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Université de Lorraine • Poznan University of Technology

Research Paper

On Computational Solution of the Dynamic and Static Behaviour of a Coupled Thermoelastic Timoshenko Beam

Theddeus T. AKANO[1)]*, Akintoye O. OYELADE[2)]

[1)] *Department of Systems Engineering*
Faculty of Engineering
University of Lagos
Nigeria
*Corresponding Author e-mail: takano@unilag.edu.n

[2)] *Department of Civil and Environmental Engineering*
Faculty of Engineering
University of Lagos
Nigeria
e-mail: aoyelade@unilag.edu.ng

The Timoshenko beam theory caters for transverse shear deformations, which are more pronounced in short beams. Previous works were examined, and Hamilton's principle was used in deriving the governing equation. This research considers two dimensions (2-D): heat and displacement response. A more comprehensive mathematical expression that incorporates this 2-D model on the vibration of a coupled Timoshenko thermoelastic beam and axial deformation effect is formulated. The significance of this model will be expressed through its finite element method (FEM) formulation. The results compared favourably with those of previous works. It was re-established that the amplitude of deflections, as well as cross-sectional rotations, increases considerably as the aspect ratio of the beam decreases. In this way, for larger aspect ratios, the response of the beam is like the quasi-static heating condition. This is expected since the increase in the aspect ratio of the beam reduces its structural stiffness and consequently its natural frequencies. So, the amplitude and temporal period of its vibrations become greater. The beam under the applied thermal loading experiences thermally-induced vibrations. Also, the dynamic solution is substantially influenced by the coupling between strain and temperature fields. The results also reveal that the aspect ratio of the beam could have a significant impact on the vibratory response of the beam. Specifically, it is proportional to the amplitude and temporal period of the thermally-induced vibrations of the beam.

Key words: thermally-induced; vibrations; Timoshenko beam; finite element method.

1. Introduction

One of the most vital structural elements used in engineering is the beam. The application domain of beams ranges from civil or structural engineering to weapons technology, space exploration and aeronautical engineering. Beams have been studied extensively by researchers and engineers because of their importance in structures. Different solution algorithms have been proposed to study both the dynamic and static behaviour of beams. A numerical method for the behaviour of functionally graded beams under vibration and buckling conditions with different boundary conditions has been studied by Zhang and Wang [1], Manolis and Beskos [2], Manoach and Ribeiro [3], Giunta *et al.* [4], Wen and Yi [5], and Zhang *et al.* [6]. At the same time, analytical solutions have been employed by Trinh *et al.* [7], Boley [8], and Kidawa-Kukla [9, 10]. Three key distinct cases of thermal increase, namely uniform, linear and nonlinear, were considered in most of these studies. Their results show a marked difference between the results of the temperature-dependent solutions and the independent ones. In general, their model has shown to be effective in the analysis of vibration and buckling under the thermal and the mechanical load of functionally graded beams.

Several works [11–13] have focused on the Timoshenko beam theory assumption to solve beam problems. In the same way, thermoelastic beams have attracted the attention of researchers in recent times [14–19]. When an elastic material is subjected to cyclic deformations, there tends to be a thermoelastic dissipation [20], especially when the period of an exciting cycle and the material relaxation are close to each other [21–23]. This phenomenon is prominent during the vibrations of an elastic beam. There is a significant transformation of the mechanical work into elastic energy. However, some of the works left convert to thermal energy.

As bending occurs in engineering structures, there is extension and cooling on one side, while compression and heating occur on the other side [24]. Temperature is non-homogeneous, as such heat is being transferred from the area of higher heat flux to a lower region. This heat transfer causes an entropy in the system corresponding to the second law of thermodynamics, and lastly the loss of the mechanical energy in the form of heat. The loss in mechanical energy could come from several mechanisms, namely: energy to the structure supports [25], loss to the internal energy [26], loss due to air damping [27], and notably, thermoelastic damping [28–30].

Problems arising from thermally induced vibration in beams and other structures have been an age-long issue [31–33]. Some engineering applications such as micro-electromechanical systems (MEMS), high-speed aircraft, reactor vessels and turbines also face a thermally induced vibration-related problem. In the literature, two types of models have been remarkable to identify the vibrational problems of the beam structures that are thermally induced; they are uncoupled and coupled models [3, 10, 34, 35]. The uncoupled model is adopted in the analysis of Euler-Bernoulli beam theory and eliminates the coupling effect of the displacement field in the thermal conduction equation. On the other hand, the coupled model takes into account the coupling effect of the displacement field in the thermal conduction equation, and it is prominent in the study of the Timoshenko beam model.

Several works have employed a coupled model to analyse thermally induced vibration of a Timoshenko beam. Manoach *et al.* [36] investigated the effect of heat on the vibration of beams. The thermal and mechanical loads considered the impact of the beam dynamics, and a numerical method for solving the thermomechanical problem was demonstrated. The composite beam was studied under three different heat pulses: unheated, small heat pulse and large heat pulse. The oscillations in the results varied from having the response of the unheated beam excited by the mechanical loading close to the first natural frequency to getting irregular behaviour. This shows that the heat pulse can lead to different dynamics for the nonlinear beam. To study the coupled, thermoelastic and significant amplitude vibration of a Timoshenko beam exposed to the short heat flux and harmonic mechanical load, Manoach and Ribeiro [3] adopted an accurate and time-saving computational approach. The dynamic behaviour of a Timoshenko beam under a step heat flux at the top of the medium area was investigated by Massalas and Kalpakidis [34]. The thickness directions of the beam were linearly varied, and the heat conduction equation in 2-D condensed to one component. Based on this concept, Guo *et al.* [35] studied an axially moving beam under coupled thermoelastic conditions. They employed the differential quadrature method to find the frequency and study the divergence and flutter of the beam. It was discovered that the type of instability and the speed at which the beam moves are functions of the thermoelastic factor, the length, the height and the boundary condition(s) of the beam.

Efforts towards finding a solution to the resulting model from the thermoelastic effect have been dominated by an integral transform, which is very laborious. The thermoelastic models are functions in both space (x) and time (t) domain. Attempts at employing the numerical methods via the finite element method have also resulted in eliminating the time component using the Laplace transform. Afterward, another numerical scheme to obtain the final solution in the real-time domain through the inverse Laplace transform. Again, this is quite painstaking.

The present study aims to obtain the steady-state and dynamic responses of a coupled thermoelastic Timoshenko beam acted upon by an external force and heat flux (see Fig. 1) using the FEM with its backend, finite difference method (FDM). The set goal will be achieved through the formulation of the governing differential equation together with its finite element model. In addition to the assumption of Massalas and Kalpakidis [34], the effect of axial deformation on the entire beam responses is considered. Lastly, the static and dynamic responses will be determined using the formulated FEM.

2. Methodology

2.1. *Thermoelastic Timoshenko beam*

In practical situations, the axial strain in the thickness direction ε_{xx} is not usually constrained to be zero, but the normal tractions at the top and bottom of the beam are zero or its equivalence. Conversely, if width is much less than length, the stress component in the direction of width is zero. This means σ_{xy}, σ_{yy}, and σ_{zy} are all zero. This research work proceeds against this backdrop and with reference to the thermal strain that is prominent with a 1-D constitutive model.

In this problem, we consider a beam of length L and height h with range values of $\leq x \leq L$ and $-\frac{h}{2} \leq z \leq \frac{h}{2}$ along the length and height, respectively (Fig. 1). A heat flux of $q(x,z,t)$ is applied to the surface $z = \frac{h}{2}$, while a transverse force $g(x,z,t)$ is applied to the surface $z = \frac{h}{2}$ and an axial force $f(x,z,t)$ is applied to the surface $z = 0$.

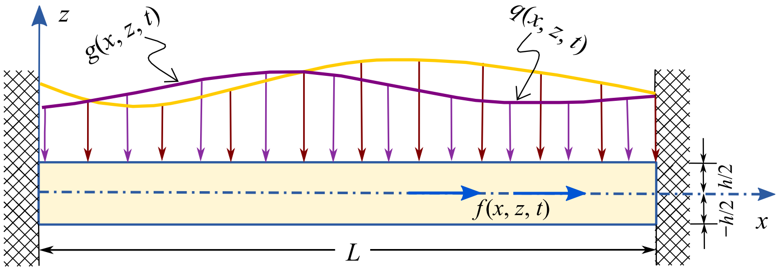

Fig. 1. Fixed Timoshenko beam under harmonic heat flux and distributed load.

From the theory of stress, the strain energy U for an elastic material with small strain is given within a region of $\varOmega$ as [37]:

$$U = \frac{1}{2} \int_{\varOmega} (\sigma_{ij}\varepsilon_{ij}) \, \mathrm{d}\varOmega. \tag{2.1}$$

The notation ε_{ij} represents the component of the strain tensor $\overline{\varepsilon}$ that could be separated into two parts, such as:

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{\mathrm{T}}, \tag{2.2}$$

where $\varepsilon_{ij}^{\mathrm{T}}$ and ε_{ij}^{e} are the thermal and elastic strains respectively given as [38]:

$$\varepsilon_{ij}^{\mathrm{T}} = \alpha \Delta \varTheta \delta_{ij}, \qquad \varepsilon_{ij}^{e} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{2.3}$$

In Eq. (2.3), α is the coefficient of linear expansion, $\Delta\Theta$ is the temperature change of the solid, and $u_{i,j}$ denotes the displacement gradient. The symmetric point of curvature tensor χ is given as [37]:

$$\chi_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}), \tag{2.4}$$

and

$$\theta_i = \frac{1}{2}\left(\mathrm{curl}(u)\right)_i \tag{2.5}$$

represents the component of the infinitesimal notation vector $\overline{\theta}$. The displacement of a Timoshenko beam is given by

$$\begin{gathered} u_x(x,y,z,t) = u(x,z,t) + z\phi_x(x,t), \qquad u_y(x,y,z,t) = 0, \\ u_z(x,y,z,t) = w(z,t), \end{gathered} \tag{2.6}$$

where (x, y, z, t) represent the coordinates of a point in the axes of the beam at a time t. While u_x, u_y, and u_z are the displacement in the Cartesian coordinate. More so, $u(x, z, t)$ stands for the axial displacement at the neutral axis. The notation ϕ_x is the angle of the beam cross-section about the z-axis, and w is the beam transverse displacement in the z-direction. Substituting Eq. (2.6) into Eq. (2.3) gives the simplified Green-Lagrange strain tensor components as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + z\frac{\partial \phi_x}{\partial x}, \qquad \varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\left(\phi_x + \frac{\partial w}{\partial x}\right), \tag{2.7}$$

where u is the n-plane displacement of particles in the x-direction. Since the shear strain ε_{xz} is not constant along the beam cross-section, it is important to introduce a correction factor k_s. Hence, Eq. (2.7) becomes

$$\varepsilon_{xz} = \frac{1}{2}k_s\left(\phi_x + \frac{\partial w}{\partial x}\right). \tag{2.8}$$

The constitutive relation for an isotropic elastic material is given as [38]:

$$\sigma_{ij} = \lambda \mathrm{tr}\overline{\varepsilon}\delta_{ij} + 2\mu\varepsilon_{ij}. \tag{2.9}$$

As a matter of fact, in real life situation, the axial strain in the thickness direction ε_{xx} is not constrained to be zero. However, the normal tractions at the bottom and top of the beam are zero or its equivalence. On the other hand, if width is much less than length, the stress component in the direction of width is zero. That is to say σ_{xy}, σ_{yy}, and σ_{zy} are all zero. Against this backdrop and with reference to the thermal strain that is prominent with 1-D constitutive relation [39] that reduces the constitutive relation as a function of ε_{xx} and $\Delta\Theta$:

$$\sigma_{xx} = E\varepsilon_{xx} - \eta\Delta\Theta, \tag{2.10}$$

where η is the thermal modulus given as $\eta = \frac{E\alpha}{(1-2\nu)}$. Here α, E, ν, and $\Delta\Theta$ are the thermal expansion coefficient, Young's modulus, Poisson's ratio, and temperature increase of the solid, respectively. By considering Eqs (2.7) and (2.10), we will arrive at the constitutive equation relating to stress, displacement, and thermal modulus, that is:

$$\sigma_{xx} = E\left(\frac{\partial u}{\partial x} + z\frac{\partial \phi_x}{\partial x}\right) - \eta\Delta\Theta. \tag{2.11}$$

After taking the variation of Eq. (2.1), it becomes:

$$\begin{aligned}\partial U &= \int_{\Omega} \left(\sigma_{xx}\delta\varepsilon_{xx} + 2\sigma_{xz}\delta\varepsilon_{xz}\right) \mathrm{d}\Omega \\ &= \int_{L} \left(-M_{xx}\frac{\partial(\delta\phi_x)}{\partial x} + Q_{zz}\left(-\delta\phi_x + \frac{\partial(\delta\phi_x)}{\partial x}\right)\right) \mathrm{d}L,\end{aligned} \tag{2.12}$$

where $M_{xx} = \int_{A} \sigma_{xx}\,\mathrm{d}A$ is the bending moment and $Q_{zz} = k_s \int_{A} \sigma_{zz}\,\mathrm{d}A$ is the shear force. If we note that at the boundaries, the variations are zero, then after integrating Eq. (2.12) by parts, we have:

$$\int_{L} \left(\left(\frac{\partial M_{xx}}{\partial x} - Q_{zz}\right)\delta\phi_x - \frac{\partial Q_{zz}}{\partial x}\delta w\right) \mathrm{d}L = 0. \tag{2.13}$$

From the lemma of the calculus of variation, Eq. (2.13) becomes

$$\frac{\partial N_{xx}}{\partial x} = 0, \qquad \frac{\partial M_{xx}}{\partial x} - Q_{zz} = 0, \qquad \frac{\partial Q_{zz}}{\partial x} = 0. \tag{2.14}$$

Considering the beam to be isotropic, uniform, and linearly elastic transforms Eqs (2.13) and (2.14) into Eqs (2.15)–(2.17):

$$EA\frac{\partial^2 u}{\partial x^2} - \frac{\partial N_A}{\partial x} = 0, \tag{2.15}$$

$$\frac{\partial}{\partial x}\left(k_s AG\frac{\partial w}{\partial x}\right) - k_s AG\frac{\partial \phi_x}{\partial x} = 0, \tag{2.16}$$

$$\frac{\partial}{\partial x}\left(EI\frac{\partial \phi_x}{\partial x}\right) - k_s AG\left(\frac{\partial w}{\partial x} + \phi_x\right) - \frac{\partial M_B}{\partial x} = 0, \tag{2.17}$$

where k_sAG and EI are the shear and bending modulus, respectively. The quantities N_A and M_B stand for the thermal axial force and bending moment, respectively denoted as:

$$N_A = b\eta \int_{-\frac{h}{2}}^{+\frac{h}{2}} \Theta \,\mathrm{d}z = 0, \qquad M_B = b\eta \int_{-\frac{h}{2}}^{\frac{h}{2}} \Theta z \,\mathrm{d}z. \tag{2.18}$$

2.2. *Dynamic equation for a Timoshenko beam*

The equation of motion for a Timoshenko beam is derived by applying Hamilton's principle [40]

$$\delta \int_{t_1}^{t_2} (T + W - U)\,\mathrm{d}t = 0. \tag{2.19}$$

Here, the kinetic energy T is

$$T = \frac{1}{2}\int_{\Omega} \rho \left(\left(\frac{\partial u_x}{\partial t}\right)^2 + \left(\frac{\partial u_y}{\partial t}\right)^2 + \left(\frac{\partial u_z}{\partial t}\right)^2\right) \mathrm{d}\Omega \tag{2.20}$$

$$= \frac{1}{2}\int_0^L \int_A \rho \left(\left(\frac{\partial u}{\partial t} + z\frac{\partial \phi_x}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right) \mathrm{d}A\,\mathrm{d}x.$$

By introducing $I = \int_A z^2\,\mathrm{d}A$, $\rho A = \int_A \rho\,\mathrm{d}A$ and $\rho I = \int_A \rho z^2\,\mathrm{d}A$ into Eq. (2.20), we have:

$$T = \frac{1}{2}\int_0^L \left(\rho A\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right) + \rho I\left(\frac{\partial \phi_x}{\partial t}\right)^2\right) \mathrm{d}x, \tag{2.21}$$

where ρ is the density of the beam material, and I stands for the moment of inertia of the cross-section of the beam. If damping force proportional to the velocity is assumed, then Eqs (2.15)–(2.17) become Eqs (2.22)–(2.24):

$$EA\frac{\partial^2 u}{\partial x^2} - \rho A\frac{\partial^2 u}{\partial t^2} - \frac{\partial N_A}{\partial x} = 0, \tag{2.22}$$

$$\rho A\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(k_sAG\left(\frac{\partial w}{\partial x} + \phi_x\right)\right) = 0, \tag{2.23}$$

$$\rho I\frac{\partial^2\phi_X}{\partial t^2}+\xi(x)\frac{\partial w}{\partial t}-\frac{\partial}{\partial x}\left(EI\frac{\partial\phi_x}{\partial x}\right)+k_sAG\left(\frac{\partial w}{\partial x}+\phi_x\right)+\frac{\partial M_B}{\partial x}=0, \tag{2.24}$$

where $\xi(x)$ is characteristic variable for rotational and translational damping, and ρA and ρI represent the mass per unit length and rotational inertia of the beam, respectively.

2.3. Thermal transport along the beam axis

The thermoelastic equation for the heat conduction through the beam could be written as [38]:

$$\rho c_v\frac{\partial}{\partial t}\left(\Theta+\tau_0\frac{\partial\Theta}{\partial t}\right)=k_c\frac{\partial^2\Theta}{\partial x_i\partial x_i}-\eta\Theta_0\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_i}+\frac{\partial\phi_x}{\partial x_i}\right), \tag{2.25}$$

where Θ_0 is the initial temperature on beam periphery, while c_v, k_c, and τ_0 are the heat capacity per unit volume, the thermal conductivity of the beam material and thermal relaxation time, respectively. By substituting the displacement relation into Eq. (2.25), we have

$$\rho c_v\left(\frac{\partial\Theta}{\partial t}+\tau_0\frac{\partial^2\Theta}{\partial t^2}\right)=k_c\left(\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial z^2}\right)-\eta\Theta_0\left(\frac{\partial^2 u}{\partial t\partial x}+z\frac{\partial^2\phi_x}{\partial t\partial x}\right). \tag{2.26}$$

Multiplying Eqs (2.26) by $b\eta z$ and integrating with respect to z within the limit of $-\frac{h}{2}$ and $+\frac{h}{2}$ produces

$$\begin{aligned}\rho c_v b\eta\int_{-\frac{h}{2}}^{+\frac{h}{2}}\left(\frac{\partial\Theta}{\partial t}+\tau_0\frac{\partial^2\Theta}{\partial t^2}\right)z\,\mathrm{d}z&=k_c b\eta\int_{-\frac{h}{2}}^{+\frac{h}{2}}\left(\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial z^2}\right)z\,\mathrm{d}z\\&\quad-b\Theta_0\eta^2\int_{-\frac{h}{2}}^{+\frac{h}{2}}\left(\frac{\partial^2 u}{\partial t\partial x}z+\frac{\partial^2\phi_x}{\partial t\partial x}z^2\right)\mathrm{d}z\end{aligned} \tag{2.27}$$

where it could be shown from Eq. (2.27) that

$$M_B=b\eta\int_{-\frac{h}{2}}^{+\frac{h}{2}}\Theta z\,\mathrm{d}z=-\frac{b}{r_x}\eta\int_{-\frac{h}{2}}^{+\frac{h}{2}}\frac{\partial^2\Theta}{\partial z^2}z\,\mathrm{d}z. \tag{2.28}$$

On the consideration that the temperature increases along the direction of the thickness h and following Eqs (2.18), (2.25), and (2.28), the heat conduction equation is derived as:

$$\rho c_v\left(\frac{\partial M_B}{\partial t}+\tau_0\frac{\partial^2 M_B}{\partial t^2}\right)=k_c\frac{\partial^2 M_B}{\partial x^2}-\frac{1}{r_x}M_B \\ -\eta\Theta_0\left(\frac{\partial^2 u}{\partial t\partial x}+z\frac{\partial^2\phi_x}{\partial t\partial x}\right)-q(x,z,t), \tag{2.29}$$

where $r_x=\frac{h^2}{12}$ is the radius of gyration of the beam cross-section and $q(x,z,t)$ is the heat flux.

2.4. The coupled thermoelastic equation

By introducing the dimensionless terms:

$$\begin{aligned} &\overline{x}=\frac{x}{L}, && \overline{z}=\frac{z}{L}, && \overline{t}=\frac{t}{L^2}\sqrt{\frac{EI}{\rho A}},\\ &\overline{\psi}_A=N_A, && \overline{\psi}_B=M_B, && \overline{\phi}_x=\phi_x,\\ &\overline{w}=\frac{w}{h}, && \overline{u}=\frac{u}{h}, && \varsigma_0=\frac{c_v h^2}{k_c L^2}\sqrt{\frac{EI\rho}{A}},\\ &\mu=\frac{k_s GAL^2}{EI}, && \beta_0=\frac{E\alpha^2\Theta_0}{\rho c_v}, && r_0=\frac{\varsigma_0}{r_x} \end{aligned} \tag{2.30}$$

the normalized forms of Eqs (2.19)–(2.21) and (2.29) become

$$\begin{aligned} &-\rho A\frac{\partial^2 u}{\partial t^2}+EA\frac{\partial^2 u}{\partial x^2}-\frac{\partial\overline{\psi}_A}{\partial x}=0,\\ &-\rho A\frac{\partial^2 w}{\partial t^2}-\xi_T(x)\frac{\partial w}{\partial t}+\frac{\partial}{\partial x}\left(k_sAG\left(\frac{\partial w}{\partial x}+\phi_x\right)\right)=0,\\ &-\rho I\frac{\partial^2\overline{\phi}_x}{\partial t^2}+\frac{\partial}{\partial x}\left(EI\frac{\partial\overline{\phi}_x}{\partial x}\right)-\xi_R(x)\frac{\partial w}{\partial t}-k_sAG\left(\frac{\partial w}{\partial x}+\overline{\phi}_x\right)-\frac{\partial\overline{\psi}_B}{\partial x}=0,\\ &-\rho c_v\left(\frac{\partial\overline{\psi}_B}{\partial t}+\tau_0\frac{\partial^2\overline{\psi}_B}{\partial t^2}\right)+k_c\frac{\partial^2\overline{\psi}_B}{\partial x^2}-\frac{1}{r_x}\overline{\psi}_B+z\eta\Theta_0\left(u+\frac{\partial^2\overline{\phi}_x}{\partial t\partial x}\right)-q(x,t)=0. \end{aligned} \tag{2.31}$$

Here, μ represents the shear stiffness parameter, and β_0 characterises the thermomechanical coupling parameter. Supposing the beam is under the action of a time-harmonic excitation, the axial extension, the transverse displacement, the rotation angle, the shear deformation, the temperature and heat flux are expressed as follows:

$$\overline{u}(x,t) = \widehat{u}(x)\,e^{i\omega\overline{t}}, \qquad \overline{w}(x,t) = \widehat{w}(x)\,e^{i\omega\overline{t}},$$

$$\overline{\phi}_x(x,t) = \widehat{\phi}_x(x)\,e^{i\omega\overline{t}}, \qquad \overline{\psi}_A(x,t) = \widehat{\psi}_A(x)\,e^{i\omega\overline{t}}, \tag{2.32}$$

$$\overline{\psi}_B(x,t) = \widehat{\psi}_B(x)\,e^{i\omega\overline{t}}.$$

Substituting Eq. (2.32) into Eq. (2.31) leads to

$$\omega^2\widehat{u} + \widehat{u}'' - \widehat{\psi}'_A = 0,$$

$$\omega^2\widehat{w} + \mu\left(\widehat{w}'' + \widehat{\phi}'_x\right) = 0,$$

$$\overline{\phi}''_x + \omega^2\widehat{\phi}_x - i\,\omega\,\xi\,\widehat{w} - \mu\left(\widehat{w}' + \overline{\phi}_x\right) - \widehat{\psi}'_B = 0, \tag{2.33}$$

$$\widehat{\psi}''_B + \varsigma_0\left(i\,\omega\,\widehat{\psi}_B - \omega^2\,\tau_0\,\widehat{\psi}_B\right) - r_0\,\overline{\psi}_B + i\,\omega\,\varsigma_0\,\beta_0\left(\widehat{u} + \overline{z}\,\widehat{\phi}'_x\right) + q\,\varsigma_0 = 0.$$

2.5. *Stiffness, mass, and damping matrices*

In the formulation of the stiffness, mass, and damping matrices, the matrices from the linear varying curvature as a result of bending and constant curvature due to shear are considered. The beam will be described relative to the axial displacement $(\overline{u}_1, \overline{u}_4)$ and transverse displacement $(\overline{w}_2, \overline{w}_3, \overline{w}_5, \overline{w}_6)$, as shown in Fig. 2.

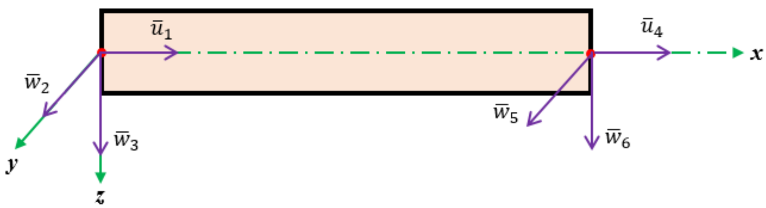

FIG. 2. A beam element showing the nodal displacements at both ends.

These deformations change with time according to the following equations:

$$\overline{u}(x,t) = \sum_{k=1}^{2} \mathcal{N}_{(e)k}(x)\overline{u}_k(t) \quad \text{with} \quad \overline{u}_1, \overline{u}_2 \equiv \overline{u}_1, \overline{u}_4, \tag{2.34}$$

$$\overline{w}(x,t) = \sum_{k=1}^{4} \mathcal{N}_{(b)k}(x)\overline{w}_k(t) + \sum_{k=1}^{4} \mathcal{N}_{(s)}k(x)\overline{\phi}_{x_k}(t), \tag{2.35}$$

$$\text{with} \quad \overline{w}_1, \overline{w}_2, \overline{w}_3, \overline{w}_4 \equiv \overline{w}_2, \overline{w}_3, \overline{w}_5, \overline{w}_6,$$

$$\overline{\psi}_B(x,t) = \sum_{k=1}^{4} \mathcal{N}_k(x)\overline{\psi}_k(t)\overline{\psi}_B(x,t) = \sum_{k=1}^{4} \mathcal{N}_k(x)\overline{\psi}_k(t), \tag{2.36}$$

where $\mathcal{N}_{(e)}$, $\mathcal{N}_{(b)}$, $\mathcal{N}_{(s)}$, $\mathcal{N}$ are the shape functions expressed in the Appendix. The subscript k represents the quantity evaluated at the k-th element of the discretised beam. The element stiffness matrix for the thermoelastic beam could now be formulated by employing thermoelastic strain energy in the form:

$$\mathcal{U}=\frac{1}{2}\int_0^L EA\left(\sum_{k=1}^{2}\mathcal{N}'_{(a)k}(x)\overline{u}_k\right)^2\mathrm{d}x+\frac{1}{2}\int_0^L EI\left(\sum_{k=1}^{4}\mathcal{N}''_{(b)k}(x)\overline{w}_k\right)^2\mathrm{d}x+\frac{1}{2}\int_0^L \kappa GA\left(\sum_{k=1}^{4}\mathcal{N}'_{(s)k}(x)\overline{\phi}_{x_k}\right)^2\mathrm{d}x+\frac{1}{2}\int_0^L k_c\left(\sum_{k=1}^{4}\mathcal{N}'_{k}(x)\overline{\phi}_{x_k}\right)^2\mathrm{d}x \tag{2.37}$$

giving the stiffness coefficient in extension as:

$$\mathcal{K}_{ij}^{(e)}=\frac{\partial}{\partial\overline{u}_1}\frac{\partial}{\partial\overline{u}_j}\frac{1}{2}\int_0^L EA\left(\sum_{k=1}^{2}\mathcal{N}'_{(e)k}(x)\overline{u}_k\right)^2\mathrm{d}x \tag{2.38}$$

and that in bending, shear, and thermal as:

$$\mathcal{K}_{ij}^{(b)}=\frac{\partial}{\partial\overline{w}_i}\frac{\partial}{\partial\overline{w}_j}\frac{1}{2}\int_0^L EI\left(\sum_{k=1}^{4}\mathcal{N}''_{(b)k}(x)\overline{w}_k\right)^2\mathrm{d}x+\frac{\partial}{\partial\overline{m}_i}\frac{\partial}{\partial\overline{m}_j}\frac{1}{2}\int_0^L k_c\left(\sum_{k=1}^{4}\mathcal{N}'_{k}(x)\overline{m}_k\right)^2\mathrm{d}x+\frac{\partial}{\partial\overline{\phi}_{x_i}}\frac{\partial}{\partial\overline{\phi}_{x_j}}\frac{1}{2}\int_0^L \kappa GA\left(\sum_{k=1}^{4}\mathcal{N}'_{(s)k}(x)\overline{\phi}_{x_k}\right)^2\mathrm{d}x. \tag{2.39}$$

By adopting the same principle as above, the kinetic energy associated with the extension is derived as:

$$\mathcal{T}=\frac{\partial}{\partial\dot{\overline{u}}_1}\frac{\partial}{\partial\dot{\overline{u}}_j}\frac{1}{2}\int_0^L \rho A\left(\sum_{k=1}^{2}\mathcal{N}_{(e)k}(x)\dot{\overline{u}}_k\right)^2\mathrm{d}x. \tag{2.40}$$

3. Results and discussion

The preceding solutions will be demonstrated numerically for values of material constants equivalent to aluminum alloy. The beam geometry with the boundary condition and a unit step thermal loading applied to the beam are shown in Figs 3 and 4, respectively. As a validation study to the present work, the results presented here are compared with the results presented in Massalas and Kalpakidis [34]. They gave an analytical solution for an aluminum beam with an initial temperature, and the beam was subjected to a step heat flux.

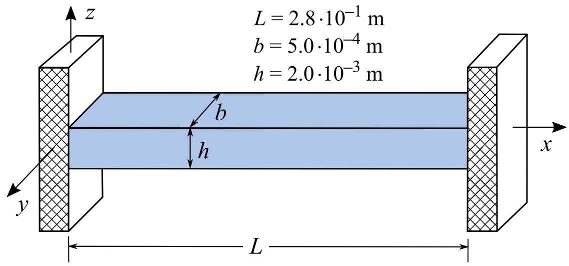

Fig. 3. The beam geometry with length L, breadth b, and height h having the clamped – clamped boundary condition.

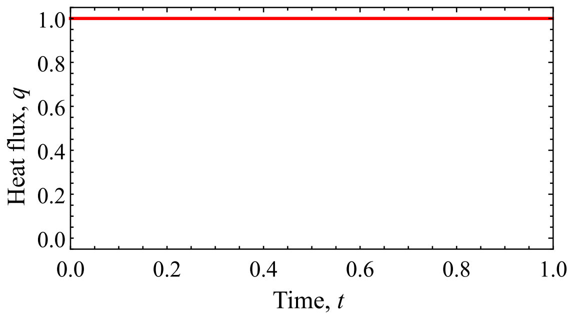

Fig. 4. A unit step heat flux applied at time $t = 0$.

This study is conducted for deflection and cross-sectional rotation at the midpoint of the beam and the temperature variation in the thickness direction of the beam. Figure 5 shows the deflection history of the heated beam and the disparity of the deflection when the coupling between temperature and strain fields is considered. From the results presented in Fig. 5, it can be seen that the solution oscillates about the quasi-static response, establishing a steady-state mode of vibration and decays with increasing time. Figure 6 indicates the history of cross-sectional rotation of the heated beam. The variation in Fig. 6 is analogous to that of deflection. Similarly, the history of temperature change at the midpoint of the heated beam is shown in Fig. 7. These results are in agreement with those from Massalas and Kalpakidis [34], thereby validating the formulations and methods used in the present work.

Figure 8 shows the deflection history of the beam. The result depicts the thermally-induced vibrations for the beam under the applied thermal loading. It is observed that the amplitude of deflections increases considerably as the as-

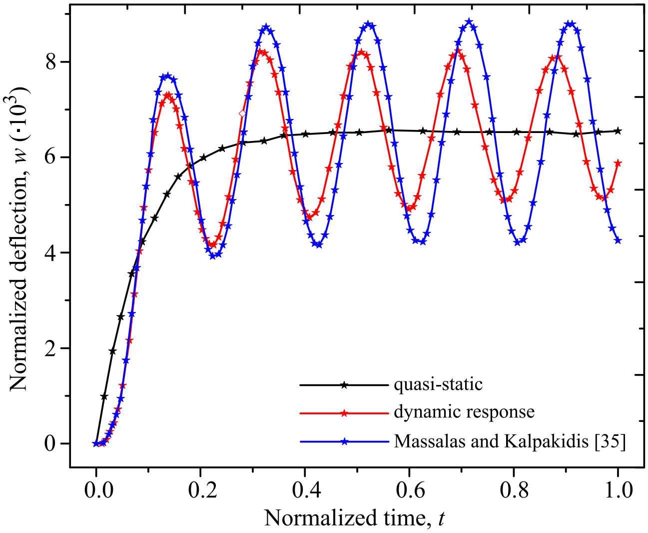

FIG. 5. Deflection (w) history at the midpoint of the thermoelastic Timoshenko beam.

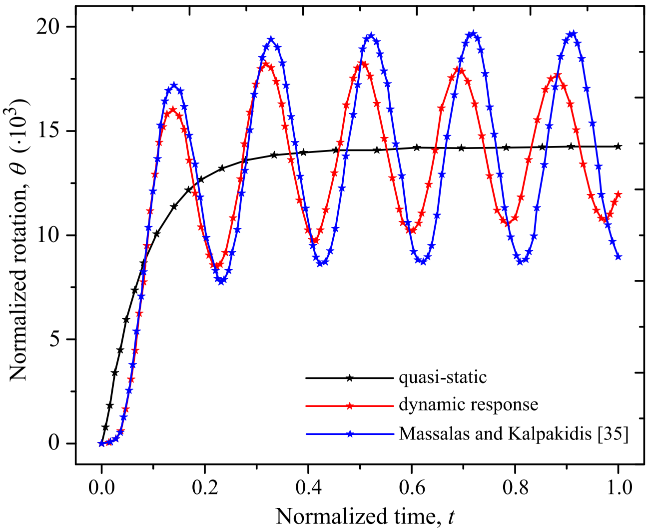

FIG. 6. Rotation history at the midpoint of the thermoelastic Timoshenko beam.

pect ratio of the beam decreases. In this way, for larger aspect ratios, the response of the beam is similar to the quasi-static heating condition. This is expectable since the increase in the aspect ratio of the beam reduces its structural stiffness and consequently its natural frequencies. So, the amplitude and temporal period of its vibrations become greater.

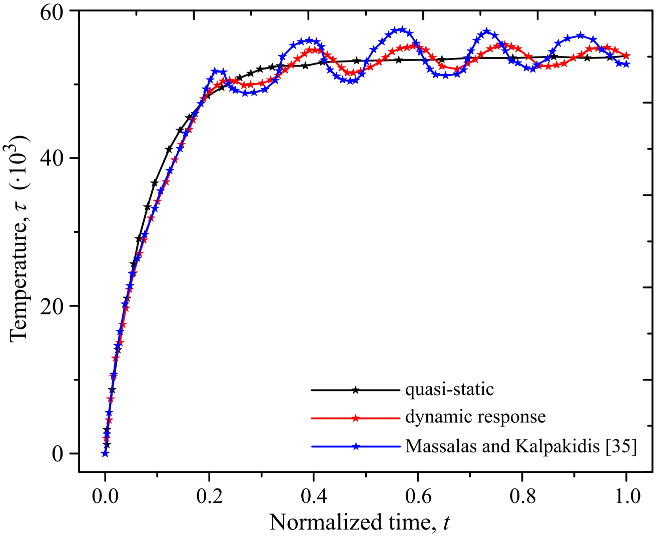

FIG. 7. Temperature variation of the thermoelastic Timoshenko beam.

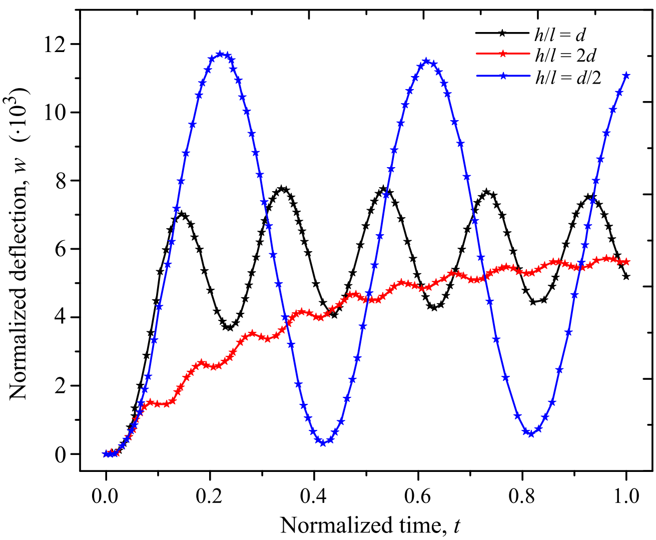

FIG. 8. Lateral deflection history at the midpoint of the beam for different aspect ratios with respect to base ratio r.

The conclusion from the results above is that the inertia plays a vital role in the deflection, rotation and temperature variation of the thermoelastic Timoshenko beam. Also, the dynamic solution is considerably affected by the coupling between temperature and strain fields.

4. Conclusion

This work presented a finite element method for the generalised coupled thermoelastic response of a Timoshenko beam subjected to thermal loading. The beam follows the shear deformation model of the first order and Hooke's law, respectively, for its displacement field and constitutive behaviour. Temperature variations through the beam have been considered to be governed by the Lord-Shulman energy equation. Displacement and temperature variations through the beam have been assumed to be infinitesimal. The FEM was applied to solve the problem in the space domain, whereas the FDM dealt with the time-domain feature. The problem wassolved, and its results have been validated with the known result in the literature. The beam under the applied thermal loading experiences thermally-induced vibrations. It is found that the aspect ratio of the beam could have a considerable effect on the vibratory response of the beam. Specifically, it is proportional to the amplitude and temporal period of thermally-induced vibrations of the beam.

The simplicity in implementing the solution algorithm and the completeness in the formulation of the governing equation make this work a benchmark for subsequent works in this area. Further research could consider geometric nonlinearity and/or nonlinear material constitutive models of the thermoelastic beam.

Appendix
Shape functions for various components in Eqs (2.34)–(2.36)

The shape functions for the extension of the neutral axis are:

$$\mathcal{N}_{(e)1} = 1 - \overline{x},$$
$$\mathcal{N}_{(e)4} = \overline{x}.$$

The bending components of the shape functions are:

$$\mathcal{N}_{(b)2} = \frac{1}{\Psi + 1}(1 - 3\overline{x}^2 + 2\overline{x}^3),$$
$$\mathcal{N}_{(b)3} = \frac{L}{\Psi + 1}\left(\overline{x}(1+\Psi) - 2\overline{x}^2 + \overline{x}^3 - \frac{\Psi}{2}\overline{x}^2\right),$$
$$\mathcal{N}_{(b)5} = \frac{1}{\Psi + 1}(3\overline{x}^2 - 2\overline{x}^3),$$
$$\mathcal{N}_{(b)6} = -\frac{L}{\Psi + 1}\left(\overline{x}^2 - \overline{x}^3 - \frac{\Psi}{2}\overline{x}^2\right),$$

The shear components of the shape functions are:

$$\mathcal{N}_{(s)2} = \frac{\Psi}{\Psi+1}(1-\overline{x}),$$

$$\mathcal{N}_{(s)3} = -\frac{\Psi L}{\Psi+1}\left(\frac{1}{2}\overline{x}\right),$$

$$\mathcal{N} = \frac{1}{\Psi+1}(3\overline{x}^2 - 2\overline{x}^3),$$

$$\mathcal{N}_{(s)6} = -\frac{\Psi L}{\Psi+1}\left(\frac{1}{2}\overline{x}\right).$$

The shape functions for a Timoshenko beam that is used to approximate the heat flux are:

$$\mathcal{N}_2 = \frac{1}{\Psi+1}(1 - 3\overline{x}^2 + 2\overline{x}^3 + (1-\overline{x})\Psi),$$

$$\mathcal{N}_3 = \frac{L}{\Psi+1}\left(\overline{x} - 2\overline{x}^2 + \overline{x}^3 - \frac{\Psi}{2}\left(\overline{x}^2 - \overline{x}\right)\right),$$

$$\mathcal{N}_5 = \frac{1}{\Psi+1}\left(3\overline{x}^2 - 2\overline{x}^3 + \Psi\overline{x}\right),$$

$$\mathcal{N}_6 = -\frac{L}{\Psi+1}\left(\overline{x}^2 - \overline{x}^3 + \frac{\Psi}{2}\left(\overline{x} - \overline{x}^2\right)\right).$$

Here,

$$\Psi = \frac{12EI}{GAL^2} \quad \text{represents the ratio of bending to shear stiffnesses,}$$

$$\overline{x} = \frac{x}{L}.$$

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