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ASYMPTOTIC METHODS IN THE THEORY OF PERFORATED MEMBRANE OF NONHOMOGENEOUS STRUCTURE (*)

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Homogenization equations are constructed for perforated membranes. Asymptotic methods are used for solving the cell problem.

1. Introduction

Homogenization theory for perforated media has been recently developed by many authors [1–7]. The main problem in this field consists in solving the so-called cell (or local) problem. This problem has been usually treated by numerical methods [6]. We have used asymptotic methods for solving the cell problem and we have constructed analytical approach in this paper. The paper deals with the following problems:

1. Deformation of membrane with periodic circular perforations.
2. Eigenvalue problem for perforated membrane.
3. Deformation of membrane with periodic inclusions.
4. Deformation of membrane with square perforations.

2. Deformation of membrane with periodic circular perforations

We consider the Poisson equation

$$\nabla^2 u = f(x, y) \tag{2.1}$$

in domain Ω which consists of a perforated medium with a large number of circular holes arranged in a periodic pattern with period ε (Fig. 1).

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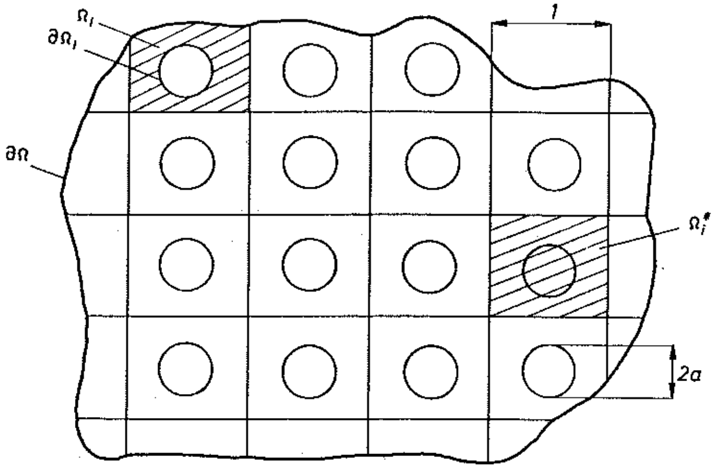

FIG. 1.

Here ε – small parameter, characterizing the ratio of structure period to the typical size Ω of the region, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ – Cartesian notation for the Laplace operator.

Let the boundaries of holes $\partial\Omega_i$ be free (Neumann problem), so that

$$\frac{\partial u}{\partial n_i} = 0 \qquad \text{on} \quad \partial\Omega_i, \tag{2.2}$$

here n_i – outer normal to hole's contour.

Boundary conditions (without any loss of generality) along the domain boundary $\partial\Omega$ may be assumd to be

$$u = 0 \qquad \text{on} \quad \partial\Omega. \tag{2.3}$$

The method used here is a variant of multiscaling technique used in the books [2, 3] (se also [4, 5, 8]). Let us introduce new "fast" variables

$$\xi = x/\varepsilon, \qquad \eta = y/\varepsilon.$$

The solution is written in the form of a formal expansion

$$u = u_0(x,y) + \varepsilon u_1(x,y,\xi,\eta) + \varepsilon^2 u_2(x,y,\xi,\eta) + \dots \,. \tag{2.4}$$

Period 1 with respect to variables ξ, η for u_j $(j = 1, 2, \dots)$ is assumed.

The operators $\partial/\partial x$ and $\partial/\partial y$ applied to a function have the form

$$\partial/\partial x = \partial/\partial x + \varepsilon^{-1}\partial/\partial\xi, \qquad \partial/\partial y = \partial/\partial y + \varepsilon^{-1}\partial/\partial\eta. \tag{2.5}$$

Substituting series (2.4) into the boundary-value problem (2.1)–(2.3), taking into account relations (2.5) and splitting it according to the powers of ε, one obtains the recurrent sequence of boundary value problems

$$\frac{\partial^2 u_1}{\partial \xi^2} + \frac{\partial^2 u_1}{\partial \eta^2} = 0 \qquad \text{in} \quad \Omega_i,$$

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + 2\left(\frac{\partial^2 u_1}{\partial x \partial \xi} + \frac{\partial^2 u_1}{\partial y \partial \eta}\right) + \frac{\partial^2 u_2}{\partial \xi^2} + \frac{\partial^2 u_2}{\partial \eta^2} = f, \tag{2.6}$$

$$\frac{\partial u_1}{\partial \overline{n}} + \frac{\partial u_0}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega_i;$$

$$\begin{aligned} u_0 &= 0 \qquad \text{on} \quad \partial\Omega, \\ u_1 &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.7}$$

Here $\overline{n}$ – outer normal expressed in "fast" variables.

We consider the averaging operator defined upon the variables (ξ, η) of a periodic function $\Phi(x, y, \xi, \eta)$

$$\widetilde{\Phi}(x, y) = \frac{1}{|\Omega_i^*|} \iint\limits_{\Omega_i} \Phi(x, y, \xi, \eta)\, d\xi\, d\eta. \tag{2.8}$$

Here Ω_i – area of the region occupied by periodically repeating cell of the structure; Ω_i^* – cell without a hole (Fig. 1).

This is easily obtained from (2.6) by applying the averaging operator defined by (2.8)

$$\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} - f\right)|\Omega_i| + \iint\limits_{\Omega_i} \left(\frac{\partial^2 u_1}{\partial x \partial \xi} + \frac{\partial^2 u_1}{\partial y \partial \eta}\right) d\xi\, d\eta = 0. \tag{2.9}$$

We now consider the cell problem (here x, y are parameters)

$$\frac{\partial^2 u_1}{\partial \xi^2} + \frac{\partial^2 u_1}{\partial \eta^2} = 0 \qquad \text{in} \quad \Omega_i\,, \tag{2.10}$$

$$\frac{\partial u_1}{\partial \overline{n}} + \frac{\partial u_0}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega_i\,, \tag{2.11}$$

and conditions of periodic continuation

$$\begin{aligned} \left\{u_1, \xi\right\} &= 0, \qquad \xi \rightleftarrows \eta, \\ \left\{\frac{\partial u_1}{\partial \xi}, \xi\right\} &= 0, \qquad \xi \rightleftarrows \eta, \end{aligned} \tag{2.12}$$

Here the following notations have been introduced:

$$\left\{\Phi,\,\xi\right\} = \Phi\Big|_{\xi=0.5} - \Phi\Big|_{\xi=-0.5}.$$

The asymptotic method of domain perturbation [8, 9] is to be used for solving the boundary value problem (2.10)–(2.12).

Let us consider the case, when hole diameter $2a$ is small in comparison with cell' size 1 (case $a \approx 1$ is considered in [1]). Then, in the first approximation for the function $u_1^{(1)}$ one obtains the boundary value problem for the hole in the infinite plane

$$\frac{\partial^2 u_1^{(1)}}{\partial \xi^2} + \frac{\partial^2 u_1^{(1)}}{\partial \eta^2} = 0 \qquad \text{in} \qquad \Omega_i, \tag{2.13}$$

$$\frac{\partial u_1^{(1)}}{\partial \overline{n}} = -\frac{\partial u_0}{\partial n} \qquad \text{on} \qquad \partial\Omega_i, \tag{2.14}$$

$$u_1^{(1)} \to 0 \qquad \text{when} \qquad \xi^2 + \eta^2 \to \infty. \tag{2.15}$$

Condition (2.15) should follow from condition $a \ll 1$ and means that we are not taking into account, in the first approximation, the boundaries $\xi, \eta = \pm 0.5$. In the polar coordinates, the boundary value problem (2.13)–(2.15) may be written in the form

$$\frac{\partial^2 u_1^{(1)}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u_1^{(1)}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u_1^{(1)}}{\partial \theta^2} = 0, \tag{2.16}$$

$$\frac{\partial u_1^{(1)}}{\partial \rho}\Bigg|_{\rho=a} = -\frac{\partial u_0}{\partial x}\cos\theta - \frac{\partial u_0}{\partial y}\sin\theta, \tag{2.17}$$

$$u_1^{(1)} \to 0 \qquad \text{when} \qquad \rho \to \infty. \tag{2.18}$$

Solution of the boundary value problem (2.16)–(2.18) may be written in polar coordinates in the following form:

$$u_1^{(1)} = \frac{a^2}{\rho}\left(\frac{\partial u_0}{\partial x}\cos\theta + \frac{\partial u_0}{\partial y}\sin\theta\right).$$

Function $u_1^{(1)}$ does not satisfy the conditions of periodicity, and that is why we obtain, in the second approximation, the following problem for the function $u_1^{(2)}$ (we take into account only the principal terms of the series)

$$\Delta u_1^{(2)} = 0 \qquad \text{in} \quad \Omega_i^*, \tag{2.19}$$

$$\left\{u_1^{(1)},\xi\right\} + \left\{u_1^{(2)},\xi\right\} = 0, \qquad \xi \rightleftarrows \eta, \tag{2.20}$$

$$\left\{\frac{\partial u_1^{(1)}}{\partial \xi},\xi\right\} + \left\{\frac{\partial u_1^{(2)}}{\partial \xi},\xi\right\} = 0, \qquad \xi \rightleftarrows \eta. \tag{2.21}$$

We consider problem (2.19) – (2.21) in the simply connected domain Ω_i^* ($|\xi| \leq 0.5$; $|\eta| \leq 0.5$).

Let us represent $u_1^{(2)}$ as

$$u_1^{(2)} = \overline{u}_1^{(2)} + \overline{\overline{u}}_1^{(2)}, \tag{2.22}$$

where $\overline{u}_1^{(2)}$ is the solution of the following boundary value problem:

$$\Delta \overline{u}_1^{(2)} = 0 \quad \text{in} \quad \Omega_i^*; \tag{2.23}$$

$$\left\{\overline{u}_1^{(2)}, \xi\right\} = 0, \qquad \left\{\frac{\partial \overline{u}_1^{(2)}}{\partial \xi}, \xi\right\} = 0; \tag{2.24}$$

$$\begin{gathered} \left\{\overline{u}_1^{(2)}, \eta\right\} + \left\{u_1^{(1)}, \eta\right\} = 0, \\ \left\{\frac{\partial \overline{u}_1^{(2)}}{\partial \eta}, \eta\right\} + \left\{\frac{\partial u_1^{(1)}}{\partial \eta}, \eta\right\} = 0. \end{gathered} \tag{2.25}$$

One easily obtains the solution of (2.23) – (2.25),

$$\begin{aligned} \overline{u}_1^{(2)} = A_0 + B_0 \eta + \sum_{n=1}^{\infty} [(A_n \operatorname{ch} 2\pi n \eta + B_n \operatorname{sh} 2\pi n \eta) \cos 2\pi n \xi \\ + (C_n \operatorname{ch} 2\pi n \eta + D_n \operatorname{sh} 2\pi n \eta) \sin 2\pi n \xi], \end{aligned} \tag{2.26}$$

where arbitrary constants A_n, B_n, C_n, D_n, $n = 0, 1, \ldots$ are obtained from the boundary conditions.

Solution in the form (2.26) satisfies the boundary conditions (2.24). To satisfy the conditions (2.25), rewrite them in the form

$$\begin{aligned} \left\{\overline{u}_1^{(2)}, \eta\right\} &= -\frac{\partial u_0}{\partial y} a^2 (\xi^2 + 0.25)^{-1}, \\ \left\{\frac{\partial \overline{u}_1^{(2)}}{\partial \eta}, \eta\right\} &= 2 \frac{\partial u_0}{\partial x} a^2 \xi (\xi^2 + 0.25)^{-2}. \end{aligned} \tag{2.27}$$

Expanding right-hand sides of the Eqs. (2.27) into the Fourier series and equating the corresponding coefficients, one obtains

$$\begin{aligned} A_n &= D_n = 0, \qquad n = 0, 1, \ldots, \\ B_0 &= \frac{\partial u_0}{\partial y} B_0^* = -\frac{\partial u_0}{\partial y} \pi a^2, \\ B_n &= \frac{\partial u_0}{\partial y} B_n^* = -\frac{\partial u_0}{\partial y} \frac{2a^2}{\operatorname{sh} \pi n} \Big(e^{-\pi n} \operatorname{Im} E_1(\pi n (i - 1)) \\ &\qquad - e^{\pi n} \operatorname{Im} E_1(\pi n (i + 1)) \Big), \\ C_n &= B_n \left(\frac{\partial u_0}{\partial y} \Rightarrow \frac{\partial u_0}{\partial x} \right), \qquad n = 0, 1, 2, \ldots . \end{aligned}$$

Here E_1 – exponential integral; $i = \sqrt{-1}$; $\mathrm{Im}(\ldots) \equiv$ imaginary part of $(\ldots)$.

So, for $\overline{u}_1^{(2)}$ we have

$$\overline{u}_1^{(2)} = \frac{\partial u_0}{\partial y} B_0^* \eta + \sum_{n=1}^{\infty} B_n^* \left(\frac{\partial u_0}{\partial y} \operatorname{sh} 2\pi n\eta \cos 2\pi n\xi + \frac{\partial u_0}{\partial x} \operatorname{ch} 2\pi n\eta \sin 2\pi n\xi \right). \tag{2.28}$$

Function $\overline{\overline{u}}_1^{(2)}$ is obtained in the same way,

$$\overline{\overline{u}}_1^{(2)} = \frac{\partial u_0}{\partial x} B_0^* \xi + \sum_{n=1}^{\infty} B_n^* \left(\frac{\partial u_0}{\partial x} \operatorname{sh} 2\pi n\xi \cos 2\pi n\eta + \frac{\partial u_0}{\partial y} \operatorname{ch} 2\pi n\xi \sin 2\pi n\eta \right). \tag{2.29}$$

Substituting solutions of the cell boundary value problems $u_1 = u_1^{(1)} + u_1^{(2)}$ into Eq.(2.9), one obtains the homogenized equation

$$q \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} \right) = \overline{q}\, f. \tag{2.30}$$

Here the homogenized coefficient q equals

$$\begin{aligned} q &= 1 - 2\pi a^2 + \pi^2 a^4 + 8\pi^2 a^4 \\ &\times \sum_{n=1}^{\infty} \frac{n}{\operatorname{sh} \pi n} \left(e^{-\pi n} \operatorname{Im} E_1(\pi n(i-1)) - e^{\pi n} \operatorname{Im} E_1(\pi n(i+1)) \right), \\ \overline{q} &= 1 - \pi a^2. \end{aligned} \tag{2.31}$$

Series in expression (2.31) is absolutely convergent with rapidly decreasing terms ($|a_{n-1}/a_n| \simeq exp(-\pi n)$).

The homogenized boundary conditions are:

$$u_0 = 0 \qquad \text{on} \quad \partial\Omega.$$

For $\Omega_i/\Omega_i^* = 8/9$, the homogenized coefficient, obtained by the method mentioned above, is $q = 0.79$. We note that the results obtained with the use of the above-mentioned asymptotic procedure are in good agreement with the results of numerical computations [6].

For estimation of the reliability of solution (2.31) we will solve the boundary value problem (2.10)–(2.12) in another way. Actually, we will use the variational method proposed by GALERKIN [10]. Expansion for u_1 is written in the following form:

$$u_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(A_{1mn}\sin 2m\pi\xi\cos 2n\pi\eta + A_{2mn}\cos 2m\pi\xi\sin 2n\pi\eta + A_{3mn}\cos 2m\pi\xi\cos 2n\pi\eta + A_{4mn}\sin 2m\pi\xi\sin 2n\pi\eta). \tag{2.32}$$

Here, constants A_{kmn} ($k = 1-4$) are obtained from the condition of vanishing variation of Galerkin's functional. Assumption of function u_1 in the form (2.32) enables us to fulfill the boundary conditions (2.12) on the opposite sides of the cell. Boundary conditions on the contour of the holes will be fulfilled only in the average.

Variation of Galerkin's functional is written in the following form:

$$\iint_{\Omega_i}\Delta u_1\delta u_1\,ds + \int_{\partial\Omega_i}\left(\frac{\partial u_1}{\partial \overline{n}} + \frac{\partial u_0}{\partial n}\right)\delta u_1\,dl = 0.$$

From the conditions of symmetry we have

$$A_{3mn} = A_{4mn} = 0.$$

Coefficients A_{1mn} and A_{2mn} are obtained by using Galerkin's procedure [10]

$$A_{1mn} = a\frac{\partial u_0}{\partial x}A^*_{mn}, \qquad A_{2mn} = a\frac{\partial u_0}{\partial y}A^*_{mn}, \qquad A^*_{mn} - \text{const}.$$

After substitution of the expression (2.32) into Eq. (2.9), and after some transformations, one obtains the homogenized equation in the form (2.30), where the homogenized coefficient is

$$q = 1 - \pi a^2 - 2\pi a^2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A^*_{mn}\left(\sqrt{m^2+n^2}\right)^{-1}I_1\left(2\pi a\sqrt{m^2+n^2}\right).$$

Here I_1 – Bessel function.

For $a = 1/(3\sqrt{\pi})$, the homogenized cofficient is equal to 0.826, and this result is slightly different from the previously obtained value.

3. EIGENVALUE PROBLEM FOR A PERFORATED MEMBRANE

Using the notations introduced in Sec. 1, consider the following eigenvalue problem:

$$\nabla^2 u + \lambda u = 0, \tag{3.1}$$

where λ – square of the natural frequency.

For Eq. (3.1) we may formulate the boundary conditions (2.2), (2.3).

We represent eigenvalue λ and eigenfunction u in the following form

$$u = u_0(x,y) + \varepsilon\left(u_{10}(x,y) + u_1(x,y,\xi,\eta)\right) + \varepsilon^2\left(u_{20}(x,y) + u_2(x,y,\xi,\eta)\right) + \ldots, \tag{3.2}$$

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots. \tag{3.3}$$

Substituting expansions (3.2), (3.3) into equation (3.1) and boundary conditions (2.2)–(2.3) and splitting it into the powers of ε, one obtains the recurrent systems of boundary value problems. The first step of solving is the same that above – solving the local problem (2.10)–(2.12). It means that eigenvalue problem for perforated membrane is quasi-static.

Homogenized eigenvalue problem may be obtained by applying the averaging operator defined by (2.8)

$$q\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right) + \overline{q}\lambda_0 u_0 = 0.$$

This equation must be supplied by homogenized boundary conditions (2.7).

For rectangular membrane ($0 \le x \le l_1$, $0 \le y \le l_2$), solution of the eigenvalue problem is

$$u_0 = \sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2}, \tag{3.4}$$

$$\lambda_0 = \pi^2\left[\left(\frac{m}{l_1}\right)^2 + \left(\frac{n}{l_2}\right)^2\right]\frac{q}{\overline{q}}. \tag{3.5}$$

Calculation of the first correctness term to the frequency square needs obtaining function u_2 due to the following boundary value problem

$$\frac{\partial^2 u_2}{\partial \xi^2} + \frac{\partial^2 u_2}{\partial \eta^2} = -\frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial y^2} - 2\frac{\partial^2 u_1}{\partial x \partial \xi} - 2\frac{\partial^2 u_1}{\partial y \partial \eta} - \lambda_0 u_0 \quad \text{in} \quad \Omega_i, \tag{3.6}$$

$$\frac{\partial u_2}{\partial \overline{n}} = -\frac{\partial u_1}{\partial n} - \frac{\partial u_{10}}{\partial n} \quad \text{on} \quad \partial\Omega_i, \tag{3.7}$$

with conditions of periodic continuation, analogous to (2.12).

Using for solution of the boundary value problem (3.6), (3.7), any of the methods mentioned above (variational Galerkin procedure, iterations), one can obtain function u_2. It may be written (taking into account only the principal terms) as

$$u_2 = u_1\left(\frac{\partial u_0}{\partial x} \Rightarrow \frac{\partial u_{10}}{\partial x};\ \frac{\partial u_0}{\partial y} \Rightarrow \frac{\partial u_{10}}{\partial y}\right).$$

After averaging of the equation

$$\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial^2 u_{10}}{\partial x^2}+\frac{\partial^2 u_{10}}{\partial y^2}+2\left(\frac{\partial^2 u_2}{\partial x\partial\xi}+\frac{\partial^2 u_2}{\partial y\partial\eta}\right)$$
$$+\frac{\partial^2 u_3}{\partial\xi^2}+\frac{\partial^2 u_3}{\partial\eta^2}+\lambda_1 u_0+\lambda_0(u_1+u_{10})=0,$$

we obtain the homogenized equation

$$q\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right)+\overline{q}(\lambda_0 u_{10}+\lambda_1 u_0)=0 \tag{3.8}$$

with the boundary condition

$$u_{10}=-\widetilde{u}_1 \qquad \text{on} \quad \partial\Omega. \tag{3.9}$$

Here $\widetilde{u}_1$ is obtained by applying the averaging operator defined by (2.8) to the function u_1.

Routine procedure of the perturbation method (multiplying relation (3.8) by u_0 and integrating by parts over the region Ω^*, and taking into account the boundary conditions (3.9) [9]) leads to the following expressions for λ_1

$$\lambda_1=\frac{q}{\overline{q}}\frac{\int_0^{l_2}\varphi\,dy+\int_0^{l_1}\psi\,dx}{\int_0^{l_1}\int_0^{l_2}u_0^2\,dx\,dy}. \tag{3.10}$$

Here

$$\varphi=\frac{\partial u_0}{\partial x}u_{10}\Big|_{x=0}^{x=l_1}, \qquad \psi=\frac{\partial u_0}{\partial y}u_{10}\Big|_{y=0}^{y=l_2}.$$

One obtains:

if $\widetilde{u}_1=0$, then $\lambda_1=0$ and $\lambda=\lambda_0+O(\varepsilon^2)$,
if $\widetilde{u}_1\neq 0$, then $\lambda=\lambda_0+O(\varepsilon)$.

4. Membrane with periodic inclusions

Now we consider the boundary value problem for a composite membrane with periodic circular elastic inclusions. In the domain Ω_i^+ (matrix) we have equation (2.1), in the domain Ω_i^- (inclusion)

$$\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f.$$

The conditions of continuity may be written in the form

$$u^+ = u^-, \qquad \frac{\partial u^+}{\partial n} = \alpha \frac{\partial u^-}{\partial n} \qquad \text{on} \quad \partial \Omega_i.$$

The cell boundary value problem may be written in the following form:

$$\frac{\partial^2 u_1^+}{\partial \xi^2} + \frac{\partial^2 u_1^+}{\partial \eta^2} = 0 \qquad \text{in} \quad \Omega_i^+, \tag{4.1}$$

$$\frac{\partial^2 u_1^-}{\partial \xi^2} + \frac{\partial^2 u_1^-}{\partial \eta^2} = 0 \qquad \text{in} \quad \Omega_i^-, \tag{4.2}$$

$$u^+ = u^-, \qquad \frac{\partial u_1^+}{\partial \overline{n}} - \alpha \frac{\partial u_1^-}{\partial \overline{n}} = \frac{\partial u_0}{\partial n}(\alpha - 1) \quad \text{on} \quad \partial \Omega_i. \tag{4.3}$$

After solution Eqs. (4.1) and (4.2), and assuming that the inclusion is small, we satisfy the continuity conditions (4.3) and obtain u_1^+, u_1^-

$$u_1^+ = -\frac{\alpha - 1}{\alpha + 1} \frac{a^2}{\rho} \left(\frac{\partial u_0}{\partial x} \cos \theta + \frac{\partial u_0}{\partial y} \sin \theta \right), \tag{4.4}$$

$$u_1^- = -\frac{\alpha - 1}{\alpha + 1} \rho \left(\frac{\partial u_0}{\partial x} \cos \theta + \frac{\partial u_0}{\partial y} \sin \theta \right), \tag{4.5}$$

and the homogenized equation

$$q \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} \right) = f.$$

Here

$$q = 1 - \pi a^2 + \frac{2\alpha}{\alpha + 1} \pi a^2. \tag{4.6}$$

Constructing for function u_1^+ the second approximation (2.22), (2.28), (2.29), one obtains the homogenized coefficient

$$\begin{aligned} q = {} & 1 - 2\pi a^2 + \pi^2 a^4 + 2\pi a^2 \alpha / (\alpha + 1) \\ & + 8\pi^2 a^4 \sum_{n=1}^{\infty} \frac{n}{\operatorname{sh} \pi n} \Big(e^{-\pi n} \operatorname{Im} E_1(\pi n(i - 1)) - e^{\pi n} \operatorname{Im} E_1(\pi n(i + 1)) \Big). \end{aligned} \tag{4.7}$$

Various asymptotics for the limiting values α may be obtained.

If $\alpha \sim \varepsilon$, relation for the homogenized coefficient coincides with that obtained earlier for a perforated membrane (2.31).

For $\alpha \sim \varepsilon^{-1}$ (absolutely rigid inclusions) we have

$$q = 1 + \pi^2 a^4 + 8\pi^2 a^4 \sum_{n=1}^{\infty} \frac{n}{\operatorname{sh} \pi n}\Big(e^{-\pi n} \operatorname{Im} E_1(\pi n(i-1)) - e^{\pi n} \operatorname{Im} E_1(\pi n(i+1))\Big).$$

Numerical results obtained on the basis of formula (4.6) are in good agreement with the known numerical results [6].

The eigenvalue problem for a membrane with small circular elastic inclusions may be reduced to the following problem

$$\begin{aligned} \nabla^2 u^+ + \lambda u^+ &= 0 \quad \text{in} \quad \Omega^+, \\ \nabla^2 u^- + \alpha\lambda u^- &= 0 \quad \text{in} \quad \Omega^-, \\ u^+ = u^-, \qquad \frac{\partial u^+}{\partial n} &= \alpha \frac{\partial u^-}{\partial n} \quad \text{on} \quad \partial\Omega_i. \end{aligned}$$

Boundary condition on the outer contour are written as follows:

$$u = 0 \qquad \text{on} \quad \partial\Omega.$$

Expansion of every function u^+, u^- and eigenvalue λ are analogous to (3.2) and (3.3), respectively; then the solution of the local problem, on using the above obtained relations (4.4), (4.5), (2.22), (2.28), (2.29), and the homogenized equation, may be written as

$$q\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right) + \overline{q}\lambda_0 u_0 = 0,$$

where

$$\overline{q} = 1 - \pi a^2 + \alpha\pi a^2. \tag{4.8}$$

Homogenized coefficient q is defined by formula (4.7).

The following solution may be obtained using the procedure written for perforation in Sec. 3. Main terms of the eigenfunctions and frequencies for a rectangular membrane are given by relations (3.4), (3.5), and first corrections to the frequency λ_1 – by formula (3.10), account being taken of relations (4.7), (4.8) for the homogenized parameters q, $\overline{q}$. Table 1 presents the eigenvalues for a rectangular membrane clamped on the outer contour, with periodic inclusions for various sizes and rigidities.

Table 1.

(m,n)		(1,1)	(1,2;2,1)	(2,2)	(1,3;3,1)
α	a				
	0	19.7392	49.3480	78.9568	98.6960
0	1/12	19.2898	48.2246	77.1593	96.4491
	1/6	18.0170	45.0425	72.0680	90.0850
1/10	1/12	19.3270	48.3175	77.3079	96.6349
	1/6	18.4954	46.2385	73.9816	92.4770
10	1/12	16.4258	41.0645	65.7032	82.1290
	1/6	12.6800	31.7225	50.7560	63.4450
100	1/12	6.2426	15.6066	24.9705	31.2131
	1/6	2.05953	5.1488	8.2381	10.2976

5. Deformation of a membrane with square perforations

Only final results of the analysis are presented here.

Boundary conditions on the square hole boundaries $2a$ may be reduced to the form

$$\left.\frac{\partial u_1}{\partial \xi}\right|_{\xi=\pm a} = -\frac{\partial u_0}{\partial x}, \qquad \begin{matrix} \xi \rightleftarrows \eta \\ x \rightleftarrows y \end{matrix}.$$

We use Galerkin procedure and represent the solution of the cell problem by trigonometric series (2.32). Using the Galerkin procedure, one obtains an infinite system of algebraic equations for $A_{1mn} = a\frac{\partial u_0}{\partial x}A^*_{mn}$:

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{l=0}^{\infty} A_{1kl}\Bigg[\beta_{1kl}
&+\frac{(k^2+l^2)}{l+n}\left(\frac{\sin 2\pi a(k-m)}{k-m}-\frac{\sin 2\pi a(k+m)}{k+m}\right)\sin 2\pi a(n+l)\\
&+\frac{2k}{l+n}\cos 2\pi ak \sin 2\pi am \sin 2\pi a(n+l)\\
&-2l\left(\frac{\sin 2\pi a(k-m)}{k-m}-\frac{\sin 2\pi a(k+m)}{k+m}\right)\sin 2\pi al \cos 2\pi an\Bigg]\\
&= -\frac{2a}{\pi n}\frac{\partial u_0}{\partial x}\sin 2\pi am \sin 2\pi an, \quad m=1,2,\ldots, \quad n=0,1,2,\ldots .
\end{aligned}$$

Here

$$\beta_{1kl} = \begin{cases} -2\pi^2, & m = k,\ n = l = 0, \\ -\pi^2, & m = k,\ n = l \neq 0, \\ 0, & m \neq k\,. \end{cases}$$

Analogous system of equations may be written for $A_{2mn} = a\dfrac{\partial u_0}{\partial y}A^*_{mn}$.
The homogenized equation may be represented as

$$q\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right) = \overline{q}\, f.$$

Here

(5.1) $$q = 1 - 4a^2 - \frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A^*_{mn} \sin 2m\pi a \sin 2n\pi a,$$

(5.2) $$\overline{q} = 1 - 4a^2.$$

For $a = 1/6$, the homogenized coefficient (obtained by the method mentioned above) is 0.823; the numerical result obtained by the finite element method was 0.81 [6].

Eigenvalue problem for a rectangular membrane with periodic square perforations may be written as

$$\nabla^2 u + \lambda u = 0,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_i,$$

$$u = 0 \quad \text{for} \quad x = 0,\ l_1; \quad y = 0,\ l_2.$$

For solution of a local problem, the Galerkin variational procedure with function u_1, u_2 written in the form (2.32) was used. Homogenized equation and relations for u_0, λ_0, λ_1 are of forms analogous to Eqs. (3.4), (3.5), (3.10) written above. Here one must take into account relations (5.1), (5.2) for the homogenized parameters q, $\overline{q}$. In Table 2 are displayed squares of the natural frequencies of a rectangular membrane, for various sizes of square perforations.

Table 2.

(m, n) / a	$(1, 1)$	$(1, 2; 2, 1)$	$(2, 2)$	$(1, 3; 3, 1)$
0	19.7392	49.3480	78.9568	98.6960
1/3	18.2805	45.7012	73.1219	91.4024
2/3	14.2513	35.6282	57.0053	71.2566

6. Conclusion

It is seen that application of the homogenization procedure with asymptotic methods enables us to obtain simple analytic expressions for effective parameters of a periodically nonhomogeneous membrane.

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