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STRESSES DISTRIBUTION IN A MAGNETO-THERMOELASTIC MEDIUM WITH CYLINDRICAL HOLE

D. CHAND (GURDASPUR)

The distributions of elastic and thermal stresses in a thermally and electrically conducting infinitely extended homogeneous isotropic medium with a cylindrical hole, in the presence of uniform magnetic field, have been studied due to: (i) step in stress with zero temperature change and (ii) – step in temperature with zero stress at the boundary of the hole, in the context of the generalised theory of thermoelasticity. The Laplace transform technique has been used to obtain the small time solutions. As the "second sound" effects are of short duration, so the small time approximations have been considered. The standard results obtained have also been discussed at the wavefronts. The jumps and their particular cases obtained theoretically have been computed numerically and are represented graphically for carbon steel material.

1. Introduction

Kaliski and Nowacki [1] studied the magneto-thermoelastic waves in a perfectly conducting elastic half-space in contact with vacuum, due to applied thermal disturbances acting on the plane boundary, in the absence of coupling between temperature and strain fields. Massalas and Dalmangas [2, 3] also studied the same problem by taking into account the thermo-mechanical coupling. The problem [2] was extended to generalised thermo-elasticity developed by Green and Lindsay [4] and was also studied by Chatterjee and Roychoudhuri [5]. Sharma and Chand [6] studied the transient magneto-thermoelastic waves in the context of generalised theories of thermoelasticity [4, 7]. Sharma *et al.* [8] considered the distribution of displacement, temperature, and stresses due to a thermal shock in a homogeneous transversely isotropic medium with a cylindrical hole, in the context of generalised theories of thermoelasticity [4, 7]. Sharma and Chand [9, 10] analysed the thermoelastic waves in a homogeneous isotropic elastic plate due to suddenly punched hole in the context of generalised theories thermoelasticity [4, 7]. Noda *et al.* [11] studied the generalised thermoelasticity in an infinite solid with a hole.

These types of problems are important in view of their relevance to various industrial machines subject to heating and rotating components in the presence of electric and magnetic fields. These types of problems are also important in exploration of geomagnetic fields. Such problems also arise in quenching studies, the analysis of experimental data and measurements of aerodynamic heating. The present paper deals with the study of distributions of deformation, temperature, and stresses in a thermally and electrically conducting, infinitely extended homogeneous isotropic medium with a cylindrical hole due to (i) step in stress with zero temperature change, and (ii) – step in temperature with zero stress at the boundary of the hole in the context of generalised theory of thermoelasticity [7]. The Laplace transform technique [14] has been employed to obtain the small time solutions. As the "second sound" effects are short-lived, so the small time approximations have been considered. The jumps obtained theoretically have been computed numerically and are represented graphically for carbon steel material.

2. Basic equations

The basic governing magneto-thermoelastic interactions in a homogeneous isotropic solid consist of the following:

a) Maxwell's equations

$$\nabla \times \mathbf{h} = 4\pi \mathbf{J}/c, \qquad \nabla \times \mathbf{E} = -\mu_0 \dot{\mathbf{h}}/c,$$
$$\nabla \cdot \mathbf{h} = 0, \qquad \mathbf{E} = -\mu_0 (\dot{\mathbf{u}} \times \mathbf{H}_0)/c; \tag{2.1}$$

b) strain-displacement relations

$$e_{ij} = \frac{1}{2}(u_{ij} + u_{ji}), \qquad i,j = 1 \text{ to } 3; \tag{2.2}$$

c) stress-strain-temperature relations

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} - \beta\theta\delta_{ij}; \tag{2.3}$$

d) equation of motion

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \frac{\mu_0}{4\pi}[(\nabla \times \mathbf{h}) \times \mathbf{H}_0] - \beta \nabla \theta = \varrho \ddot{\mathbf{u}}; \tag{2.4}$$

e) energy equation

$$\varrho c_v (\dot{\theta} + \tau_0 \ddot{\theta}) + \beta T_0 \nabla \cdot (\dot{\mathbf{u}} + \tau_0 \ddot{\mathbf{u}}) = K\theta_{,ij}, \tag{2.5}$$

where

$\mathbf{H}_0$ initial magnetic field,
τ_0 thermal relaxation time,
T_0 initial temperature,
$\mathbf{h}$ perturbation of the magnetic field,

J electric current density,
E electric field,
μ_0 magnetic permeability,
H magnetic field,
c velocity of light,
θ temperature change,
e_{ij} components of strain tensor,
σ_{ij} components of stress tensor,
$\mathbf{u} = (u_1, u_2, u_3)$ the displacement vector,
ϱ density of the medium,
C_v specific heat at constant volume,
$K = \lambda_T/\varrho C_e$, λ_T the coefficient of heat conduction,
λ, μ Lamé's constants,
C_e specific heat at constant strain,
$\beta = (3\lambda + 2\mu)\alpha_T$, α_T the coefficient of linear expansion,
δ_{ij} Kronecker's delta; and $\dot{u} = \partial u/\partial t$.

3. The problem and its solution

We consider an infinitely extended, thermally as well as electrically conducting, homogeneous isotropic elastic medium at uniform initial temperature T_0 having an infinite cylindrical hole of radius a. We suppose that an initial magnetic field $\mathbf{H}_0 = (0, 0, H)$ is acting along the z-axis. We choose the origin of the cylindrical coordinate system (r, θ, z) at the axis of cylindrical hole. We also consider the case of radial symmetry, so that the non-zero displacement $u(r, t)$ and temperature $T(r, t)$ satisfy the following equations:

$$\mathbf{E} = \mu_0 H(0, \dot{u}, 0)/c, \qquad \mathbf{h} = H(0, 0, u_{,r} + r^{-1}u), \qquad \mathbf{J} = c(0, -h, r, 0)/4\pi, \tag{3.1}$$

$$(\lambda + 2\mu + a_0^2\varrho)\left[u_{,rr} + r^{-1}u_{,r} - r^{-2}u\right] - \beta\theta_{,r} = \varrho\ddot{u}, \tag{3.2}$$

$$K(\theta_{,rr} + r^{-1}\theta_{,r}) - \varrho C_v(\dot{\theta} + \tau_0\ddot{\theta}) = T_0\beta\left[\dot{u}_{,r} + r^{-1}\dot{u} + \tau_0(\ddot{u}_{,r} + r^{-1}\ddot{u})\right], \tag{3.3}$$

where $a_0^2 = \mu_0 H^2/4\pi\varrho$, a_0 is the Alfvén wave velocity.

The components of Maxwell's stress tensor T_{11} and stress σ_{11} in the elastic medium are given by

$$T_{11} = -\mu_0 hH/4\pi = -\mu_0 H^2(u_{,r} + r^{-1}u)/4\pi = -a_0^2\varrho(u_{,r} + r^{-1}u) \tag{3.4}$$

and

$$\sigma_{11} = (\lambda + 2\mu)u_{,r} + \lambda r^{-1}u - \beta\theta. \tag{3.5}$$

We assume that the medium is at rest and undisturbed initially. Therefore the initial conditions can be written as

$$u = 0 = \theta, \qquad \frac{\partial u}{\partial t} = 0, \qquad \text{at} \quad t = 0, \quad r \geq a. \tag{3.6}$$

We take two types of boundary conditions.

CASE 1. Normal load acting at the boundary of the hole

$$\begin{aligned} \sigma_{11} + T_{11} &= \sigma_0 H(t), \\ E_2 &= 0, \\ \theta(0,t) &= 0 \quad \text{at} \quad r = a. \end{aligned} \tag{3.7}$$

CASE 2. Thermal shock applied at the boundary of the hole

$$\begin{aligned} \sigma_{11} + T_{11} &= 0, \\ E_2 &= 0, \\ \theta(0,t) &= \theta_0 H(t) \qquad \text{at} \quad r = a, \end{aligned} \tag{3.8}$$

where $H(t)$ is a Heaviside function of time. σ_0, θ_0, and E_2 are the step in stress, step in temperature, and component of electric field along the y-axis.

We define the quantities, to make the equations non-dimensional,

$$\begin{aligned} r' &= w^* r/c_0, & t' &= w^* t, & u' &= \varrho w^* c_0 u/T_0\beta, & \\ T' &= \theta/T_0, & \tau_0' &= w^*\tau_0, & \varepsilon &= T_0\beta/\varrho^2 c_0^2 C_e, & C_e &= \varrho C_v, \\ w^* &= \varrho C_e c_0^2/K, & c_1^2 &= (\lambda + 2\mu)/\varrho, & c_2^2 &= \mu/\varrho, & c_0^2 &= c_1^2 + a_0^2. \end{aligned} \tag{3.9}$$

w^* is the characteristic frequency and ε is the coupling constant. Using quantities (3.9) in Eqs. (3.1) and (3.2), we have

$$u_{,rr} + r^{-1}u_{,r} - r^{-2}u - \ddot{u} = T_{,r}, \tag{3.10}$$

$$T_{,rr} + r^{-1}T_{,r} - (\dot{T} + \tau_0\ddot{T}) = \varepsilon\left[\dot{u}_{,r} + r^{-1}\dot{u} + \tau_0(\ddot{u}_{,r} + r^{-1}\ddot{u})\right], \tag{3.11}$$

where dashes have been disregarded for convenience and comma denotes spatial derivative.

The boundary of the cylindrical hole, i.e. $r = a$, is given by

$$r = w^* a/c_0 = \eta \ \text{(say)}.$$

The initial conditions (3.6) become

(3.12) $$u(\eta,0)=0,\qquad T(\eta,0)=0,\qquad \frac{\partial u}{\partial\tau}=0.$$

The boundary conditions (3.7) and (3.8) become:

CASE 1

(3.13) $$\begin{aligned} S_{rr}(\eta,t) &= \sigma_0 H(t)/T_0\beta,\\ E_2 &= 0,\\ T(\eta,t) &= 0 \qquad \text{at} \quad r=\eta, \end{aligned}$$

and

CASE 2

(3.14) $$\begin{aligned} S_{rr}(\eta,t) &= 0,\\ E_2 &= 0,\\ T(\eta,t) &= \theta_0 H(t) \qquad \text{at} \quad r=\eta, \end{aligned}$$

where

(3.15) $$S_{rr}(\eta,t)=u_{,r}+br^{-1}u-T,\qquad b=(1-2c_2^2c_0^{-2}).$$

$S_{rr}(\eta,t)$ is the dimensionless form of the stress in the radial direction.

Applying the Laplace transform defined by

(3.16) $$\overline{f}(r,p)=\int_0^\infty f(r,t)e^{-pt}\,dt$$

to Eqs. (3.10) and (3.11), we obtain

(3.17) $$\left[D(D+r^{-1})-p^2\right]\overline{u}=D\overline{T},$$

(3.18) $$\left[(D+r^{-1})D-p^2\tau^*\right]\overline{T}=\varepsilon p(D+r^{-1})\overline{u},$$

where $D=d/dr$, $\tau^*=(\tau_0+p^{-1})$.

Simplifying Eqs. (3.17) and (3.18), we get

(3.19) $$\left[\{D(D+r^{-1})\}^2-(m_1^2+m_2^2)D(D+r^{-1})+m_2^1m_2^2\right]\overline{u}=0,$$

(3.20) $$\left[\{(D+r^{-1})D\}^2-(m_1^2+m_2^2)(D+r^{-1})D+m_1^2m_2^2\right]\overline{T}=0,$$

where m_i^2 $(i=1,2)$ are the roots [6] of the equation

(3.21) $$m^4-p(\lambda_1+\lambda_2p)m^2+\tau^*p^4=0,$$

and $\lambda_1=1+\varepsilon$, $\lambda_2=1+\tau_0+\varepsilon\tau_0$.

On solving Eqs. (3.19) and (3.20), we get

(3.22) $$\overline{u} = A_1 K_1(m_1 r) + A_2 K_1(m_2 r),$$

(3.23) $$\overline{T} = B_1 K_0(m_1 r) + B_2 K_0(m_2 r),$$

where $K_1(m_i r)$ and $K_0(m_i r)$ $i = 1, 2$ are the modified Bessel functions of the first and zero order, respectively.

From equations (3.15), (3.17), (3.22) and (3.23) we get

(3.24) $$B_i = (p^2 - m_i^2) A_i / m_i\,, \qquad i = 1, 2.$$

Equations (3.15), (3.22), (3.23) and (3.24), provide us with

(3.25) $$\overline{T} = A_1 N_1(m_1 r) + A_2 N_2(m_2 r),$$

(3.26) $$\overline{S}_{rr} = A_1 M_1(m_1 r) + A_2 M_2(m_2 r),$$

where

(3.26′) $$\begin{aligned} N_1(m_1 r) &= (p^2 - m_1^2) K_0(m_1 r)/m_1\,, \\ N_2(m_2 r) &= (p^2 - m_2^2) K_0(m_2 r)/m_2\,, \\ M_1(m_1 r) &= -\left[2r^{-1} K_1(m_1 r) + m_1^{-1}(p^2 - 2m_1^2) K_0(m_1 r)\right], \\ M_2(m_2 r) &= -\left[2r^{-1} K_1(m_2 r) + m_2^{-1}(p^2 - 2m_2^2) K_0(m_2 r)\right]. \end{aligned}$$

Applying boundary conditions (3.13) and (3.14) to Eqs. (3.25) and (3.26), we get:

CASE 1

(3.27) $$\begin{aligned} A_1 &= \sigma_0 N_2(m_2 \eta)/T_0 \beta p \Delta, \\ A_2 &= -\sigma_0 N_1(m_1 \eta)/T_0 \beta p \Delta. \end{aligned}$$

CASE 2

(3.28) $$\begin{aligned} A_1 &= -\theta_0 M_2(m_2 \eta)/p\Delta, \\ A_2 &= \theta_0 M_1(m_1 \eta)/p\Delta, \end{aligned}$$

where $\Delta = M_1(m_1\eta) N_2(m_2\eta) - M_2(m_2\eta) N_1(m_1\eta)$.

Using Eqs. (3.27) and (3.28) in Eqs. (3.22), (3.25), and (3.26), we get:

CASE 1

(3.29) $$\overline{u} = \sigma_0 [K_1(m_1 r) N_2(m_2\eta) - K_1(m_2 r) N_1(m_1\eta)]/\beta T_0 p \Delta,$$

(3.30) $$\overline{T} = \sigma_0 [N_1(m_1 r) N_2(m_2\eta) - N_2(m_2 r) N_1(m_1\eta)]/\beta T_0 p \Delta,$$

(3.31) $$\overline{S}_{rr} = \sigma_0 [M_1(m_1 r) N_2(m_2\eta) - M_2(m_2 r) N_1(m_1\eta)]/\beta T_0 p \Delta.$$

CASE 2

(3.32) $$\overline{u} = \theta_0 [K_1(m_2 r) M_1(m_1\eta) - K_1(m_1 r) M_2(m_2\eta)]/p\Delta,$$

(3.33) $$\overline{T} = \theta_0 [M_1(m_1\eta) N_2(m_2 r) - M_2(m_2\eta) N_1(m_1 r)]/p\Delta,$$

(3.34) $$\overline{S}_{rr} = \theta_0 [M_1(m_1\eta) M_2(m_2 r) - M_1(m_2 r) M_2(m_2\eta)]/p\Delta.$$

4. Small time approximations

The dependence of m_1, m_2 on p is complicated, and thus inversion of the Laplace transform is very difficult. These difficulties can, however, be reduced if we use approximate methods. As the thermal effects are short-lived [12], we confine the discussion to small time approximations, i.e. we take p large. A similar approach was used by SHARMA [13] to study the thermal problem in the generalised theory of thermoelasticity. The roots m_1, m_2 of Eq. (3.21) when expanded binomially in powers of p, lead to

$$m_i = pv_i^{-1} + \phi_i + O(p^{-1}), \qquad i = 1, 2, \tag{4.1}$$

where

$$v_{1,2}^{-1} = \left[\lambda_2 \pm (\lambda_2^2 - 4\tau_0)^{1/2}\right] / \sqrt{2}, \tag{4.2}$$

$$\phi_{1,2} = \left[\lambda_1 \pm (\lambda_1\lambda_2 - 2)/(\lambda_2^2 - 4\tau_0)^{1/2}\right] / 2\sqrt{2}\left[\lambda_2 \pm (\lambda_2^2 - 4\tau_0)^{1/2}\right]. \tag{4.3}$$

From the above analysis, it can be established that there exist three types of waves, namely an elastic wave, a thermal wave, and an Alfvén acoustic wave travelling with velocity v_1, v_2, and a_0, respectively, with $v_1 < v_2$. The elastic wave follows the thermal wave. The modified Bessel function $K_n(z)$ has the asymptotic expansion [14]

$$K_n(z) = (\pi/2z)^{1/2} e^{-z}\left[1 + \frac{(4n^2 - 1^2)}{\lfloor 1 (8z)^2} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{\lfloor 2 (8z)^2} + \ldots\right]. \tag{4.4}$$

Equations (3.4), (3.29)–(3.34) and (4.4) after straightforward but lengthy algebra, leads to

CASE 1

$$\overline{u}(R,p) = \sigma_0(\eta/r)^{1/2}\Big[(\lambda_3 p^{-2} + \lambda_4 p^{-3})e^{-m_1 R} - (\lambda_3' p^{-2} + \lambda_4' p^{-3})e^{-m_2 R}\Big]/\beta T_0, \tag{4.5}$$

$$\overline{T}(R,p) = \sigma_0(\eta/r)^{1/2}\Big[(\lambda_5 p^{-1} + \lambda_6 p^{-2})e^{-m_1 R} - (\lambda_5' p^{-1} + \lambda_6' p^{-2})e^{-m_2 R}\Big]/\beta T_0, \tag{4.6}$$

$$\overline{S}_{rr}(R,p) = \sigma_0(\eta/r)^{1/2}\Big[-(\lambda_7 p^{-1} + \lambda_8 p^{-2})e^{-m_1 R} + (\lambda_7' p^{-1} + \lambda_8' p^{-2})e^{-m_2 R}\Big]/\beta T_0, \tag{4.7}$$

$$\overline{T}_{11}(R,p) = -\sigma_0 a_0^2 \varrho(\eta/r)^{1/2}\Big[(\lambda_9 p^{-1} + \lambda_{10} p^{-2})e^{-m_1 R} - (\lambda_9' p^{-1} + \lambda_{10}' p^{-2})e^{-m_2 R}\Big]/\beta T_0. \tag{4.8}$$

CASE 2

$$(4.9)\qquad \overline{u}(R,p) = \theta_0(\eta/r)^{1/2}\Big[(\lambda_{11}p^{-2} + \lambda_{12}p^{-3})e^{-m_1R} - (\lambda'_{11}p^{-2} + \lambda_{12}p^{-3})e^{-m_2R}\Big],$$

$$(4.10)\qquad \overline{T}(R,p) = \theta_0(\eta/r)^{1/2}\Big[(\lambda_{13}p^{-1} + \lambda_{14}p^{-2})e^{-m_1R} - (\lambda'_{13}p^{-1} + \lambda_{14}p^{-2})e^{-m_2R}\Big],$$

$$(4.11)\qquad \overline{S}(R,p) = \theta_0(\eta/r)^{1/2}\Big[-(\lambda_{15}p^{-1} + \lambda_{16}p^{-2})e^{-m_1R} - (\lambda'_{15}p^{-1} + \lambda'_{16}p^{-2})e^{-m_2R}\Big],$$

$$(4.12)\qquad \overline{T}_{11}(R,p) = 2a_0^2\beta_0^2\varrho\theta_0(\eta/r)^{1/2}\Big[(\lambda_{17}p^{-1} + \lambda_{18}p^{-2})e^{-m_1R} - (\lambda'_{17}p^{-1} + \lambda'_{18}p^{-2})e^{-m_2R}\Big]/\eta,$$

where

$$R = (r-\eta),$$

$$\lambda_3 = v_1(v_2^2-1)/(v_2^2-v_1^2),$$

$$\lambda'_3 = v_2(v_1^2-1)/(v_2^2-v_1^2),$$

$$\begin{aligned}\lambda_4 = {} & v_1v_2(v_2^2-v_1^2)^{-1}\left[v_2^{-1}(v_2^2-1)\left(\frac{3v_1}{8r}-\frac{v_2}{8\eta}\right)-\phi_2(v_2^2+3)\right] \\ & + v_2^{-1}(v_2^2-1)(v_2^2-v_1^2)^{-2}\Big[v_1v_2(v_1+v_2)(8\eta)^{-1}(v_2^2-v_1^2) \\ & - 2\eta^{-1}\beta_0^2v_1v_2\{v_2(v_1^2-1)+v_1(v_2^2-1)\} \\ & + 2v_1^2v_2^2(\phi_1v_2-\phi_2v_1) - v_1v_2(v_2^2-v_1^2)(\phi_1v_1+\phi_2v_2)\Big],\end{aligned}$$

$$\begin{aligned}\lambda'_4 = {} & v_1v_2(v_2^2-v_1^2)^{-1}\left[v_1^{-1}(v_1^2-1)\left(\frac{3v_2}{8r}-\frac{v_1}{8\eta}\right)-(\phi_1(v_1^2+3)\right] \\ & - v_1^{-1}(v_1^2-1)(v_2^2-v_1^2)^{-2}\Big[v_1v_2(v_1+v_2)(8\eta)^{-1}(v_2^2-v_1^2) \\ & - 2\eta^{-1}\beta_0^2v_1v_2\{v_1(v_2^2-1)+v_2(v_1^2-1)\} + 2v_1^2v_2^2(\phi_1v_2-\phi_2v_1) \\ & - v_1v_2(v_2^2-v_1^2)(\phi_1v_1+\phi_2v_2)\Big],\end{aligned}$$

$$\lambda_5 = (v_1^2v_2^2 - v_1^2 - v_2^2 + 1)/(v_2^2-v_1^2) = \lambda'_5,$$

$$\begin{aligned}\lambda_6 = {} & v_1v_2(v_2-v_1)^{-1}\Big[(v_1v_2)^{-1}(\phi_1v_1+\phi_2v_2)(v_1^2v_2^2-v_2^2-v_1^2+1) \\ & - 2(\phi_1v_2+\phi_2v_1) + 2(v_1v_2)^{-1}(\phi_1v_1+\phi_2v_2) - \left(\frac{v_1}{8r}+\frac{3v_2}{8\eta}\right) \\ & \times(v_1v_2)^{-1}(v_1^2v_2^2-v_2^2-v_1^2+1)\Big] + (v_1v_2)^{-1}(v_1^2v_2^2-v_1^2-v_2^2+1)Q,\end{aligned}$$

$$\lambda_6' = P\Big[(v_1v_2)^{-1}(\phi_1v_1+\phi_2v_2)(v_1^2v_2^2-v_1^2-v_2^2+1)-2(\phi_1v_2+\phi_2v_1) + 2(v_1v_2)^{-1}(\phi_1v_1+\phi_2v_2)-\left(\frac{v_1}{8\eta}+\frac{3v_2}{8r}\right)(v_1v_2)^{-1}(v_1^2v_2^2-v_1^2-v_2^2+1)\Big] + (v_1v_2)^{-1}(v_1^2v_2^2-v_1^2-v_2^2+1)Q,$$

$$\lambda_7 = (v_1^2v_2^2-2v_2^2-v_1^2+2)/(v_2^2-v_1^2),$$

$$\lambda_7' = (v_1^2v_2^2-2v_1^2-v_2^2+2)/(v_2^2-v_1^2),$$

$$\lambda_8 = P\Big[4(v_1v_2)^{-1}(\phi_1v_1+\phi_2v_2)-(4\phi_1v_2+2\phi_2v_1)-(v_1v_2)^{-1}(v_1^2v_2^2 - 2v_2^2-v_1^2+2)-(v_1v_2)^{-1}(v_1^2v_2^2-2v_2^2-v_1^2+2)\left(\frac{v_1}{8r}+\frac{v_2}{8\eta}\right)\Big] + (v_1v_2)^{-1}(v_1^2v_2^2-2v_1^2-v_2^2+2)Q,$$

$$\lambda_8' = P\Big[4(v_1v_2)^{-1}(\phi_1v_1+\phi_2v_2)-(4\phi_1v_2+2\phi_2v_1)-(v_1v_2)^{-1}(v_1^2v_2^2 - 2v_1^2-v_2^2+2)-(v_1v_2)^{-1}(v_1^2v_2^2-2v_1^2-v_2^2+2)\left(\frac{v_1}{8\eta}+\frac{v_2}{8r}\right)\Big] + (v_1v_2)^{-1}(v_1^2v_2^2-2v_1^2-v_2^2+2)Q,$$

$$\lambda_9 = (v_2^2-1)/(v_2^2-v_1^2), \qquad \lambda_9' = (v_1^2-1)/(v_2^2-v_1^2),$$

$$\lambda_{10} = P\Big[v_2^{-2}(v_2^2-1)\{(\phi_1v_2-\phi_2v_1)-2\phi_2v_1^{-1}\}-(v_1v_2)^{-1}(v_2^2-1)\left(\frac{v_1}{8r}+\frac{v_2}{8\eta}\right)\Big] + (v_1v_2)^{-1}(v_2^2-1)Q,$$

$$\lambda_{10}' = P\Big[v_1^{-2}(v_1^2-1)\{(\phi_2v_1-\phi_1v_2)-2\phi_2v^{-1}\}-(v_1v_2)^{-1}(v_1^2-1)\left(\frac{v_1}{8\eta}+\frac{v_2}{8r}\right)\Big] + (v_1v_2)^{-1}(v_1^2-1)Q,$$

$$\lambda_{11} = v_1(v_2^2-1)/(v_2^2-v_1^2), \qquad \lambda_{11}' = v_2(v_1^2-1)/(v_2^2-v_1^2),$$

$$\lambda_{12} = \left[2\beta_0^2\eta^{-1}-\phi_2(v_2^2+3)+v_2^{-1}(v_1^2-1)\left(\frac{3v_1}{8r}-\frac{v_2}{8\eta}\right)\right]P+v_1^{-1}(v_2^2-1)Q,$$

$$\lambda'_{12} = \left[2\beta_0^2\eta^{-1} - \phi_1(v_1^2+3) + v_1^{-1}(v_2^2-1)\left(\frac{3v_2}{8r} - \frac{v_1}{8\eta}\right)\right] P + v_2^{-1}(v_1^2-1)Q,$$

$$\lambda_{13} = (v_1^2 v_2^2 - 2v_2^2 - v_1^2 + 2)/(v_2^2 - v_1^2),$$

$$\lambda'_{13} = (v_1^2 v_2^2 - 2v_1^2 - v_2^2 + 2)/(v_2^2 - v_1^2),$$

$$\lambda_{14} = \Big[(v_1v_2)^{-1}\{4(\phi_1 v_2 + \phi_2 v_1) - 8(\phi_1 v_1 + 2\phi_2 v_2)\} - (v_1v_2)^{-1}(v_1^2v_2^2 - 2v_2^2 - v_1^2 + 2) - (v_1v_2)^{-1}(v_1^2v_2^2 - 2v_2^2 - v_1^2 + 2)\left(\frac{v_1}{8r} + \frac{v_2}{8\eta}\right)\Big] P + (v_1v_2)^{-1}(v_1^2v_2^2 - 2v_2^2 - v_1^2 + 2)Q,$$

$$\lambda'_{14} = \Big[(v_1v_2)^{-1}\{4(\phi_2 v_1 + \phi_1 v_2) - 8(\phi_1 v_1 + \phi_2 v_2)\} - (v_1v_2)^{-1}(v_1^2v_2^2 - 2v_1^2 - v_2^2 + 2) - (v_1v_2)^{-1}(v_1^2v_2^2 - 2v_1^2 - v_2^2 + 2)\left(\frac{v_1}{8\eta} + \frac{v_2}{8r}\right)\Big] P + (v_1v_2)^{-1}(v_1^2v_2^2 - 2v_1^2 - v_2^2 + 2)Q,$$

$$\lambda_{15} = (v_1^2v_2^2 - 2v_1^2 - 2v_2^2 + 4)/(v_2^2 - v_1^2) = \lambda'_{15},$$

$$\lambda_{16} = \Big[4v_1v_2(\phi_1 v_2 + \phi_2 v_1) - 16(\phi_1 v_1 + \phi_2 v_2) - (v_1^2-1)(v_2^2-1) \cdot \left\{1 + \left(\frac{v_1}{8r} + \frac{v_2}{8\eta}\right)\right\}\Big](v_1v_2)^{-1}P + (v_1v_2)^{-1}(v_1^2-1)(v_2^2-1)Q,$$

$$\lambda'_{16} = \Big[4v_1v_2(\phi_1 v_2 + \phi_2 v_2) - 16(\phi_1 v_1 + \phi_2 v_2) - (v_1^2-1)(v_2^2-1) \cdot \left\{1 + \left(\frac{v_1}{8\eta} + \frac{v_2}{8r}\right)\right\}\Big](v_1v_2)^{-1}P + (v_1v_2)^{-1}(v_1^2-1)(v_2^2-1)Q,$$

$$\lambda_{17} = (v_2^2 - 2)/(v_2^2 - v_1^2), \qquad \lambda'_{17} = (v_1^2 - 2)/(v_2^2 - v_1^2),$$

$$\lambda_{18} = \left[1 - \phi_2(v_2^2+6) - v_2^{-1}(v_2^2-2)\left(\frac{v_1}{8r} + \frac{v_2}{8\eta}\right)\right] P + v_2^{-1}(v_2^2-2)Q,$$

$$\lambda'_{18} = \left[1 - \phi_1(v_1^2+6) - v_1^{-1}(v_1^2-2)\left(\frac{v_1}{8\eta} + \frac{v_2}{8r}\right)\right] P + v_1^{-1}(v_1^2-2)Q,$$

$$P = v_1 v_2/(v_2 - v_1),$$

$$Q = -\Big[2v_1v_2(\phi_1 v_2 - \phi_2 v_1) - (8\eta)^{-1}(v_1+v_2)(v_2^2 - v_1^2) - (\phi_1 v_1 + \phi_2 v_2)(v_2^2 - v_1^2) + 2\beta_0^2\eta^{-1}\left\{v_2(v_1^2-1) + v_1(v_2^2-1)\right\}\Big] v_1 v_2 (v_2^2 - v_1^2)^{-2}.$$

On inverting the Laplace transform in Eqs. (4.5)–(4.12), we get:

CASE 1

(4.13) $$U(R,\tau) = \sigma_0(\eta/r)^{1/2}\Big[\{\lambda_3 + \lambda_4(\tau - R/v_1)\}(\tau - R/v_1)H(\tau - R/v_1)e^{-\phi_1 R} - \{\lambda_3' + \lambda_4'(\tau - R/v_2)\}(\tau - R/v_2)H(\tau - R/v_2)e^{-\phi_2 R}\Big]/\beta T_0,$$

(4.14) $$T(R,\tau) = \sigma_0(\eta/r)^{1/2}\Big[\{\lambda_5 + \lambda_6(\tau - R/v_1)\}H(\tau - R/v_1)e^{-\phi_1 R} - \{\lambda_5' + \lambda_6'(\tau - R/v_2)\}H(\tau - R/v_2)e^{-\phi_2 R}\Big]/\beta T_0,$$

(4.15) $$S_{rr}(R,\tau) = \sigma_0(\eta/r)^{1/2}\Big[-\{\lambda_7 + \lambda_8(\tau - R/v_1)\}H(\tau - R/v_1)e^{-\phi_1 R} + \{\lambda_7' + \lambda_8'(\tau - R/v_2)\}H(\tau - R/v_2)e^{-\phi_2 R}\Big]/\beta T_0,$$

(4.16) $$T_{11}(R,\tau) = \sigma_0 a_0^2\varrho(\eta/r)^{1/2}\Big[-\{\lambda_9 + \lambda_{10}(\tau - R/v_1)\}H(\tau - R/v_1)e^{-\phi_1 R} + \{\lambda_9' + \lambda_{10}'(\tau - R/v_2)\}H(\tau - R/v_2)e^{-\phi_2 R}\Big]/\beta T_0.$$

CASE 2

(4.17) $$U(R,\tau) = \theta_0(\eta/r)^{1/2}\Big[\{\lambda_{11} + \lambda_{12}(\tau - R/v_1)\}(\tau - R/v_1)H(\tau - R/v_1)e^{-\phi_1 R} - \{\lambda_{11}' + \lambda_{12}'(\tau - R/v_2)\}(\tau - R/v_2)H(\tau - R/v_2)e^{-\phi_2 R}\Big],$$

(4.18) $$T(R,\tau) = \theta_0(\eta/r)^{1/2}\Big[-\{\lambda_{13} + \lambda_{14}(\tau - R/v_1)\}H(\tau - R/v_1)e^{-\phi_1 R} + \{\lambda_{13}' + \lambda_{14}'(\tau - R/v_2)\}H(\tau - R/v_2)e^{-\phi_2 R}\Big],$$

(4.19) $$S_{rr}(R,\tau) = \theta_0(\eta/r)^{1/2}\Big[-\{\lambda_{15} + \lambda_{16}(\tau - R/v_1)\}H(\tau - R/v_1)e^{-\phi_1 R} + \{\lambda_{15}' + \lambda_{16}'(\tau - R/v_2)\}H(\tau - R/v_2)e^{-\phi_2 R}\Big],$$

(4.20) $$T_{11}(R\tau) = 2\theta_0\beta_0^2\varrho a_0^2(\eta/r)^{1/2}\Big[\{\lambda_{17} + \lambda_{18}(\tau - R/v_1)\}H(\tau - R/v_1)e^{-\phi_1 R} - \{\lambda_{17}' + \lambda_{18}'(\tau - R/v_2)\}H(\tau - R/v_2)e^{-\phi_2 R}\Big].$$

5. Long time solutions

The long time solutions can be obtained by expanding the values of m_i^2 ($i = 1, 2$) of Eq. (3.21) for small values of p in the Taylor series. The roots m_1, m_2 can be obtained as

$$m_1 = [1+\varepsilon]^{1/2}\sqrt{p} + 0(p^{3/2}),$$
$$m_2 = [1+\varepsilon]^{-1/2}p + 0(p^2).$$

The above expressions for the roots do not contain the thermal relaxation times up to the first order, which confirms the fact that the "second sound" effects are short-lived. Hence short-time solutions are more important than long-time solutions. However, the expressions for displacement, temperature, and stress can easily be obtained by using these values of m_i in different relevant equations.

6. Discussions of the results

The detailed analysis in the previous sections shows that there exist three types of waves, i.e. the dilatational wave, the thermal wave, and Alfvén wave travelling with velocities v_1, v_2 and a_0, respectively. The analysis also shows that the expressions containing $H(\tau - R/v_1)$, $H(\tau - R/v_2)$ and $H(\tau - \beta\eta')$ [6] represent the contributions of the dilatational wave, thermal wave and the Alfvén acoustic wave in the vicinity of their wavefronts $R = v_1\tau$, $R = v_2\tau$ and $\eta' = \tau/\beta$, respectively.

The displacement is found to be continuous but the temperature and stresses are found to be discontinuous in both the cases, and they are given by:

Case 1

$$(T^+ - T^-)_{R=v_1\tau} = \sigma_0(\eta/r)^{1/2}(v_1^2v_2^2 - v_1^2 - v_2^2 + 1)e^{-\phi_1 v_1\tau}/\beta T_0(v_2^2 - v_1^2), \tag{6.1}$$
$$(T^+ - T^-)_{R=v_2\tau} = -\sigma_0(\eta/r)^{1/2}(v_1^2v_2^2 - v_1^2 - v_2^2 + 1)e^{-\phi_2 v_2\tau}/\beta T_0(v_2^2 - v_1^2), \tag{6.2}$$
$$(S_{rr}^+ - S_{rr}^-)_{R=v_1\tau} = -\sigma_0(\eta/r)^{1/2}(v_1^2v_2^2 - v_1^2 - 2v_2^2 + 2)e^{-\phi_1 v_1\tau}/\beta T_0(v_2^2 - v_1^2), \tag{6.3}$$
$$(S_{rr}^+ - S_{rr}^-)_{R=v_2\tau} = \sigma_0(\eta/r)^{1/2}(v_1^2v_2^2 - v_2^2 - 2v_1^2 + 2)e^{-\phi_2 v_2\tau}/\beta T_0(v_2^2 - v_1^2), \tag{6.4}$$
$$(T_{11}^+ - T_{11}^-)_{R=v_1\tau} = -\sigma_0\varrho a_0^2(\eta/r)^{1/2}(v_2^2 - 1)e^{-\phi_1 v_1\tau}/\beta T_0(v_2^2 - v_1^2), \tag{6.5}$$
$$(T_{11}^+ - T_{11}^-)_{R=v_2\tau} = \sigma_0\varrho a_0^2(\eta/r)^{1/2}(v_1^2 - 1)e^{-\phi_2 v_2\tau}/\beta T_0(v_2^2 - v_1^2). \tag{6.6}$$

Case 2

$$(T^+ - T^-)_{R=v_1\tau} = \theta_0(\eta/r)^{1/2}(v_1^2v_2^2 - v_1^2 - 2v_2^2 + 2)e^{-\phi_1 v_1\tau}/(v_2^2 - v_1^2), \tag{6.7}$$
$$(T^+ - T^-)_{R=v_2\tau} = -\theta_0(\eta/r)^{1/2}(v_1^2v_2^2 - v_1^2 - 2v_2^2 + 2)e^{-\phi_2 v_2\tau}/(v_2^2 - v_1^2), \tag{6.8}$$

(6.9) $(S_{rr}^{+}-S_{rr}^{-})_{R=v_1\tau} = -\theta_0(\eta/r)^{1/2}(v_1^2v_2^2-2v_1^2-2v_2^2+4)e^{-\phi_1 v_1 \tau}/(v_2^2-v_1^2),$

(6.10) $(S_{rr}^{+}-S_{rr}^{-})_{R=v_2\tau} = \theta_0(\eta/r)^{1/2}(v_1^2v_2^2-2v_1^2-2v_2^2+4)e^{-\phi_2 v_2 \tau}/(v_2^2-v_1^2),$

(6.11) $(T_{11}^{+}-T_{11}^{-})_{R=v_1\tau} = 2\theta_0\varrho\beta_0^2a_0^2(\eta/r)^{1/2}(v_2^2-2)e^{-\phi_1 v_1 \tau}/\eta(v_2^2-v_1^2),$

(6.12) $(T_{11}^{+}-T_{11}^{-})_{R=v_2\tau} = -2\theta_0\varrho\beta_0^2a_0^2(\eta/r)^{1/2}(v_1^2-2)e^{-\phi_2 v_2 \tau}/\eta(v_2^2-v_1^2).$

From the above expressions, it is clear that the discontinuities decay exponentially with time.

7. Particular cases

i. If $\tau_0 = 0$, i.e. in the case of conventional coupled thermoelasticity

$$\lambda = 1+\varepsilon, \qquad \lambda_2 = 1, \qquad v_1 = 1, \qquad v_2 \to \infty, \qquad \phi_1 = \varepsilon/2, \qquad \phi_2 \to \infty.$$

From Eqs. (4.7)–(4.9) and (4.11)–(4.13), it is observed that the temperature at both the wavefronts in Case 1 and at the thermal wavefront in Case 2 become continuous. The temperature in Case 2 and stresses in both the cases suffer a finite jump at the elastic wavefront, given by

(7.1) $(S_{rr}^{+} - S_{rr}^{-})_{R=v_1\tau} = \sigma_0(\eta/r)^{1/2}\{e^{-\varepsilon\tau/2}\}/\beta T_0,$

(7.2) $(T_{11}^{+} - T_{11}^{-})_{R=v_1\tau} = -\sigma_0\varrho a_0^2(\eta/r)^{1/2}\{e^{-\varepsilon\tau/2}\}/\beta T_0,$

and

(7.3) $(T^{+} - T^{-})_{R=v_1\tau} = -\theta_0(\eta/r)^{1/2}e^{-\varepsilon\tau/2},$

(7.4) $(S_{rr}^{+} - S_{rr}^{-})_{R=v_1\tau} = \theta_0(\eta/r)^{1/2}e^{-\varepsilon\tau/2},$

(7.5) $(T_{11}^{+} - T_{11}^{-})_{R=v_1\tau} = \theta_0\varrho\beta_0^2a_0^2(\eta/r)^{1/2}\{e^{-\varepsilon\tau/2}\}/\eta.$

ii. If $\varepsilon = 0$ and $\tau_0 \neq 0$, then we have

$$\lambda_1 = 1, \qquad \lambda_2 = 1, \qquad v_1 = 1, \qquad v_2 \to \infty, \qquad \phi_1 = 0, \qquad \phi_2 \to \infty.$$

It is observed that the temperature and stresses at the elastic wavefront experience finite jumps in Case 1. The stresses and temperature experience finite jumps at both the wavefronts in Case 2 and are given by

(7.6) $(S_{rr}^{+} - S_{rr}^{-})_{R=v_1\tau} = \sigma_0(\eta/r)^{1/2}/\beta T_0,$

(7.7) $(T_{11}^{+} - T_{11}^{-})_{R=v_2\tau} = -\sigma_0\varrho a_0^2(\eta/r)^{1/2}/\beta T_0,$

and

(7.8) $(S_{rr}^{+} - S_{rr}^{-})_{R=v_1\tau} = \theta_0(\eta/r)^{1/2}(1-2\tau_0^*)/(1-\tau_0^*),$

(7.9) $(S_{rr}^{+} - S_{rr}^{-})_{R=v_2\tau} = -\theta_0(\eta/r)^{1/2}(1-2\tau_0^*)\{e^{-\tau/2\tau*_0}\}/(1-\tau_0^*),$

(7.10) $(T_{11}^{+} - T_{11}^{-})_{R=v_1\tau} = 2\theta_0\varrho\beta_0^2a_0^2(\eta/r)^{1/2}(1-2\tau_0^*)/\eta(1-\tau_0^*),$

(7.11) $(T_{11}^{+} - T_{11}^{-})_{R=v_2\tau} = 2\theta_0\varrho\beta_0^2a_0^2(\eta/r)^{1/2}\{e^{-\tau/2\tau_0^*}\}/\eta(1-\tau_0^*).$

iii. If $\varepsilon = 0$, $\tau_0 = 0$, i.e. the coupling and relaxation effects are ignored then

$$\lambda_1 = 1, \quad \lambda_2 = 1, \quad v_1 = 1, \quad v_2 \to \infty, \quad \phi_1 = 0, \quad \phi_2 \to \infty.$$

The finite jumps obtained in both cases at the respective wavefronts are given by

$$(S^+_{rr} - S^-_{rr})_{R=v_1\tau} = \sigma_0(\eta/r)^{1/2}/\beta T_0, \tag{7.12}$$

$$(T^+_{11} - T^-_{11})_{R=v_2\tau} = \sigma_0 \varrho a_0^2(\eta/r)^{1/2}/\beta T_0, \tag{7.13}$$

and

$$(T^+ - T^-)_{R=v_1\tau} = -\theta_0(\eta/r)^{1/2}, \tag{7.14}$$

$$(S^+_{rr} - S^-_{rr})_{R=v_1\tau} = \theta_0(\eta/r)^{1/2}, \tag{7.15}$$

$$(T^+_{11} - T^-_{11})_{R=v_1\tau} = 2\theta_0\varrho\beta_0^2 a_0^2(\eta/r)^{1/2}. \tag{7.16}$$

iv. If the magnetic field is ignored, i.e., $H = 0$, that is when there is no coupling between the electromagnetic and strain fields, then the stress produced by the magnetic field $T_{11} = 0$.

8. Numerical results and discussion

The various jumps obtained at their respective wavefronts theoretically for temperature and stresses in the sections, are computed numerically for carbon steel [15] for which the physical data are

$$\begin{aligned}
\lambda &= 9.3 \times 10^{10}\,\mathrm{Nm^{-2}}, & \mu &= 8.4 \times 10^{10}\,\mathrm{Nm^{-2}},\\
\alpha_T &= 13.2 \times 10^{-6}\,\mathrm{deg^{-1}}, & \varrho &= 7.9 \times 10^{3}\,\mathrm{Kg\,m^{-3}},\\
C_v &= 6.4 \times 10^{2}\,\mathrm{J\,Kg^{-1}deg^{-1}}, & \sigma_0 &= 8.6 \times 10^{5}\mathrm{N},\\
\mu_0 &= 1.3 \times 10^{-6}\mathrm{Hr\,m^{-1}}, & T_0 &= 293.16^\circ\,\mathrm{K},\\
H &= 1.0\,\mathrm{Gauss}, & \varepsilon &= 0.34.
\end{aligned}$$

The variations of jumps and their particular cases are plotted with respect to time for different relaxation times, i.e. $\tau_0^* = 0.0, 0.1$, as shown in the Figs. 1 to 6. The jumps decay exponentially with time. It is observed that the decay of these jumps is more rapid at the thermal wavefront than at the elastic wavefront. It is also observed that the magnitude of these jumps, in general, is greater in case of the step in stress than in case of the step in temperature. It is also observed that an additional stress is produced in the medium due to the perturbation in the magnetic field, which vanishes in the absence of magnetic field. It is also clear from the comparison of Figs. 2–6 that the variations in magnitudes of the jumps are greater in case of the coupled theory of thermoelasticity than in the uncoupled one.

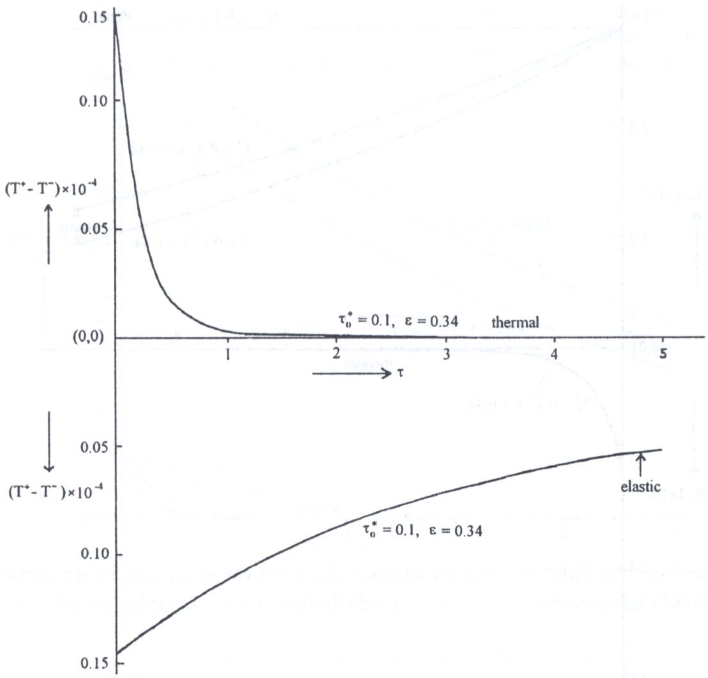

FIG. 1. Variation of jumps in temperature with respect to time for different relaxation times, at the wavefronts; — coupled theory.

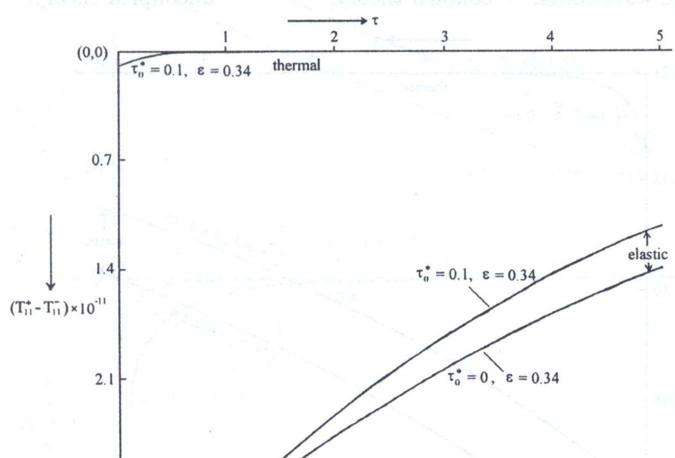

FIG. 2. Variation of jumps in stress with respect to time for different relaxation times, at the wavefronts; — coupled theory, — · — · —· uncoupled theory.

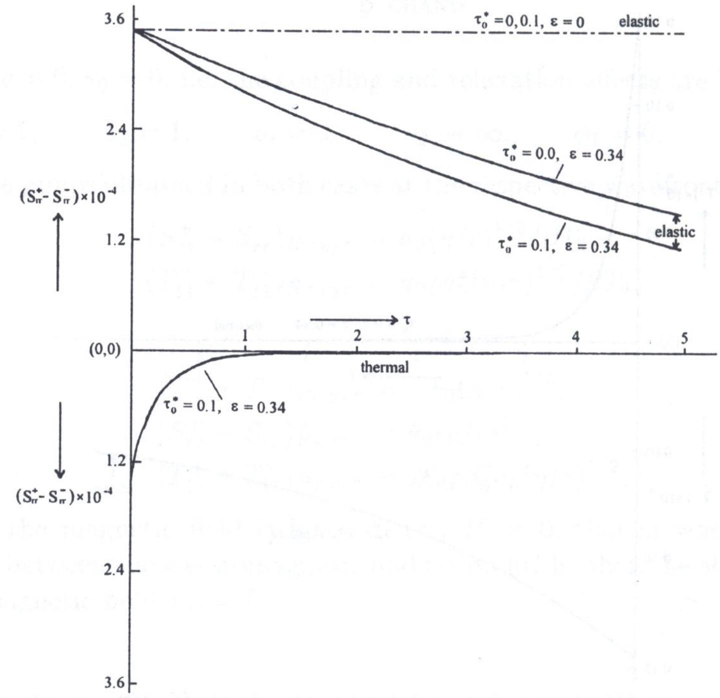

FIG. 3. Variation of jumps in stress with respect to time for different relaxation times, at the wavefronts; — coupled theory, — · — · — uncoupled theory.

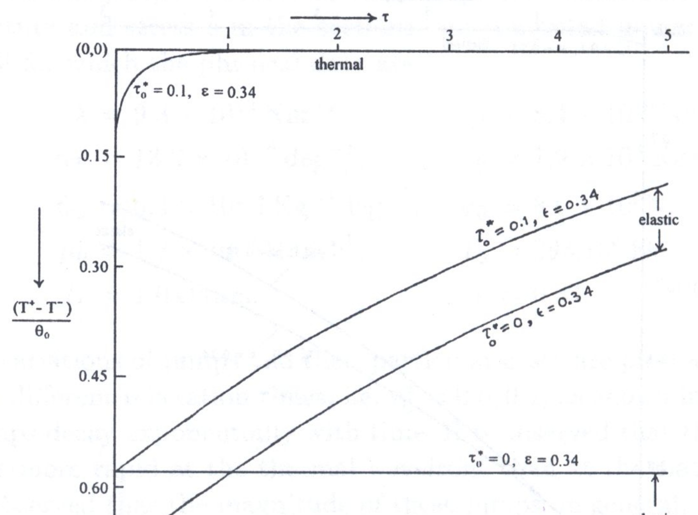

FIG. 4. Variation of jumps in temperature with respect to time for different relaxation times, at the wavefronts; — coupled theory, — · — · — uncoupled theory.

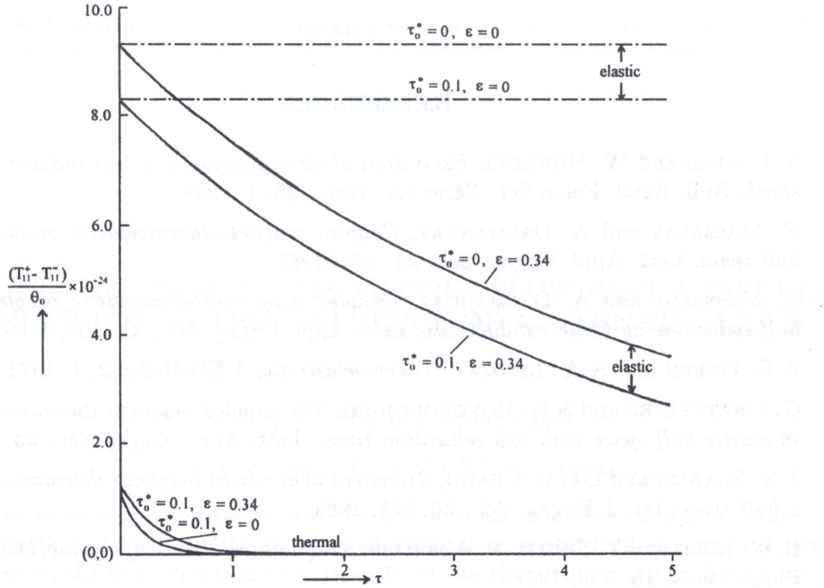

FIG. 5. Variation of jumps in stress with respect to time for different relaxation times, at the wavefvronts; — coupled theory, — · — · — uncoupled theory.

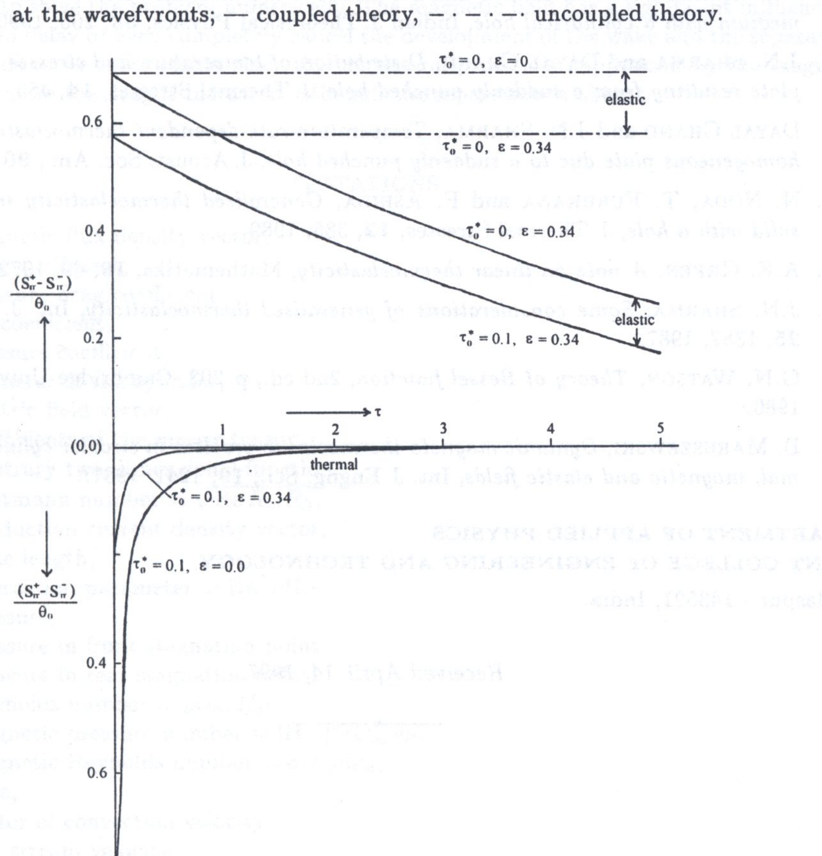

FIG. 6. Variation of jumps in stress with respect to time for different relaxation times, at the wavefronts; — coupled theory, — · — · — uncoupled theory.

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DEPARTMENT OF APPLIED PHYSICS
BEANT COLLEGE OF ENGINEERING AND TECHNOLOGY,
Gurdaspur - 143521, India.

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