problems as a superposition of the terms like (3.15) may be inefficient for two reasons: first – in practical applications the normal load is described using rather simple, piecewise linear profiles, than the trigonometric functions, thus it may happen that a large number of terms may be needed for the description of a simple situation; the second – a sophisticated approach may be necessary to provide simultaneously the proper normal load profile and stress (displacement) distribution at the end.

In the next section we shall propose an approximate method which, being in fact a generalized beam theory approch, makes it possible to take into account some additional information about the end conditions (besides the overall ones, integrated over the whole cross-section) and reproduces the observed normal stress nonlinearity at the cross-sections, remaining at the same time within the familiar area of the notions of structural mechanics.

## 4. The polynomial solutions

We are looking for a displacement function in the form of the power expansion in $x_2$, i.e. in the form

$$G(x_1, x_2) = \sum_{i=0}^{n} f_{2i}(x_1) x_2^{2i}, \tag{4.1}$$

where $f_{2i}(x_1)$, $(i = 0, \ldots, n)$ are unknown functions.

As the first question, what is usual in the case of such representations, arises the problem of the degree of the polynomial, i.e of the order of approximation. Let us notice that, in order to find all functions $f_{2i}(x_1)$ one needs $n+1$ equations. If we introduce, as usual in the beam theory, two boundary conditions at $x_2 = \pm h$ – vanishing of the tangential tractions and the condition of the normal load distribution and two integral conditions of equilibrium: for the transversal force and bending moments (8), then we obtain a system of four differential equations. This may suggest taking $n = 3$, i.e. the 6-th order polynomials. The obtained system of equations can be effectively solved producing oscillating solutions of decaying (growing) amplitude, which can hardly approximate monotonically decaying fields generated by the self-equilibrated tractions at the end cross-sections. Thus we should take at least $n = 4$, i.e. the 8-th order polynomials. Here the problem of a missing differential equation arises at once. In many problems of bending we may expect the maximal shear stress in the vicinity of the mediane $x_2 = 0$, thus we propose to adopt the condition of exact fulfilling of the second equilibrium equation $(2.4)_2$ along the axis $x_2 = 0$ as the missing equation. Thus we have the following five conditions:

$$\tag{4.2}
\begin{aligned}
\sigma_{12}\Big|_{x_2=\pm h} &= 0,\\
\sigma_{22}\Big|_{x_2=\pm h} &= -\frac{q(x_1)}{2},\\
\int_{-h}^{h} (\sigma_{11,1} + \sigma_{12,2}) x_2 \, dx_2 &= 0,\\
\int_{-h}^{h} (\sigma_{12,1} + \sigma_{22,2}) \, dx_2 &= 0,\\
(\sigma_{12,1} + \sigma_{22,2})\Big|_{x_2=0} &= 0,
\end{aligned}$$

where $q(x_1)$ denotes the total normal load (9). Substituting expressions (2.5) instead of (2.4) into $(4.2)_3$, $(4.2)_4$ and $(4.2)_5$ and expressing $\sigma_{12}$ and $\sigma_{22}$ in $(4.2)_1$ and $(4.2)_2$ by derivatives of $G(x_1, x_2)$ (10), one arrives at the following

(8) The condition of integral equilibrium of the longitudinal force is automatically fulfilled due to the symmetry properties of the normal stresses generated by even powers of $x_2$.

(9) We assume that the normal load is equally distributed between two sides $x_2 = \pm h$, in many cases this restrictive condition can be omitted by a proper choice of additional polynomial (in both variables) terms in the displacement function "shifting" the $\sigma_{22}$ field.

(10) Strictly speaking, the relation $\sigma_{12} = -G_{,1212} + \text{const.}$ holds true only if Eq. (2.6) is rigorously fulfilled, while we assume, that it holds only in the integral sense, but in the case of the expected stress profiles with high stresses at the boundaries, the condition of integral moment equilibrium is almost equivalent to the fulfilment of (2.6) in the vicinity of the outer surfaces.

set of five ordinary differential equations for five unknown functions $f_{2i}(x_1)$:

$$
\begin{aligned}
&f_2'' + 6f_4''h^2 + 15f_6''h^4 + 28f_8''h^6 = 0,\\
&f_2'''h + 2f_4'''h^3 + 3f_6'''h^5 + 4f_8'''h^7 = -\frac{q}{4},\\
&120f_6 + 672f_8h^2 + \left(\gamma_1^2+\gamma_2^2\right)\left(4f_4'' + 12f_6''h^2 + 24f_8''h^4\right)\\
&\qquad + \gamma_1^2\gamma_2^2\left(\frac{1}{3}f_2^{IV} + \frac{2}{5}f_4^{IV}h^2 + \frac{3}{7}f_6^{IV}h^4 + \frac{4}{9}f_8^{IV}h^6\right) = 0,\\
&24f_4' + 120f_6'h^2 + 336f_8'h^4\\
&\qquad +2\left(\gamma_1^2+\gamma_2^2\right)\left(f_2''' + 2f_4'''h^2 + 3f_6'''h^4 + 4f_8'''h^6\right)\\
&\qquad +\gamma_1^2\gamma_2^2\left(f_0^V + \frac{1}{3}f_2^Vh^2 + \frac{1}{5}f_4^Vh^4 + \frac{1}{7}f_6^Vh^6 + \frac{1}{9}f_8^Vh^8\right) = 0,\\
&24f_4' + 2\left(\gamma_1^2+\gamma_2^2\right)f_2''' + \gamma_1^2\gamma_2^2 f_0^V = 0.
\end{aligned}
\tag{4.3}
$$

Eliminating from the system (4.3) functions $f_0$, $f_2$ and $f_4$ as well as their derivatives, after some rearrangements one can obtain the following system of two differential equations for two functions $f_6$ an $f_8$:

$$
\begin{aligned}
&120f_6' + 672f_8'h^2 + \gamma_1^2\gamma_2^2\left(\frac{8}{35}f_6^Vh^4 + \frac{32}{45}f_8^Vh^6\right)\\
&\qquad = \gamma_1^2\gamma_2^2\frac{q''}{10h} - (\gamma_1^2+\gamma_2^2)\frac{q}{4h^3},\\
&120f_6' + 336f_8'h^2 + \gamma_1^2\gamma_2^2\left(\frac{19}{35}f_6^Vh^4 + \frac{71}{45}f_8^Vh^6\right)\\
&\qquad -2(\gamma_1^2+\gamma_2^2)(3f_6'''h^2 + 8f_8'''h^4) = \gamma_1^2\gamma_2^2\frac{9q''}{80h} - (\gamma_1^2+\gamma_2^2)\frac{q}{4h^3}.
\end{aligned}
\tag{4.4}
$$

Seeking for the solution of the homogeneous system associated with (4.4) (i.e for $q = 0$) in the following form:

$$
\begin{aligned}
f_6(x_1) &= k_6 \exp\left(\frac{\alpha}{\sqrt{\gamma_1\gamma_2}}\frac{x_1}{h}\right),\\
f_8(x_1) &= k_8 \exp\left(\frac{\alpha}{\sqrt{\gamma_1\gamma_2}}\frac{x_1}{h}\right),
\end{aligned}
\tag{4.5}
$$

by substituting expressions (4.5) into the reduced system (4.4) one obtains at once the following condition of existance of nontrival solutions (characteristic equation):

$$s^4 - \mu s^3 + 23s^2 - 11\mu s + 16 = 0, \tag{4.6}$$

where:

$$\mu \equiv \frac{24}{\sqrt{315}} \frac{\gamma^2 + 1}{\gamma}, \qquad \gamma \equiv \frac{\gamma_1}{\gamma_2}, \qquad s \equiv \frac{\alpha^2}{\sqrt{315}}.$$

Equation (4.6) for $\gamma$ large enough ($\gamma > 1.94$) has two real roots and two complex ones, i.e. $\alpha$ can assume four real values and another four complex ones. Positive real values of $\alpha_1$ and $\alpha_2$ are depicted in Fig. 4 for different values of dimensionless material constant $\gamma$. Thus in a general case we can write

$$\begin{aligned} f_6 &= \sum_{i=1}^{8} k_6^{(i)} \exp\left(\frac{\alpha_i}{\sqrt{\gamma_1\gamma_2}} \frac{x_1}{h}\right), \\ f_8 &= \sum_{i=1}^{8} k_8^{(i)} \exp\left(\frac{\alpha_i}{\sqrt{\gamma_1\gamma_2}} \frac{x_1}{h}\right), \end{aligned} \tag{4.7}$$

where constants $k_6^{(i)}$ and $k_8^{(i)}$ are not quite arbitrary, their ratio being dependent on $\alpha_i$,

$$\frac{k_8^{(i)}}{k_6^{(i)}} = -\frac{3(3\alpha_i^4 + 5 \cdot 315)}{28(\alpha_i^4 + 3 \cdot 315)h^2}. \tag{4.8}$$

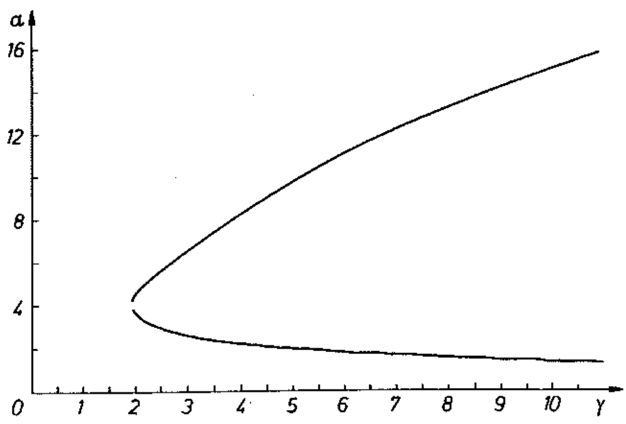


FIG. 4.

Knowing $f_6(x_1)$ and $f_8(x_1)$ and performing with care quite elementary (however rather boring) calculations one obtains the following expressions for the other three functions $f_0(x_1)$, $f_2(x_1)$, $f_4(x_1)$:

$$
\begin{aligned}
f_0(x_1) = 2h^6 \sum_{i=1}^{8} \frac{1}{\alpha_i^4} \Bigg[ 3k_6^{(i)} \left( 12 - \alpha_i^2 \frac{\gamma^2+1}{\gamma} \right) \\
+8h^2 k_8^{(i)} \left( 9 - \alpha_i^2 \frac{\gamma^2+1}{\gamma} \right) \Bigg] \exp\left( \frac{\alpha_i}{\sqrt{\gamma_1\gamma_2}} \frac{x_1}{h} \right) \\
- \frac{1}{15\gamma_1^2\gamma_2^2} \left( -\frac{A_2}{6h^2} x_1^6 + 3B_4 x_1^5 \right) + C_0 x_1^4 + D_0 x_1^3, \\
f_2(x_1) = h^4 \sum_{i=1}^{8} (3k_6^{(i)} + 8k_8^{(i)} h^2) \exp\left( \frac{\alpha_i}{\sqrt{\gamma_1\gamma_2}} \frac{x_1}{h} \right) + A_2 x_1^2 + B_2 x_1, \\
f_4(x_1) = -h^2 \sum_{i=1}^{8} (3k_6^{(i)} + 6k_8^{(i)} h^2) \exp\left( \frac{\alpha_i}{\sqrt{\gamma_1\gamma_2}} \frac{x_1}{h} \right) - \frac{A_2}{6h^2} x_1^2 + B_4 x_1 + C_4.
\end{aligned}
\tag{4.9}
$$

In expressions (4.9) we have omitted possible "void" constants which do not contribute to the expressions for displacement and stress fields.

Knowing all $f_{2i}$, $(i = 0, \ldots, 4)$ one can find the displacement field by substituting expressions (4.9) into the following equalities:

$$
\begin{aligned}
u_1 = \frac{1}{E_1} [(24 f_4 - 2\nu f_2'') x_2 + (120 f_6 - 4\nu f_4'') x_2^3 \\
+(336 f_8 - 6\nu f_6'') x_2^5 - 8\nu f_8'' x_2^7], \\
u_2 = \frac{1}{E_1} [(\gamma_1^2\gamma_2^2 f_0''' - 2\nu f_2') + (\gamma_1^2\gamma_2^2 f_2''' - 12\nu f_4') x_2^2 \\
+(\gamma_1^2\gamma_2^2 f_4''' - 30\nu f_6') x_2^4 + (\gamma_1^2\gamma_2^2 f_6''' - 56\nu f_8') x_2^6 + \gamma_1^2\gamma_2^2 f_8''' x_2^8].
\end{aligned}
\tag{4.10}
$$

Corresponding relations for the stress fields are much simpler:

$$
\begin{aligned}
\sigma_{11} &= 24(f_4' x_2 + 5 f_6' x_2^3 + 14 f_8' x_2^5), \\
\sigma_{22} &= 2(f_2''' x_2 + 2 f_4''' x_2^3 + 3 f_6''' x_2^5 + 4 f_8''' x_2^7), \\
\sigma_{12} &= -2(f_2'' + 6 f_4'' x_2^2 + 15 f_6'' x_2^4 + 28 f_8'' x_2^6).
\end{aligned}
\tag{4.11}
$$

Thus, at least in principle, we have obtained a sequence of formulae approximating the stress and displacement fields generated by the self-equilibrated end tractions (we recall here, that we have assumed no transversal load). The results presented above are complex enough to justify the opinion that the general solutions, taking into account arbitrary tranversal load profile, would be of no practical value.

## 5. Concluding remarks

Even a superficial analysis indicates that the crucial for our models, dimensionless material constant $\gamma$ can assume large values, e.g. for the elastic constants of graphite/epoxy composite quoted in [12] $\gamma$ is greater than 7, while $\gamma_1$ equals about 5 and $\gamma_2$ is approximately 0.7. This gives the decaying slope approximetely proportional to $\exp(x_1/h)$. Nonlinearity of normal stress distribution in the considered case of simply supported beam is also evident, however it is not drastic from the viewpoint of the load capacity estimate. The proposed approach (using the displacement function) makes it possible to find the beam deflection immediately from the formula (2.3). Thus the authors can recommend the following method of solving the problems of the stresses and deflections of these anisotropic beam structures by means of the plane stress (plane strain) approach: first find any solution (e.g. polynomial) satisfying the load distribution profile and the integral conditions at the end cross-sections, then introduce the necessary corrections using the rigorous or approximate form of the stress and displacements fields generated by the self-equilibrated end loads. At the present stage of our knowledge we are not able to point out any method of reasonable complexity for solving the problems of anisotropic beams using any kind of the generalized beam theory, such as e.g. the Vlasov theory of thin-walled profiles [3]. It seems that the problem of creation of such a theoretical model remains still open.

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*Received January 24, 1995.*