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# COMPUTER ANALYSIS OF DAMAGE DEVELOPMENT IN RECTANGULAR PLATES (*)

J. BIAŁKIEWICZ and P. MIKA (KRAKÓW)

A numerical analysis of the time-dependent rupture of a plate of moderate thickness is presented. The solutions focus on the time and localisation of appearance of the first macroscopic cracks as well as on the mode of rupture front propagation to the instant of failure. The damage development is coupled with the variation of the plate thickness.

## 1. INTRODUCTION

In this paper a study is made of the rupture mechanics of metal plates of moderate thickness which operate at temperatures sufficiently high to cause material deterioration due to damage evolution. The problem of particular interest is to determine the motion of the rupture front in orthotropically-damaged material, the current state of which is described by the symmetric second rank damage tensor due to MURAKAMI and OHNO [21].

The basic studies of failure modes and time predictions of structural components subject to multiaxial stress have been performed by the application of scalar damage representations. The theories adopted here generalize the classical approach to damage description introduced by KACHANOV [10] and RABOTNOV [23]. In particular, interesting proposals of various creep damage theories verified by experimental investigations with multiaxial loads have been elaborated by HAYHURST [6, 7], LECKIE and HAYHURST [14], MURAKAMI and OHNO [21], LEMAITRE [15], KRAJCINOVIC [11], LEMAITRE and CHABOCHE [16], HULT [9], GLOCKNER and SZYSZKOWSKI [5]. Some of these theories have found application in the analysis of damage of structural components. The combination of creep and the motion of the rupture surface, interpreted as a ductile creep rupture, has been performed by BIAŁKIEWICZ [1] for a rotating disk of variable thickness.

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The problem addressed in this paper is focussed on the mode of rupture propagation to the instant of the carrying section failure. As it is known, the material damage consists in reduction of the net cross-sectional area caused by nucleation and growth of fissures and grain boundary cavities. Such an internal damage process has directional character resulting also in the development of material anisotropy (cf. CHABOCHE [3]). The solutions limited to orthotropic distributions of microdefects can be investigated by means of a tensorial damage description as proposed by LITEWKA [18, 19]. However, the correctness of this analysis will require the assumption of initially isotropic materials under proportional loading (cf. KRAJCINOVIC [12, 13]).

This paper is based directly on a previous study (cf. BIAŁKIEWICZ and OLEKSY [2]), but with a more complex modelling development of the damage zones. The damage evolution in the plate cross-section is coupled with the variation of the plate thickness. This procedure will initiate the stress redistribution and accelerate the propagation of the damage front.

## 2. TENSORIAL DAMAGE MODEL

The current state for orthotropically-damaged material is described by the symmetric second rank damage tensor **D** as proposed by VAKULENKO and KACHANOV [24] and MURAKAMI, OHNO [21], with principal values $D_i$ (for $i = 1, 2, 3$) defined as follows (cf. LITEWKA, [17]):

$$D_i = s_{ci}/s_{Li}\,, \tag{1}$$

where $s_{ci}$ and $s_{Li}$ stand for the respective areas of the crack and of the material remaining on the plane orthogonal to the principal directions $x_i$. In the limit case of absence of cracking on a given plane ($s_{ci} = 0$), the appropriate principal component of damage tensor is equal to zero. In the opposite case, if the damage growth reduces the net area (the ligament between the adjacent cracks) to zero ($s_{Li} \to 0$), the respective principal value increases indefinitely causing loss of stiffness in the considered direction.

In further investigations we also use the more convenient (particularly for the equation of damage evolution, cf. MURAKAMI and SANOMURA, [22]) damage tensor $\boldsymbol{\Omega}$ whose principal values $\Omega_i$ are related to those of the tensor **D** through

$$\Omega_i = D_i/(1 + D_i). \tag{2}$$

According to LITEWKA'S suggestion [19], the one-parameter damage evolution equation written in the principal directions of damage-stress $\sigma_i$, will be assumed in the form

$$\partial_t \Omega_i = k \left(\mathbf{M}\mathbf{N}^T\right)^2 \sigma_i H(\sigma_i), \qquad i = 1, 2, 3. \tag{3}$$

Here for mathematical simplicity, the definitions of the vectors of material-damage **M** and stress-damage **N** have been introduced

$$\mathbf{M} = \begin{bmatrix} \frac{1-2\nu}{6E} & \frac{1+\nu}{2E} & \frac{\Omega_1}{2E} \end{bmatrix}, \tag{4}$$

$$\mathbf{N} = \begin{bmatrix} \operatorname{tr}^2\mathbf{T} & \operatorname{tr}\mathbf{S}^2 & \operatorname{tr}(\mathbf{T}^2\mathbf{D}) \end{bmatrix}, \tag{5}$$

where **E**, **T** and **S** are the strain and stress tensors and stress deviator, respectively, $E$ and $\nu$ are Young's modulus and Poisson's ratio of the undamaged material at current temperature. The magnitude $k$ is a temperature-dependent material constant. The Heaviside step function $H(\sigma_i)$ has been applied for elimination of damage development in the directions of compressive stresses. In this way the principal compressive stresses are assumed here to leave the already existing damage unchanged (the principal values of the damage rate tensor are equal to zero, $\partial_t \Omega_i = 0$). This description is justified by the experimental investigations indicating that damage orientation is mainly associated with the positive values of the principal stresses (cf. HAYHURST [6]; DAYSON and MCLEAN [4]).

The state of rupture in a particle of the structure identifies with a critical combination of the damage tensor components which can be determined from the criterion of failure (cf. LITEWKA and HULT [20])

$$\mathbf{C}\mathbf{R} - s_u^2 = \mathbf{0}, \tag{6}$$

where

$$\mathbf{R} = \frac{1}{s_1^2} \begin{bmatrix} (s_1 + s_2 + s_3)^2 \\ s_1^2 + s_2^2 + s_3^2 - s_2 s_1 - s_3 s_1 - s_2 s_3 \\ \Omega_1 \left( \frac{s_1 r_1^2}{s_1 - r_1 \Omega_1} + \frac{s_2 r_2^2}{s_1 - r_2 \Omega_1} + \frac{s_3 r_3^2}{s_1 - r_3 \Omega_1} \right) \end{bmatrix}, \tag{7}$$

$$\mathbf{C} = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}. \tag{8}$$

The equation (6) has been formulated in terms of dimensionless principal stresses

$$s_i = \sigma_i / \sigma_1 \qquad i = 1, 2, 3, \tag{9}$$

$$s_u = \sigma_u / \sigma_1 \tag{10}$$

where $\sigma_1$ is the maximal tensile stress, and $s_u$ denotes the dimensionless ultimate strength $\sigma_u$. The multipliers

$$r_i = s_i H(s_i), \qquad i = 1,2,3 \tag{11}$$

expressed by the Heaviside function $H(s_i)$ eliminate the influence of compressive stresses on the damage growth.

The vector of material constants $\mathbf{C}$ is dependent on the temperature and on the state of the damage growth process. Its components $C_i$ are determined by applying the Eq. (6) to three different states of stress: uniaxial tensions in the two principal directions of the damage tensor, and equal biaxial tension in those directions. This procedure leads to the following set of equations:

$$\mathbf{U}\,\mathbf{C}^T = \mathbf{I}, \tag{12}$$

where

$$\mathbf{U} = \begin{bmatrix} (1-\Omega_1)^2 & \frac{2}{3}(1-\Omega_1)^2 & (1-\Omega_1)\Omega_1 \\ (1-r_2\Omega_1)^2 & \frac{2}{3}(1-r_2\Omega_1)^2 & (1-r_2\Omega_1)r_2\Omega_1 \\ 4(1-\Omega_1)^2 & \frac{2}{3}(1-\Omega_1)^2 & 2(1-\Omega_1)\Omega_1 \end{bmatrix}, \tag{13}$$

$$\mathbf{I}^T = [1 \quad 1 \quad 1]. \tag{14}$$

The numerical analysis of the damage process will be carried out on the basis of the following set of equations: the differential equations of damage evolution (3), and the failure criterion (6) coupled with (12). The solution of this set of equations will be investigated in fixed discrete point pattern of the plate. Rupture in all the points occurs when $\Omega_i$, increasing monotonically according to damage evolution (3), satisfy the failure criterion (6), where the constants $\mathbf{C}[C_1, C_2, C_3]$ are calculated from (12). The Runge-Kutta integral procedures have been applied to solve the Equations (5). The material was assumed to be undamaged in the initial state, i.e.

$$\Omega_i(t_0) = 0 \qquad \text{for} \quad i = 1,2,3, \tag{15}$$

where $t_0$ is the instant of loading.

The principle values $\Omega_i$ corresponding to the material rupture will be assumed as the critical ones. The critical values attained at the same time in different points of the structure form the rupture front propagation surface.

The numerical solution of the plate problem is carried out by means of the Mindlin - Reissner finite elements. The plate is discretised by 9 nodal isoparametric elements of *heterosis* type, where the shape function is interpolated by Lagrange's formulas. For the numerical integration of volume the Gauss nine-point quadrature was applied. The rupture within the plate thickness is analysed using a layered approach. The details of the algorithm, including the conditions for numerical stability and accuracy, can be found in monograph by HINTON [8]. The stresses obtained here, after transforming to the principle directions, are the input data for the analysis of the damage process.

## 3. NUMERICAL ANALYSIS

To illustrate the process of a brittle rupture, appearing in the form of reduction of the plate thickness, we consider a uniformly loaded, square plate with entirely clamped edges (i.e. lateral displacement, tangential and normal edge rotations are zero, $C_h : w = \theta_t = \theta_h = 0$), Fig. 1. The plate

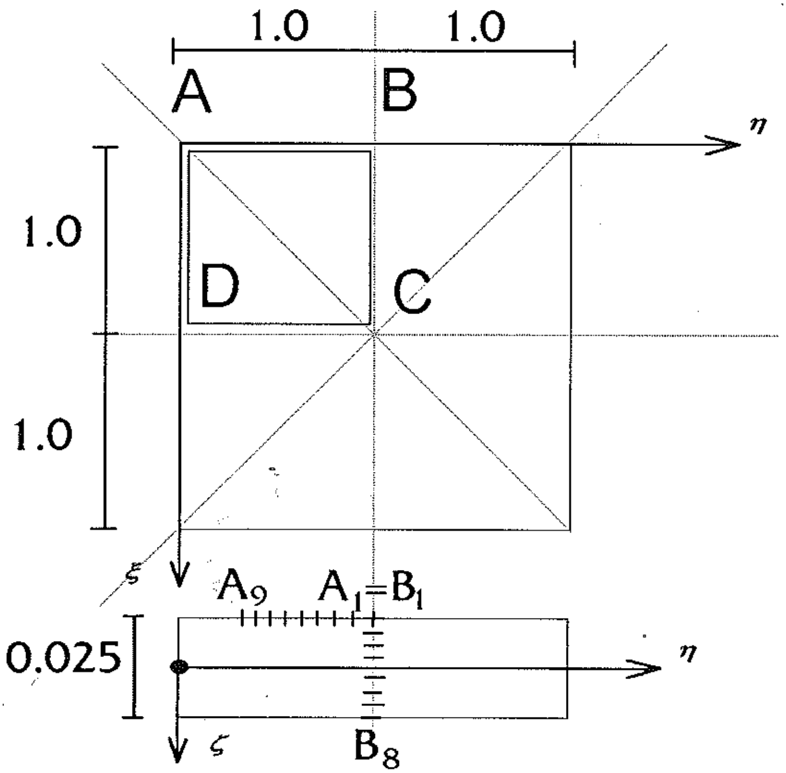


FIG. 1.

has a thickness-to-span ratio ($\overline{h} = h/a$) of 0.025. In all presentations of the numerical solutions, in case of symmetry we analyse a symmetric quadrant $ABCD$ in the zone $\xi \in \langle 0, 1 \rangle$, $\eta \in \langle 0, \rangle$ in which the functions sought for are symmetric with respect to the diagonal $AC$. The following dimensionless quantities have also been assumed: uniform load $\overline{q} = q/\sigma_u = 1.7 \cdot 10^{-3}$, Young's modulus $\overline{E} = E/\sigma_u = 416.7$, Poissons ratio $\nu = 0.47$, material constants of damage evolution (5) $\overline{k} = k\sigma_u^3\tau = 6.81 \cdot 10^6$, where $\tau = 1\,[\mathrm{h}]$ is unit time, and $\sigma_u = 288\,[\mathrm{MPa}]$. These material data correspond to the carbon steel AISI at a temperature of 811 [K] (cf. LITEWKA and HULT [20].

The distribution of the principal values of damage tensor $\mathbf{\Omega}_1$ at the instant of the first appearance of cracks in points $B$ and $D$ in the upper surface of the plate is illustrated in Fig. 2. The compressive stresses in vicinity of

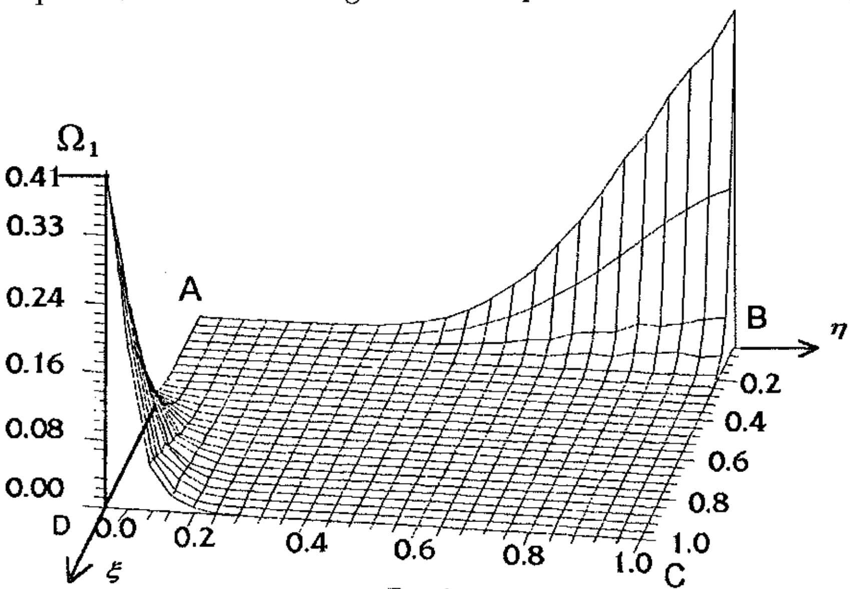


FIG. 2.

the mid-point $C$ are related to the zero principal values of the damage tensor. The very steep gradients of the principal stresses at the clamped zone in the upper surface of the plate (disregarding the neighbourhood of point $A$) indicate a high intensity of the damage growth in the narrow edge layer (directions $DA$ and $BA$). The first macroscopic cracks will decrease the thickness of the plate in points $B$ and $D$. The changes of the plate thickness are here computed by means of layer elements. The plate is divided into eight layers with constant dimensionless thickness $\Delta\overline{h} = 3.125 \cdot 10^{-3}$ (points $B_i$, for $i = 1, 2...8$, Fig. 1). The changes in the plate thickness are associated with the redistribution (increase) of stresses accelerating the damage development.

The redistribution of stresses $s_1$ in points $A_i$ (for $i = 1...9$) of the damage front, forming along the clamped edge $BA$ in the upper plate surface, is shown by dashed lines in Fig. 3a. The dimensionless time is related to the time of the first cracks $t_1$ ($\bar{t} = t/t_1$) arriving in point $A_1$. The stress state in an undamaged state is represented by the initial value of functions at $\bar{t} = 1$. Rupture will occur in points $A_i$ which define the critical curve in the $s_1 - \bar{t}$ plane (continuous line).

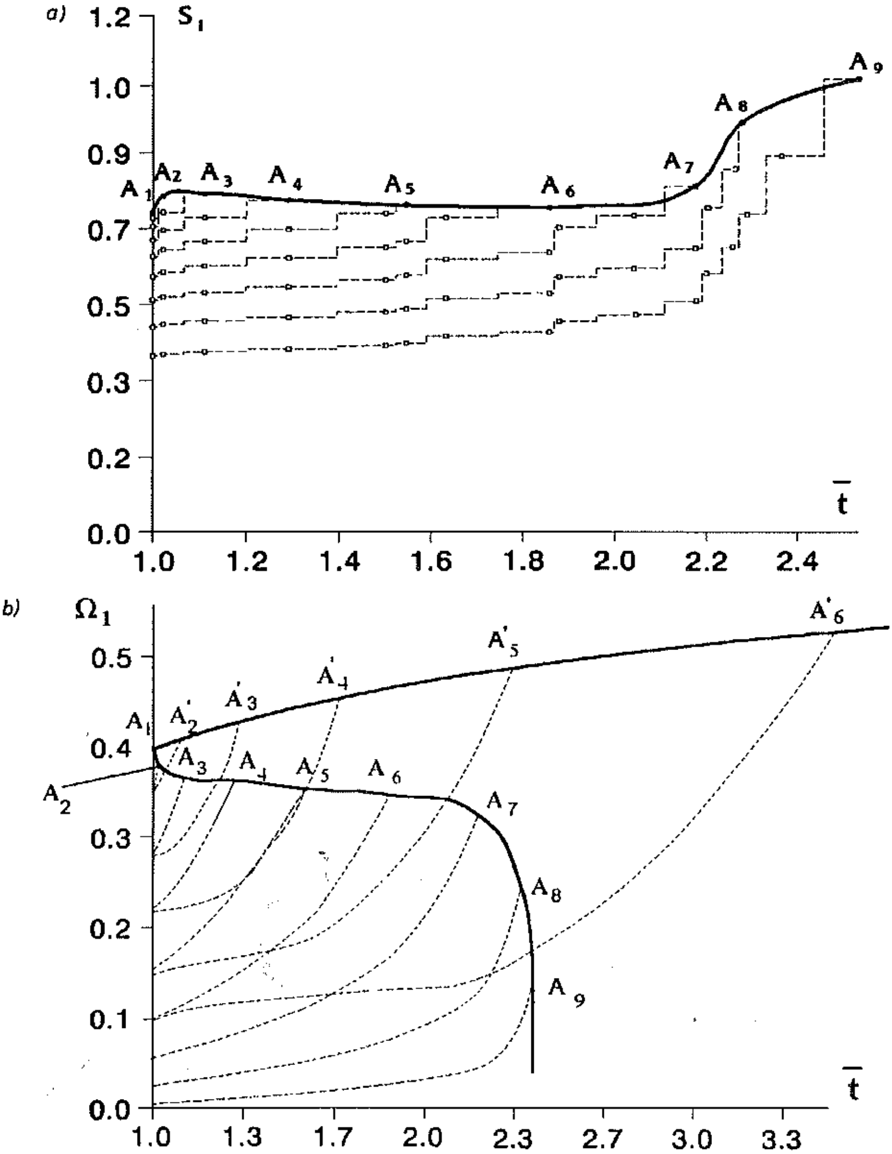


FIG. 3.

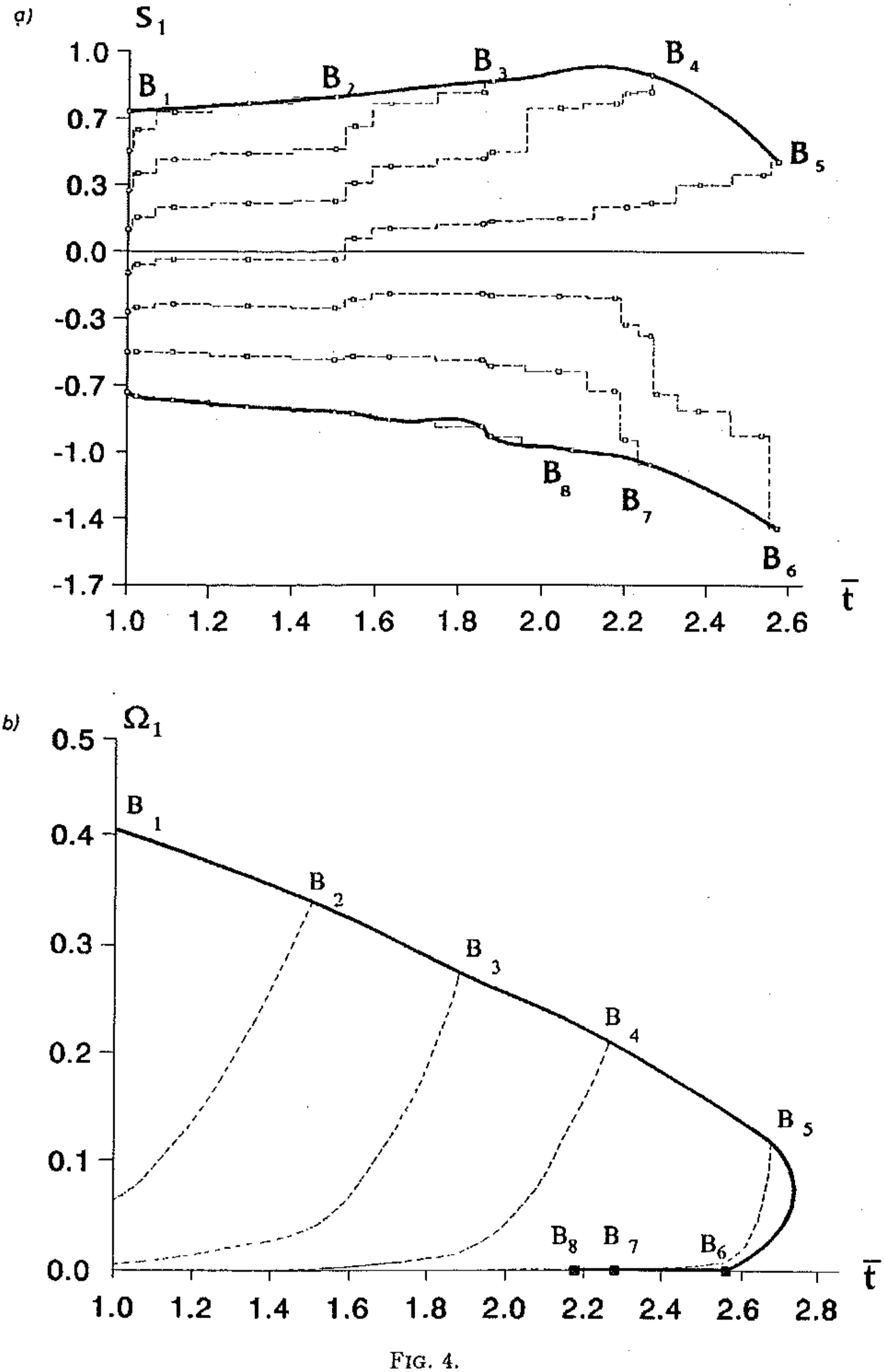
a)
S1
1.0
0.7
0.3
0.0
-0.3
-0.7
-1.0
-1.4
-1.7
B1
B2
B3
B4
B5
B6
B7
B8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
t̄
b)
Ω1
0.5
0.4
0.3
0.2
0.1
0.0
B1
B2
B3
B4
B5
B6
B7
B8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
t̄

FIG. 4.

The damage state in points $A_i$ is illustrated in Fig. 3b. The starting points on the $\mathbf{\Omega}_1$ axis for dashed curves are related to the principal values of damage tensor $\mathbf{\Omega}_1$ at the instant of the first crack at point $A_1$ ($\Omega_1^A = 0.4$). A continuous line with points $A_i$ shows the variation of the principal value of the damage tensor on the front of the rupture. It is seen from the course of this line that the critical damage $\mathbf{\Omega}_1$ is decreasing in the process of rupture propagation. The opposite phenomenon can be observed when the changes in the plate thickness caused by rupture are neglected and nominal stress analysis in damage development is applied. The solution in such a case is illustrated by a continuous line with points $A_i'$ (for $i = 1...9$). The rupture propagation is associated with an increase of the principal values of the damage tensor. Also the interval of incubation time to the instant of rupture increases when compared with the results of previous analysis.

A similar analysis of damage front motion in the direction of plate thickness (points $B_i$, for $i = 1...8$) is shown in Fig. 4a-b. Here the state of rupture in points $B_6$, $B_7$ and $B_8$ appears at a time when, in the course of stress redistribution, the ultimate compressive stresses in these points are reached, Fig. 4a. The principal values of the damage tensor in these points are equal to zero, Fig. 4b.

High discrepancy between the scales of time for the analysis with the assumption of nominal stresses makes it impossible to present a suitable solution in Fig. 4b. The rupture in point $B_2$ will appear at instant $\bar{t} = 6.3$ for $\mathbf{\Omega}_1 = 0.59$.

The dimensionless function of deflection $w^c$ in point $C$ related to the instantaneous deflection $w^c(t_0)$ of undamaged plate is shown in Fig. 5. The

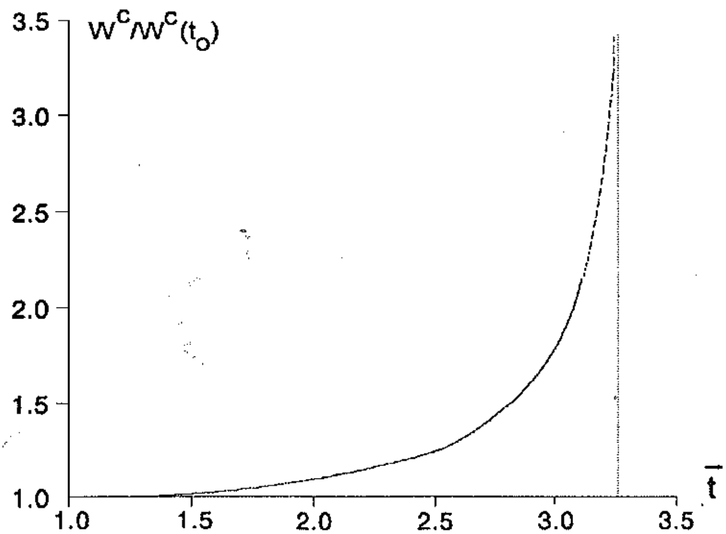


FIG. 5.

asymptotic course of this function indicates the time of rupture of the plate $\bar{t}_k \cong 3.25$. The damage process at the instant close to $\bar{t}_k$ has an avalanche course.

Two chosen stages of the damage development are shown by shaded zones in $BA$ and $BC$ cross-sections in Fig. 6a-b. Figure 6a corresponds to

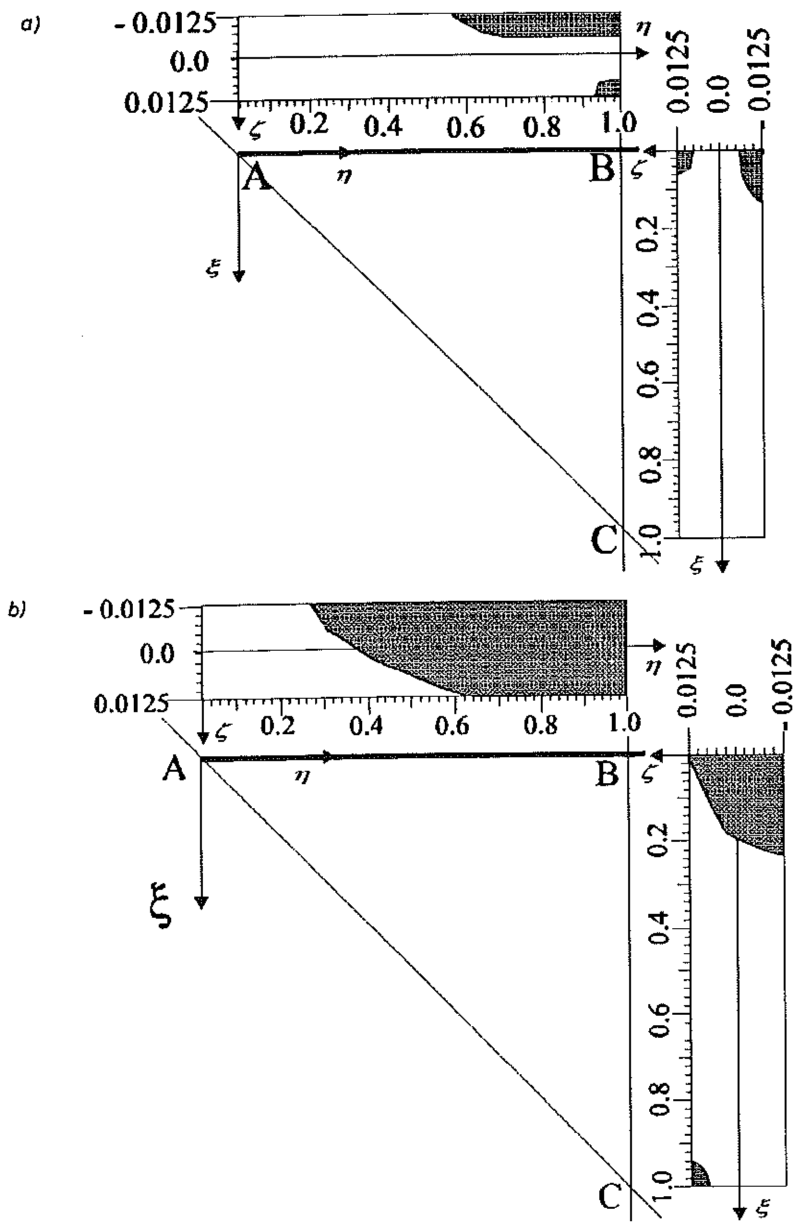


FIG. 6.

the instant $\bar{t} \cong 2.2$ when the crack appeared in point $B_8$, whereas Fig. 6b illustrates the state of rupture close to the instant $\bar{t}_k$.

## 4. Final remarks

The numerical analysis indicates a different course of the rupture process in time, depending on the assumed model of deterioration. For plates displaying no changes in thickness (uncoupled theory), rupture on the front surface occurs for increasing principal values of the damage tensor. The reverse remarks can be formulated when real stresses are used in the rupture analysis. The real stresses introduced through the variation of the plate thickness influence the acceleration of rupture front motion. The rupture will occur in a shorter time at lower principal values of the damage tensor (Fig. 3b). This indicates that the coupled theory, in which the rupture determines the current geometry of the structure, gives safer design conditions.

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CRACOW UNIVERSITY OF TECHNOLOGY
INSTITUTE OF STRUCTURAL MECHANICS, KRAKÓW.

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