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YIELD CRITERION ACCOUNTING FOR THE INFLUENCE OF THE THIRD INVARIANT OF STRESS TENSOR DEVIATOR. PART II. ANALYSIS OF CONVEXITY CONDITION OF THE YIELD SURFACE

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General form of yield condition for isotropic and homogeneous bodies is considered in the paper. In the space of principal stresses, the limit condition is graphically represented by a proper regular surface which is assumed here to be at least of C^2 class. Due to Drucker's Postulate, the yield surface should be convex. General form of convexity condition of the considered surface is derived using methods of differential geometry. Parametrization of the yield surface is given, the first and the second derivatives of the position vector with respect to the chosen parameters are calculated, what enables determination of the tangent and unit normal vectors at given point, and also determination of the first and the second fundamental form of the considered surface. Finally the Gaussian and mean curvatures, which are given by the coefficients of the first and the second fundamental form as the invariants of the shape operator, are found. Convexity condition of the considered surface expressed in general in terms of the mean and Gaussian curvatures, is formulated for any form of functions determining the character of the surface.

Key words: yield surface, convexity condition.

1. Introduction

Let us consider ideally elastic-plastic material. In the range of elastic deformation, linear constitutive law (Hooke's Law) is considered true:

$$\left\{\begin{array}{l}\boldsymbol{\sigma} = \mathbf{S}\cdot\boldsymbol{\varepsilon}\\[2mm] \boldsymbol{\varepsilon} = \mathbf{C}\cdot\boldsymbol{\sigma}\\[2mm] \mathbf{C}\circ\mathbf{S} = \mathbf{I}^S\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}\sigma_{ij} = S_{ijkl}\varepsilon_{kl}\\[2mm] \varepsilon_{ij} = C_{ijkl}\sigma_{kl}\\[2mm] S_{ijkl}C_{klmn} = \dfrac{1}{2}\left(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}\right)\end{array}\right. \qquad i,...,n = 1,2,3, \tag{1.1}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the second order symmetric Cauchy stress tensor and infinitesimal strain tensor (symmetric part of the displacement gradient) respectively,

$\mathbf{S}$ and $\mathbf{C}$ are the fourth order symmetric stiffness and compliance tensors respectively and $\mathbf{I}^S$ is the identity operator in the linear space of symmetric second order tensors. The range of validity of the Hooke's Law is defined by a general limit (yield) condition of form

$$W(\boldsymbol{\sigma}) < 0. \tag{1.2}$$

Plastic flow rule is assumed to be of the following form:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\boldsymbol{\partial}_{\boldsymbol{\sigma}} F \quad \Leftrightarrow \quad \dot{\varepsilon}_{ij}^p = \dot{\lambda}\frac{\partial F}{\partial \sigma_{ij}}, \tag{1.3}$$

where $\dot{\boldsymbol{\varepsilon}}^p$ is the plastic strain rate tensor, F is the plastic potential.

In 1950's Drucker has introduced and developed a proposition of the idea of a stable plastic material [1]. Drucker stated that the material is stable if the total work performed by the increment of load through the caused displacement is non-negative. It is always fulfilled in case of elastic deformation. If the final stress reaches the limit state determined by the yield condition, then plastic deformation occurs and the Drucker's postulate can be written in form of the following inequality

$$dL = \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^0\right) \cdot d\boldsymbol{\varepsilon}^p \geqslant 0. \tag{1.4}$$

In particular, if the stress increment is infinitesimal, one can write simply:

$$dL = d\boldsymbol{\sigma} \cdot d\boldsymbol{\varepsilon}^p \geqslant 0. \tag{1.5}$$

Let us consider that the initial stress state is the limit stress-state. All limit states, which are given by the yield condition of general form $W(\boldsymbol{\sigma}) = 0$ are represented in the space of principal stresses as a three-dimensional surface. If no hardening is assumed, then any stress increment vector $d\boldsymbol{\sigma}$ connects two points, both of which belong to the surface. In case of an infinitesimal increment of stress at the limit state this is a tangent vector to the yield surface at the considered point. It can be shown that at current assumptions on the model of material, validity of Drucker's postulate requires that the flow rule (1.3) must be associated with the limit condition (1.2), namely $F(\boldsymbol{\sigma}) = W(\boldsymbol{\sigma})$ – then the infinitesimal strain increment vector corresponding to the considered stress increment is represented by a vector perpendicular to the yield surface. In this case, the Drucker's postulate (1.5) can be interpreted as a requirement of non-negativeness of the scalar product of tangent and normal vector at any point of the yield surface. It is equivalent to the statement that whole surface is non-concave. If the flow rule is not associated with the yield condition, convexity of the yield surface is often assumed as well.

The aim of this paper is to show the application of the general convexity analysis of three-dimensional surfaces in mechanics of solids. In particular, the convexity of a new yield surface for isotropic homogeneous solids proposed in the first part of the paper [3] is analyzed. As a result, a general form of convexity condition for arbitrary chosen form of pressure and Lode angle influence functions appearing in the proposed yield surface formulation is derived. Those conditions may be applied as an inequality constraints in an optimization problem of material identification. Material parameters determined in the process of fitting the results obtained from the simulation using assumed model to those obtained from experiment with use of the aforementioned condition, guarantee convexity of the determined yield surface.

General methodology of the differential geometry in the analysis of convexity of the given surface $\mathscr{S}$ in E^3 is as follows:

- Surface parametrization

$$\mathbf{x} \in \mathscr{S} \ \rightarrow \ \mathbf{x} = \mathbf{x}(\alpha, \beta). \tag{1.6}$$

- Finding the tangent vectors (first derivatives of the position vector with respect to the chosen surface parameters) at any point

$$\begin{aligned} \mathbf{x}_\alpha &= \frac{\partial}{\partial \alpha}\mathbf{x}(\alpha, \beta), \\ \mathbf{x}_\beta &= \frac{\partial}{\partial \beta}\mathbf{x}(\alpha, \beta). \end{aligned} \tag{1.7}$$

- Finding the second derivatives of the position vector with respect to the chosen surface parameters at any point

$$\begin{aligned} \mathbf{x}_{\alpha\alpha} &= \frac{\partial^2}{\partial \alpha^2}\mathbf{x}(\alpha, \beta), \\ \mathbf{x}_{\beta\beta} &= \frac{\partial^2}{\partial \beta^2}\mathbf{x}(\alpha, \beta), \\ \mathbf{x}_{\alpha\beta} &= \frac{\partial^2}{\partial \alpha \partial \beta}\mathbf{x}(\alpha, \beta). \end{aligned} \tag{1.8}$$

- Finding the external unit normal vector at any point as a scaled cross-product of the tangent vectors

$$\boldsymbol{\nu} = \pm \frac{\mathbf{x}_\alpha \times \mathbf{x}_\beta}{|\mathbf{x}_\alpha \times \mathbf{x}_\beta|}. \tag{1.9}$$

- Determination of the coefficients of the first fundamental form referring to the tangent vectors

$$E = \mathbf{x}_\alpha \cdot \mathbf{x}_\alpha, \qquad F = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta, \qquad G = \mathbf{x}_\beta \cdot \mathbf{x}_\beta. \tag{1.10}$$

- Determination of the coefficients of the second fundamental form referring to the normal vectors and second derivatives of the position vector

$$e = \boldsymbol{\nu} \cdot \mathbf{x}_{\alpha\alpha}, \qquad f = \boldsymbol{\nu} \cdot \mathbf{x}_{\alpha\beta}, \qquad g = \boldsymbol{\nu} \cdot \mathbf{x}_{\beta\beta}. \tag{1.11}$$

- Determination of the shape operator and its invariants – Gaussian curvature κ_G and mean curvature κ_M – in terms of the coefficients of the first and the second fundamental form

$$\kappa_G = \frac{eg - f^2}{EG - F^2}, \qquad \kappa_M = \frac{eG - 2fF + gE}{2(EG - F^2)}. \tag{1.12}$$

- Formulation of the convexity condition in terms of Gaussian and mean curvatures:

$$\kappa_M < 0, \qquad \kappa_G > 0. \tag{1.13}$$

2. Convexity condition for proposed surface

2.1. *Proposition of yield condition for isotropic bodies*

Let us recall a general yield condition discussed in Part I of the paper:

$$W\colon \widetilde{\eta}_f \varPhi_f + \widetilde{\eta}_v \varPhi_v - 1 = 0, \tag{2.1}$$

where $\varPhi_f$ – density of energy of distortion, $\varPhi_v$ – density of energy of volume change and $\widetilde{\eta}_f$ and $\widetilde{\eta}_v$ are certain stress state dependent functions called *influence functions*. Certain assumptions made on those functions, discussed in details in Part I, enable rewriting (2.1) in the following form:

$$W\colon \eta_f(\theta) q^2 + \eta_p(p) - 1 = 0, \tag{2.2}$$

where p – hydrostatic stress, q – deviatoric stress, θ – Lode angle. Parameters p, q, θ (Haigh-Westergaard coordinates/Lode parameters) are proportional to cylindrical coordinates (with the specified axis parallel to the p axis) in the space of principal stresses. There exists the one-to-one relation between "Cartesian" coordinates and Lode parameters:

$$\sigma_i = \sigma_i(p,q,\theta) \ \Rightarrow \begin{cases} \sigma_1 = p + \sqrt{\dfrac{2}{3}} q\cos(\theta)\,, \\ \sigma_2 = p + \sqrt{\dfrac{2}{3}} q\cos\left(\theta - \dfrac{2\pi}{3}\right), \\ \sigma_3 = p + \sqrt{\dfrac{2}{3}} q\cos\left(\theta + \dfrac{2\pi}{3}\right), \end{cases} \qquad i = 1,2,3 \tag{2.3}$$

or equivalently:

$$\begin{cases} p = \dfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3), \qquad p \in (-\infty;\infty), \\ q = \sqrt{\dfrac{1}{3}\left[(\sigma_3-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_1-\sigma_2)^2\right]}, \qquad q > 0, \\ \theta = \dfrac{1}{3}\arccos\dfrac{\sqrt{2}\,(2\sigma_1-\sigma_2-\sigma_3)\,(2\sigma_2-\sigma_3-\sigma_1)\,(2\sigma_3-\sigma_1-\sigma_2)}{\left[(\sigma_3-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_1-\sigma_2)^2\right]^{3/2}} \\ \qquad = \operatorname{arc\,tan}\dfrac{\sqrt{3}(\sigma_2-\sigma_3)}{2\sigma_1-\sigma_2-\sigma_3}, \qquad \theta \in (0;2\pi). \end{cases} \tag{2.4}$$

The difference between the influence functions denoted with and without tilde, is only in constant scaling parameters which are proportional to the stiffness moduli – shear modulus for $\widetilde{\eta}_f$ and bulk modulus for $\widetilde{\eta}_v$. Further distinction between $\widetilde{\eta}_v$ and η_p is that η_p involves already term p^2 which is proportional to Φ_v – it has no influence on the derivation of the convexity condition since $\widetilde{\eta}_v$ is otherwise only pressure-dependent. This simple substitutions simplify the derivation in great extent.

A general form of convexity condition for any form of influence functions will be derived in the paper. The derivation will base on classical methods of differential geometry, namely – convexity analysis of three-dimensional surfaces.

2.2. Surface parametrization

Typical methods mentioned above require calculating both the tangent and normal vectors of the surface. Tangent vectors can be obtained through differentiating position vector of a point on the surface. Three-dimensional regular surface is in fact a two-dimensional differentiable manifold, thus position of any point belonging to the surface can be explicitly determined by at most two independent parameters. To avoid differentiating in curvilinear coordinate system, we should express the position vector in "Cartesian" coordinates of principal stresses, however each component of this vector should be expressed by two parameters determining the surface. Let us assume that those parameters are p and θ:

$$\sigma_i \in W \;\rightarrow\; \sigma_i = \sigma_i(p, \theta), \qquad i = 1, 2, 3. \tag{2.5}$$

Then using condition (2.2) and relations (2.4) and remembering that q as the norm of the stress tensor deviator has to be positive, we can write:

$$q = \sqrt{\frac{1 - \eta_p(p)}{\eta_f(\theta)}}. \tag{2.6}$$

Physical interpretation of η_f should be used now – since distortional strains and shearing stresses which correspond with q are in the greatest extent responsible for material effort, one should expect that $\forall_\theta\, \eta_f(\theta) > 0$ for any given p – it agrees with intuition and it is confirmed by experiments.

It should be also assumed that $\eta_p(p) \leqslant 1$. Indeed, since η_f is assumed to be positive then any stress state σ corresponding with arbitrarily chosen value of q is a limit state (it belongs to the limit surface W) only when $\eta_p(p) \leqslant 1$. If there exists such value of p equal p_0 for which $\eta_p(p_0) > 1$, then there exists no real q for which Eq. (2.2) is fulfilled and there is no point on the surface corresponding with such value of parameter p – only part of infinite domain of $p \in (-\infty; \infty)$ is used to parametrize the surface. Coordinates of any point belonging to W can be thus written as follows:

$$W : \begin{cases} \sigma_1(p;\theta) = p + \sqrt{\dfrac{2}{3}}\sqrt{\dfrac{1-\eta_p(p)}{\eta_f(\theta)}}\cos(\theta)\,, \\[2ex] \sigma_2(p;\theta) = p + \sqrt{\dfrac{2}{3}}\sqrt{\dfrac{1-\eta_p(p)}{\eta_f(\theta)}}\cos\left(\theta - \dfrac{2\pi}{3}\right), \\[2ex] \sigma_3(p;\theta) = p + \sqrt{\dfrac{2}{3}}\sqrt{\dfrac{1-\eta_p(p)}{\eta_f(\theta)}}\cos\left(\theta + \dfrac{2\pi}{3}\right). \end{cases} \tag{2.7}$$

2.3. First and second derivatives of the position vector, tangent and normal vectors

Since $W \in C^2$, having the position vector expressed by surface parameters, we can calculate now components of vectors tangent to the surface:

$$\frac{\partial \sigma_i}{\partial p} = 1 - \frac{1}{\sqrt{6\eta_f[1-\eta_p]}} \frac{\partial \eta_p}{\partial p} \cos(\theta + \alpha_i), \tag{2.8}$$

$$\frac{\partial \sigma_i}{\partial \theta} = -\sqrt{\frac{2}{3}[1-\eta_p]}[\eta_f]^{3/2} \cdot \left[\eta_f \sin(\theta + \alpha_i) + \frac{1}{2}\frac{\partial \eta_f}{\partial \theta} \cos(\theta + \alpha_i)\right], \tag{2.9}$$

where $i = 1, 2, 3$ and $\alpha_1 = 0$, $\alpha_2 = -2\pi/3$, $\alpha_3 = 2\pi/3$. Further derivatives of position vector are equal

$$\frac{\partial^2 \sigma_i}{\partial p^2} = -\frac{\cos(\theta + \alpha_i)}{\sqrt{6\eta_f}[1-\eta_p]^{3/2}} \left[\frac{1}{2}\left(\frac{\partial \eta_p}{\partial p}\right)^2 + [1-\eta_p]\frac{\partial^2 \eta_p}{\partial p^2}\right], \tag{2.10}$$

$$\begin{aligned} \frac{\partial^2 \sigma_i}{\partial \theta^2} = -\sqrt{\frac{2[1-\eta_p]}{3[\eta_f]^5}} \cdot \Bigg[& -\eta_f \frac{\partial \eta_f}{\partial \theta} \sin(\theta + \alpha_i) \\ & + \frac{1}{4}\left(2\eta_f \frac{\partial^2 \eta_f}{\partial \theta^2} - 3\left(\frac{\partial \eta_f}{\partial \theta}\right)^2 + 4[\eta_f]^2\right) \cos(\theta + \alpha_i)\Bigg], \end{aligned} \tag{2.11}$$

$$\frac{\partial^2 \sigma_i}{\partial p \partial \theta} = \frac{1}{\sqrt{6[\eta_f]^3[1-\eta_p]}} \cdot \frac{\partial \eta_p}{\partial p} \cdot \left[\eta_f \sin(\theta + \alpha_i) + \frac{1}{2}\frac{\partial \eta_f}{\partial \theta} \cos(\theta + \alpha_i)\right]. \tag{2.12}$$

Normal to the surface at a fixed point is perpendicular to any tangent vector at this point (see Fig. 1). Since tangent plane is a two-dimensional one, a basis

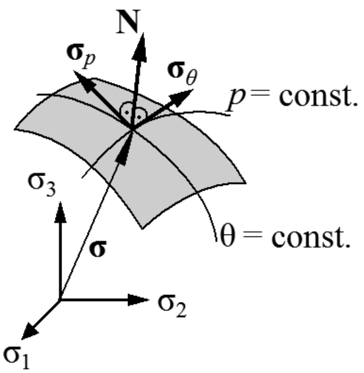

FIG. 1. Tangent and normal vectors at the given point of the surface.

in this space consists of two tangent vectors – i.e. those given by Eqs. (2.8) and (2.9). So the unit normal vector $\mathbf{N}$ is parallel to the cross-product of those two:

$$\mathbf{N} = \frac{\nabla W}{|\nabla W|} = \frac{\boldsymbol{\sigma}_\theta \times \boldsymbol{\sigma}_p}{|\boldsymbol{\sigma}_\theta \times \boldsymbol{\sigma}_p|}, \tag{2.13}$$

where $\boldsymbol{\sigma}_X$ denotes partial derivative of $\boldsymbol{\sigma}$ with respect to parameter X. The formula (2.13) specified with use of (2.8) and (2.9) leads after trigonometric simplification to the following final form. Just to make the notation clear, numerator and denominator are written separately. It is convenient to write the cross-product in the numerator as a sum of two vectors, one of which is parallel to the p axis:

$$\begin{aligned} \boldsymbol{\sigma}_\theta \times \boldsymbol{\sigma}_p = \frac{\partial \eta_p}{\partial p} \cdot \frac{1}{2\sqrt{3}\eta_f}[1,1,1] \\ + \sqrt{\frac{1-\eta_p}{2[\eta_f]^3}} \left[\left(-\frac{\partial \eta_f}{\partial \theta} \sin(\theta) + 2\eta_f \cos(\theta) \right), \right. \\ \left\{ \left(\sqrt{3}\eta_f + \frac{1}{2}\frac{\partial \eta_f}{\partial \theta} \right) \sin(\theta) + \left(-\eta_f + \frac{\sqrt{3}}{2}\frac{\partial \eta_f}{\partial \theta} \right) \cos(\theta) \right\}, \\ \left. \left\{ \left(-\sqrt{3}\eta_f + \frac{1}{2}\frac{\partial \eta_f}{\partial \theta} \right) \sin(\theta) - \left(\eta_f + \frac{\sqrt{3}}{2}\frac{\partial \eta_f}{\partial \theta} \right) \cos(\theta) \right\} \right], \end{aligned} \tag{2.14}$$

$$\begin{aligned} L_N &= |\boldsymbol{\sigma}_\theta \times \boldsymbol{\sigma}_p| \\ &= \frac{1}{\eta_f^{3/2}} \sqrt{3(1-\eta_p)\left[{\eta_f}^2 + \frac{1}{4}\left(\frac{\partial \eta_f}{\partial \theta} \right)^2 \right] + \frac{1}{4}\left(\frac{\partial \eta_p}{\partial p} \right)^2 \eta_f} = L_N > 0. \end{aligned} \tag{2.15}$$

Please note that sequence of tangent vectors in the above vector cross-product influences the orientation of resultant normal vector (cross-product is bilinear skew-symmetric operation) which has significant role in convexity condition formulation. Interior of the yield surface (area of safe stress states) should be determined. The $(0,0,0)$ point should always be in the surface's interior. Let's check the orientation of a normal vector given by (2.13). At any point in the stress space (not only at those belonging to the surface), local orthonormal basis (holonomic basis respective for (p,q,θ) coordinates – see Fig. 2) can be determined, namely, these are normalized derivatives of a position vector given by (2.3) (not to be mistaken with position vector of a point belonging to the surface given by (2.7)) with respect to the corresponding parameter:

$$
\begin{aligned}
\mathbf{e}_p &= \frac{1}{\sqrt{3}}[1,1,1], \\
\mathbf{e}_q &= \sqrt{\frac{2}{3}}\left[\cos(\theta),\ \cos\left(\theta - \frac{2\pi}{3}\right),\ \cos\left(\theta + \frac{2\pi}{3}\right)\right], \\
\mathbf{e}_\theta &= \sqrt{\frac{2}{3}}\left[-\sin(\theta),\ -\sin\left(\theta - \frac{2\pi}{3}\right),\ -\sin\left(\theta + \frac{2\pi}{3}\right)\right],
\end{aligned}
\tag{2.16}
$$

$$
\mathbf{e}_K \cdot \mathbf{e}_L = \delta_{KL}, \qquad K, L = p, q, \theta.
$$

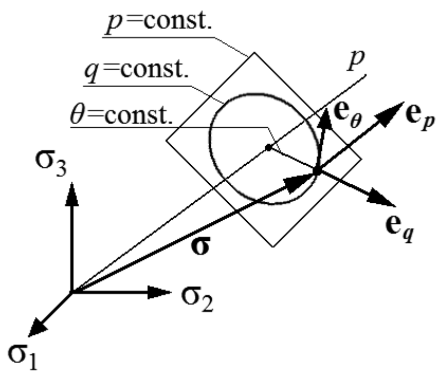

FIG. 2. Local holonomic basis respective for the Lode parameters p, q, θ.

The orientation of those vectors is already given and shown in the Fig. 2 and it is a consequence of definition (2.3) – $\mathbf{e}_p$ is oriented along the hydrostatic stress axis pointing positive values of p – $\mathbf{e}_q$ is oriented away from $(0,0,0)$ – and $\mathbf{e}_\theta$ is oriented counter-clockwise when looking at any octahedral plane (perpendicular to p axis) from the side of greater values of p. Due to the same reasons for which η_f was assumed to be positive valued, we can consider that an external normal is the one which q-component is oriented the same way as $\mathbf{e}_q$ – safe stress states (interior of yield surface) are close to $(0,0,0)$ point or – more generally speaking – the safer is the stress state, the smaller should be its deviatoric component $(q \to 0)$ and the closer should it be placed to the p axis. In such situation $\mathbf{N} \cdot \mathbf{e}_q > 0$ should be fulfilled. Indeed:

$$
\mathbf{N} \cdot \mathbf{e}_q = \frac{\sqrt{3}}{L_N}\sqrt{\frac{(1-\eta_p)}{\eta_f}}, \tag{2.17}
$$

which is always positive due to the assumed $1 \geqslant \eta_p$ and positiveness of L_N, η_f, so $\mathbf{N}$ defined by (2.13) is an external normal.

2.4. First and second fundamental form, shape operator

Once the components of the normal vector and the tangent ones and their derivatives are calculated, one can obtain coefficients of the first and the second fundamental form and finally the values of curvatures which allow us to determine the convexity condition – typical methods of differential geometry shown e.g. in [2] will be used. The first fundamental form I is an inner (scalar) product of any two tangent vectors at a given point. As it was said before, tangent plane is a two-dimensional space in which vectors $\boldsymbol{\sigma}_p$ and $\boldsymbol{\sigma}_\theta$ form a basis (not necessarily an orthogonal or normalized one), so any two tangent vectors can be expressed as a linear combination of the two mentioned: $\mathbf{v}_1 = v_{1p}\boldsymbol{\sigma}_p + v_{1\theta}\boldsymbol{\sigma}_\theta \mathbf{v}_2 = v_{2p}\boldsymbol{\sigma}_p + v_{2\theta}\boldsymbol{\sigma}_\theta$. Their scalar product is equal

$$I(\mathbf{v}_1,\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{v}_2 = v_{1p}v_{2p}(\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_p) + (v_{1p}v_{2\theta} + v_{1\theta}v_{2p})(\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_\theta) + v_{1\theta}v_{2\theta}(\boldsymbol{\sigma}_\theta\cdot\boldsymbol{\sigma}_\theta), \tag{2.18}$$

what can be rewritten in such matrix form:

$$I(\mathbf{v}_1,\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{v}_2 = \mathbf{v}_1\cdot\mathbf{I}\cdot\mathbf{v}_2 = \begin{bmatrix} v_{1p} \\ v_{1\theta} \end{bmatrix}^T \cdot \begin{bmatrix} E & F \\ F & G \end{bmatrix} \cdot \begin{bmatrix} v_{2p} \\ v_{2\theta} \end{bmatrix}. \tag{2.19}$$

Symmetric operator $\mathbf{I}$ can be considered as a metric tensor in the space of tangent vectors. Arc length of an infinitesimal section of a curve belonging to the surface is given as follows:

$$ds^2 = Edp^2 + 2Fdpd\theta + Gd\theta^2. \tag{2.20}$$

Coefficients of the first fundamental form are equal to

$$\begin{aligned} E &= |\boldsymbol{\sigma}_p|^2 = \boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_p = 3 + \left(\frac{\partial\eta_f}{\partial p}\right)^2 \cdot \frac{1}{3\eta_f(1-\eta_p)}, \\ F &= \boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_\theta = \frac{1}{4(\eta_f)^2}\cdot\frac{\partial\eta_p}{\partial p}\cdot\frac{\partial\eta_f}{\partial\theta}, \\ G &= |\boldsymbol{\sigma}_\theta|^2 = \boldsymbol{\sigma}_\theta\cdot\boldsymbol{\sigma}_\theta = (1-\eta_p)\left[\frac{1}{\eta_f} + \frac{1}{4(\eta_f)^3}\left(\frac{\partial\eta_f}{\partial\theta}\right)^2\right]. \end{aligned} \tag{2.21}$$

Second fundamental form, just as the first one, is a bilinear form on tangent vectors at a given point of the surface defined as follows:

$$II(\mathbf{v}_1,\mathbf{v}_2) = \mathrm{S}(\mathbf{v}_1)\cdot\mathbf{v}_2, \tag{2.22}$$

where $\mathbf{S}$ is shape operator defined as:

$$\mathbf{S} = \left(\mathbf{I}\,\mathbf{II}^{-1}\right)^T = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}: \qquad \mathrm{S}(\mathbf{v}) = -\mathbf{N}_{\mathbf{v}} = -\nabla\mathbf{N}\cdot\mathbf{v}. \tag{2.23}$$

Shape operator (Weingarten map, second fundamental tensor) describes variation of a unit normal of the surface with the change of direction of the tangent vector $\mathbf{v}$. Eigenvalues of the shape operator are equal principal (extremal) curvatures of the surface at a given point, while the corresponding eigenvectors indicate the directions of those curvatures. Invariants of the shape operator – determinant and half of the trace – are equal Gaussian and mean curvatures respectively. Second fundamental form II can be written as:

$$\begin{aligned} II(\mathbf{v}_1, \mathbf{v}_2) &= (-\mathbf{N}_{\mathbf{v}_1})\cdot\mathbf{v}_2 \\ &= -(v_{1p}\mathbf{N}_p + v_{1\theta}\mathbf{N}_\theta)\cdot(v_{2p}\boldsymbol{\sigma}_p + v_{2\theta}\boldsymbol{\sigma}_\theta) = -v_{1p}v_{2p}(\mathbf{N}_p\cdot\boldsymbol{\sigma}_p) \\ &\quad - [v_{1p}v_{2\theta}(\mathbf{N}_p\cdot\boldsymbol{\sigma}_\theta) + v_{1\theta}v_{2p}(\mathbf{N}_\theta\cdot\boldsymbol{\sigma}_p)] - v_{1\theta}v_{2\theta}(\mathbf{N}_\theta\cdot\boldsymbol{\sigma}_\theta) \end{aligned} \tag{2.24}$$

or it can rewritten in the following matrix form:

$$II(\mathbf{v}_1\cdot\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{II}\cdot\mathbf{v}_2 = \begin{bmatrix} v_{1p} \\ v_{1\theta} \end{bmatrix}^T \cdot \begin{bmatrix} e & f \\ f & g \end{bmatrix} \cdot \begin{bmatrix} v_{2p} \\ v_{2\theta} \end{bmatrix}, \tag{2.25}$$

where the coefficients of the second fundamental form are given by following relations:

$$\begin{aligned} e &= -\mathbf{N}_p\cdot\boldsymbol{\sigma}_p = \mathbf{N}\cdot\boldsymbol{\sigma}_{pp} = -\frac{\sqrt{3}}{2L_N\eta_f}\left[\frac{\partial^2\eta_p}{\partial p^2} + \frac{1}{2}\left(\frac{\partial\eta_p}{\partial p}\right)^2\frac{1}{(1-\eta_p)}\right], \\ f &= -\mathbf{N}_p\cdot\boldsymbol{\sigma}_\theta = -\mathbf{N}_\theta\cdot\sigma_p = \mathbf{N}\cdot\boldsymbol{\sigma}_{p\theta} = \mathbf{N}\cdot\boldsymbol{\sigma}_{\theta p} = 0, \\ g &= -\mathbf{N}_\theta\cdot\boldsymbol{\sigma}_\theta = \mathbf{N}\cdot\boldsymbol{\sigma}_{\theta\theta} = -\frac{\sqrt{3}(1-\eta_p)}{4L_N[\eta_f]^3}\left[4(\eta_f)^2 + 2\eta_f\frac{\partial^2\eta_f}{\partial\theta^2} - \left(\frac{\partial\eta_f}{\partial\theta}\right)^2\right]. \end{aligned} \tag{2.26}$$

Components of $\mathbf{S}$ can be expressed in terms of coefficients of the first and second fundamental form (Weingarten equations):

$$\begin{aligned} S_{11} &= \frac{eG - fF}{EG - F^2}, & S_{12} &= \frac{fG - gF}{EG - F^2}, \\ S_{21} &= \frac{fE - eF}{EG - F^2}, & S_{22} &= \frac{gE - fF}{EG - F^2}. \end{aligned} \tag{2.27}$$

2.5. *Invariants of* $\boldsymbol{S}$*, principal (extremal), mean and Gaussian curvatures*

Since components of $\mathbf{S}$ depend on the quantities chosen for parametrization, they are not as important as invariants of $\mathbf{S}$ – its eigenvalues (principal curvatures – extremal of all possible values of curvature), its trace (which is proportional to the mean of the principal curvatures – mean curvature κ_M) and its determinant (product of principal curvatures – Gaussian curvature κ_G). Eigenvalues of $\mathbf{S}$ can be found from the characteristic polynomial of S:

$$\det(\mathbf{S} - \kappa \mathrm{I}) = \kappa^2 - I_1(\mathbf{S})\kappa + I_2(\mathbf{S}) = 0 \tag{2.28}$$

or

$$\kappa^2 - 2\kappa_M \kappa + \kappa_G = 0, \tag{2.29}$$

where

$$\begin{aligned} \kappa_M &= \frac{1}{2} I_1(\mathbf{S}) = \frac{1}{2}\mathrm{tr}(\mathbf{S}) = \frac{1}{2}\mathbf{I}\cdot(\mathbf{II})^{-1} = \frac{1}{2}(S_{11}+S_{22}) \\ &= \frac{eG - 2fF + gE}{2(EG-F^2)} \ \text{– mean curvature,} \\ \kappa_G &= I_2(\mathbf{S}) = \det(\mathbf{S}) = \frac{\det(\mathbf{I})}{\det(\mathbf{II})} = S_{11}\cdot S_{22} \\ &= \frac{eg - f^2}{EG - F^2} \ \text{– Gaussian curvature.} \end{aligned} \tag{2.30}$$

Values of mean and Gaussian curvatures:

$$\begin{aligned} \kappa_M = -&\frac{\sqrt{3}}{4L_N\eta_f\left[3(1-\eta_p)\left(\left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4(\eta_f)^2\right) + \eta_f\left(\frac{\partial\eta_p}{\partial p}\right)^2\right]} \\ &\cdot\left[6(1-\eta_p)\eta_f\left(2\eta_f\frac{\partial^2\eta_f}{\partial\theta^2} - \left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4(\eta_f)^2\right)\right. \\ &+\left.(1-\eta_p)\frac{\partial^2\eta_p}{\partial p^2}\left(\left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4(\eta_f)^2\right) + \eta_f\left(\frac{\partial\eta_p}{\partial p}\right)^2\left(\frac{\partial^2\eta_f}{\partial\theta^2} + 4\eta_f\right)\right], \end{aligned} \tag{2.31}$$

$$\kappa_G = \frac{3\left[2(1-\eta_p)\frac{\partial^2\eta_p}{\partial p^2} + \left(\frac{\partial\eta_p}{\partial p}\right)^2\right]\cdot\left[2\eta_f\frac{\partial^2\eta_f}{\partial\theta^2} - \left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4(\eta_f)^2\right]}{4L_N\eta_f\left[3(1-\eta_p)\left(\left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4(\eta_f)^2\right) + \left(\frac{\partial\eta_p}{\partial p}\right)^2\eta_f\right]}. \tag{2.32}$$

When there is a need of calculating extremal (the greatest and the smallest) values of curvature of all curvatures of any arbitrary chosen curves containing the given point and belonging to the surface, principal curvatures can be derived from mean and Gaussian curvatures. Since they are roots of characteristic polynomial, they are equal:

$$\kappa_{1/2} = \kappa_M \pm \sqrt{\kappa_M^2 + \kappa_G^2}. \tag{2.33}$$

These quantities are rather complex and it seems that writing of the full expression for principal curvatures for this very general case is to some extent useless – they can be calculated for certain forms of influence functions. Using numerical computations makes the problem even easier.

2.6. Convexity condition

If the yield surface is oriented by the aforementioned unit normal $\mathbf{N}$ pointing exterior of the surface, then the surface will be convex (non-concave) if and only if all possible curvatures of the curves belonging to it are negative (non-positive). Since principal curvatures κ_1, κ_2 (as the eigenvalues of $\mathbf{S}$) are extremal (maximal and minimal), the values of curvature at given point then all curvatures will be negative if both of the principal ones are negative:

$$\begin{cases} \kappa_1 < 0 \\ \kappa_2 < 0 \end{cases} \Rightarrow \begin{cases} \kappa_1 + \kappa_2 < 0 \\ \kappa_1 \cdot \kappa_2 > 0 \end{cases} \Rightarrow \begin{cases} \kappa_M < 0 \\ \kappa_G > 0 \end{cases}. \tag{2.34}$$

Finally we obtain:

$$\begin{gathered}
\left[6\eta_f\left(2\eta_f\frac{\partial^2\eta_f}{\partial\theta^2} - \left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4(\eta_f)^2\right)\right.\\
\left.+\frac{\partial^2\eta_p}{\partial p^2}\left(\left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4(\eta_f)^2\right) + \frac{\eta_f}{(1-\eta_p)}\left(\frac{\partial\eta_p}{\partial p}\right)^2\left(\frac{\partial^2\eta_f}{\partial\theta^2} + 4\eta_f\right)\right] > 0,\\
\left[2(1-\eta_p)\frac{\partial^2\eta_p}{\partial p^2} + \left(\frac{\partial\eta_p}{\partial p}\right)^2\right] \cdot \left[2\eta_f\frac{\partial^2\eta_f}{\partial\theta^2} - \left(\frac{\partial\eta_f}{\partial\theta}\right)^2 + 4(\eta_f)^2\right] > 0.
\end{gathered} \tag{2.35}$$

These inequalities are general conditions which have to be fulfilled by the chosen influence functions, so that yield surface determined by them was convex. Since the yield condition (2.2) is formulated in a very general way, it allows us to use above conditions in the most cases of commonly used yield conditions and also to specify any new surface, since the form of influence functions can be chosen in an almost arbitrary way. Numerical analysis of positiveness of the above expressions with respect to all constant parameters of influence functions, defines a domain of values of those parameters for which specified yield surface is convex (see similar analysis performed by Raniecki and Mróz [4]).

As the process of material identification (determining functions and parameters describing properly material's behavior) is often considered as an optimization problem of fitting results of simulation using the assumed model to the data obtained from experiments (for certain objective function), conditions given by inequalities (2.35) can be used in the optimization process as the inequality constraints that have to be fulfilled by resultant optimal solution.

2.7. Pressure insensitive materials

Considering that the condition (2.2) describes pressure insensitive materials – what means that $\eta_p(p) = \text{const.}$ and all its derivatives are equal 0 – we can see that $\kappa_G = 0$. Resultant surface is a cylindrical shaped surface with its axis parallel to p-axis and its cross-section deformed by $\eta_f(\theta)$ influence function. Convexity condition is equivalent to the statement that mean curvature $\kappa_M < 0$:

$$2\eta_f \frac{\partial^2 \eta_f}{\partial \theta^2} - \left(\frac{\partial \eta_f}{\partial \theta}\right)^2 + 4(\eta_f)^2 > 0. \tag{2.36}$$

Specific form of yield condition for pressure insensitive materials was considered by Raniecki and Mróz in [4], namely $qf(y) - 1 = 0$ where $y = \cos(3\theta)$. We can obtain it by substituting $\eta_f(\theta) = [f(\cos(3\theta))]^2$ in (2.36). After proper differentiation we obtain:

$$f''(1 - y) - f'y + \frac{f}{9} > 0, \tag{2.37}$$

which is the form of convexity condition precisely analyzed by Raniecki and Mróz in case of certain two-parameter power and exponential influence functions η_f.

For pressure insensitive materials it is common that yield condition is defined only on the octahedral plane thus the surface convexity condition requires only convexity of a function given on that plane. Since in many cases polar coordinates are convenient in use, sometimes yield condition has the form: $q = r(\theta)$, which can be obtained by substituting $\eta_f(\theta) = [r(\theta)]^{-2}$ into (2.36). After proper differentiation we obtain:

$$\kappa = \frac{r^2 + 2(r')^2 - rr''}{[(r')^2 + r^2]^{3/2}} > 0, \tag{2.38}$$

which is exactly the same as the classical expression for curvature of a function given in polar coordinates.

3. Summary

Three-dimensional surface given by an equation in Haigh-Westergaard coordinates/Lode parameters (general form of yield condition for isotropic bodies) was considered. Condition of its convexity (being a consequence of Drucker postulate) was analyzed. Proper inequalities were formulated for arbitrary forms of influence functions using classical methods of differential geometry. Various forms of convexity condition were proposed depending on yield condition formulation and on various assumptions on properties of the influence function.

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