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Vibrations of a Double-Beam System with Intermediate Elastic Restraints Due to a Moving Force

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In this paper, we investigate the problem of the dynamic behaviour of a double-beam system with intermediate elastic restraints subjected to a moving point force. Problem is solved by replacing this type of structure with two single-span beams loaded with a given moving force and redundant forces representing reactions in the intermediate restraints. Redundant forces are obtained by solving Volterra integral equations of the second order which are compatibility equations corresponding to each redundant. Solutions for the arbitrarily supported single-span beam loaded with a moving point force and concentrated time-varying force are given. Difficulties in analytically solving Volterra integral equations are bypassed by applying a simple numerical procedure. Finally, a numerical example of a double-beam system with two elastic restraints is presented in order to show the effectiveness of the presented method.

Key words: double-beam system; vibrations; moving load; Volterra integral equations.

1. Introduction

The problem of the dynamic response of a structure subjected to a moving load is both interesting from the theoretical point of view and significant in structural designing. It occurs in the dynamics of various types of structures such as bridges, roadways, railways or runways. This problem has been analysed by many authors for many years with different structures and various types of moving load taken into account [1–4].

In this paper, we investigate the dynamic behaviour of a system of two Euler-Bernoulli beams with arbitrary boundary conditions, connected with a number of k elastic restraints of finite stiffness s_i (see Fig. 1). Beams can have different flexural rigidity EI, mass density m, damping coefficient c, and length L. One of the beams is subjected to a point force of constant magnitude P moving with

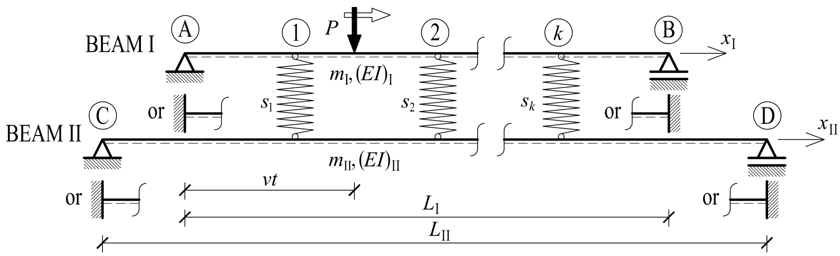

FIG. 1. Double-beam system with intermediate elastic restraints subjected to a moving force.

constant velocity v. Equations of motion describing flexural vibrations $w_{\mathrm{I}} = w(x_{\mathrm{I}}, t)$ and $w_{\mathrm{II}} = w(x_{\mathrm{II}}, t)$ of beams I and II have the form:

$$(EI)_{\mathrm{I}} w_{\mathrm{I}}^{\mathrm{IV}} + c_{\mathrm{I}} \dot{w}_{\mathrm{I}} + m_{\mathrm{I}} \ddot{w}_{\mathrm{I}} + \sum_{i=1}^{k} s_i \left[w_{\mathrm{I}} - w_{\mathrm{II}} \right] \delta \left(x_{\mathrm{I}} - x_{\mathrm{I},i} \right) = P \delta (x_{\mathrm{I}} - vt), \tag{1.1}$$

$$(EI)_{\mathrm{II}} w_{\mathrm{II}}^{\mathrm{IV}} + c_{\mathrm{II}} \dot{w}_{\mathrm{II}} + m_{\mathrm{II}} \ddot{w}_{\mathrm{II}} + \sum_{i=1}^{k} s_i \left[w_{\mathrm{II}} - w_{\mathrm{I}} \right] \delta \left(x_{\mathrm{II}} - x_{\mathrm{II},i} \right) = 0, \tag{1.2}$$

where roman numerals in the superscript denote differentiation with respect to spatial coordinate x and dots $(\dot{\ })$ denote differentiation with respect to time t. Symbol $\delta(\cdot)$ denotes the Dirac delta.

In the presented method we divide the analysed structure into two single-span beams (see Fig. 2). Vibrations of the upper and the lower beam can be described as:

$$w_{\mathrm{I}} = w_{\mathrm{I}}^{P} + \sum_{i=1}^{k} w_{\mathrm{I}}^{Xi}, \qquad w_{\mathrm{II}} = -\sum_{i=1}^{k} w_{\mathrm{II}}^{Xi}. \tag{1.3}$$

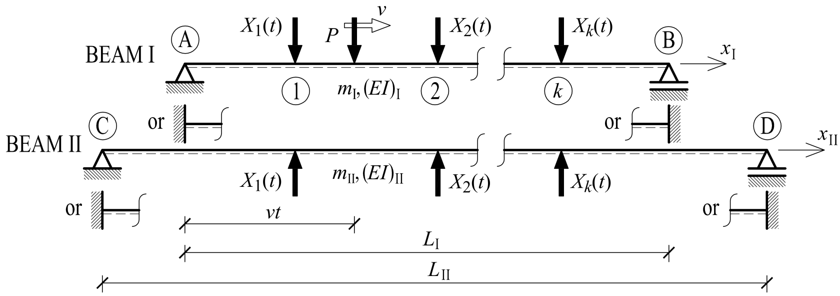

FIG. 2. Two single-span beams subjected to moving force and redundant forces.

Expression w_{I}^{P} denotes vibrations of the single-span beam resulting from the given moving force while expressions w_{I}^{Xi} and w_{II}^{Xi} are vibrations of the I and II beam resulting from the force $X_i(t)$ in the i elastic restraint. Forces $X_i(t)$ can be determined from a set of compatibility equations:

$$(1.4)\quad w_{\mathrm{I}}^{P}(x_{\mathrm{I},i},t)+\sum_{i=1}^{k} w^{Xi}(x_{\mathrm{I},i},t)+\sum_{i=1}^{k} w^{Xi}(x_{\mathrm{II},i},t)+\frac{X_i(t)}{s_i}=0, \quad i=1,2,..,k.$$

2. Vibrations of a single span beam

In the first step, we shall concentrate on a single-span uniform beam with pinned or fixed ends. This problem is well-known and was included in works [5–7]. In the following chapters cases of moving constant force and concentrated time-varying force will be analysed.

2.1. Case of a moving constant force

Let us consider a beam subjected to a vertical point force of constant magnitude P moving with a constant velocity v along the axis x (see Fig. 3). Vibrations $w^P(x,t)$ of the beam are described by the following equation:

$$(2.1)\qquad EI\left[w^P(x,t)\right]^{\mathrm{IV}} + c\dot{w}^P(x,t) + m\ddot{w}^P(x,t) = P\delta(x-vt).$$

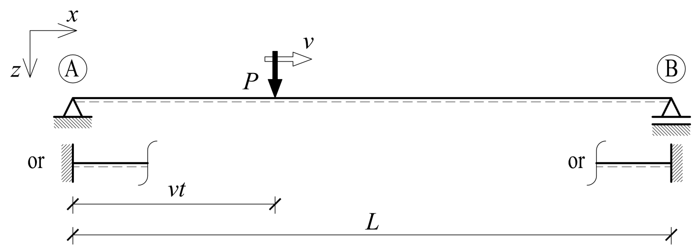

Fig. 3. Single-span beam subjected to a moving point force.

After introducing dimensionless variables:

$$(2.2)\qquad \xi=\frac{x}{L},\qquad \xi\in[0,\,1],\qquad T=\frac{vt}{L},\qquad T\in[0,\,1],$$

Eq. (2.1) takes the form:

$$(2.3)\qquad \left[w^P(\xi,T)\right]^{\mathrm{IV}} + c_0\dot{w}^P(\xi,T) + \sigma^2\ddot{w}^P(\xi,T) = P_0\delta(\xi-T),$$

where

$$c_0=\frac{cvL^3}{EI},\qquad \sigma^2=\frac{mv^2L^2}{EI},\qquad P_0=\frac{PL^3}{EI}.$$

The solution of Eq. (2.2) has the form:

$$w^P(\xi,T)=\sum_{n=1}^{\infty}Y_n(T)W_n(\xi), \tag{2.4}$$

where eigenfunctions $W_n(\xi)$ can be presented as:

$$W_n(\xi)=G_{1n}\sin\lambda_n\xi+G_{2n}\cos\lambda_n\xi+G_{3n}\sinh\lambda_n\xi+G_{4n}\cosh\lambda_n\xi. \tag{2.5}$$

Constants G_{1n}, G_{2n}, G_{3n}, G_{4n} as well as eigenvalues λ_n result from the boundary conditions and are presented in Table 1.

Table 1. Constants G_{1n}, G_{2n}, G_{3n}, G_{4n}, γ_n^2 and eigenvalues λ_n for different types of the beam.

Value	Beam type			
	pinned-pinned	pinned-fixed	fixed-pinned	fixed-fixed
λ_n	$n\pi$	3.927 for $n=1$ 7.069 for $n=2$ $(n+0.25)\pi$ for $n>2$	3.927 for $n=1$ 7.069 for $n=2$ $(n+0.25)\pi$ for $n>2$	4.730 for $n=1$ 7.853 for $n=2$ $(n+0.5)\pi$ for $n>2$
γ_n^2	0.5	0.9991 for $n=1$ 1 for $n>1$	0.9997 for $n=1$ 1 for $n>1$	1.00001 for $n=1$ 1 for $n>1$
G_{1n}	1	$\dfrac{1}{\sin\lambda_n}$	$\dfrac{\cos\lambda_n+\cosh\lambda_n}{\sin\lambda_n+\sinh\lambda_n}$	$\dfrac{\cosh\lambda_n-\cos\lambda_n}{\sinh\lambda_n-\sin\lambda_n}$
G_{2n}	0	0	-1	-1
G_{3n}	0	$-\dfrac{1}{\sinh\lambda_n}$	$-\dfrac{\cos\lambda_n+\cosh\lambda_n}{\sin\lambda_n+\sinh\lambda_n}$	$-\dfrac{\cosh\lambda_n-\cos\lambda_n}{\sinh\lambda_n-\sin\lambda_n}$
G_{4n}	0	0	1	1

Function $Y_n(T)$ can be obtained from the ordinary differential equation:

$$\ddot{Y}_n(T)+2\alpha\dot{Y}_n(T)+\omega_n^2Y_n(T)=\frac{P_0}{\gamma_n^2\sigma^2}W_n(T), \tag{2.6}$$

where $2\alpha=\frac{cL}{mv}$, $\omega_n^2=\frac{\lambda_n^4}{\sigma^2}$, and has the form:

$$\begin{aligned}Y_n(T)=A_n\sin\lambda_nT+B_n\cos\lambda_nT+C_n\sinh\lambda_nT+D_n\cosh\lambda_nT\\ +e^{-\alpha T}\left(E_n\sin\Omega_nT+F_n\cos\Omega_nT\right),\end{aligned} \tag{2.7}$$

where $\Omega_n^2 = \omega_n^2 - \alpha^2$. Constants A_n, B_n, C_n, D_n result from a set of equations:

$$
\begin{aligned}
-\lambda_n^2 A_n - 2\alpha\lambda_n B_n + \omega_n^2 A_n &= \frac{P_0}{\gamma_n^2\sigma^2} G_{1n}, \\
-\lambda_n^2 B_n + 2\alpha\lambda_n A_n + \omega_n^2 B_n &= \frac{P_0}{\gamma_n^2\sigma^2} G_{2n}, \\
\lambda_n^2 C_n + 2\alpha\lambda_n D_n + \omega_n^2 C_n &= \frac{P_0}{\gamma_n^2\sigma^2} G_{3n}, \\
\lambda_n^2 D_n + 2\alpha\lambda_n C_n + \omega_n^2 D_n &= \frac{P_0}{\gamma_n^2\sigma^2} G_{4n},
\end{aligned}
\tag{2.8}
$$

and are equal to:

$$
\begin{aligned}
A_n &= P_0 \frac{G_{1n}\left(\omega_n^2 - \lambda_n^2\right) + 2G_{2n}\alpha\lambda_n}{\gamma_n^2\sigma^2\left[\left(\omega_n^2 - \lambda_n^2\right)^2 + 4\alpha^2\lambda_n^2\right]}, \\
B_n &= P_0 \frac{G_{2n}\left(\omega_n^2 - \lambda_n^2\right) - 2G_{1n}\alpha\lambda_n}{\gamma_n^2\sigma^2\left[\left(\omega_n^2 - \lambda_n^2\right)^2 + 4\alpha^2\lambda_n^2\right]}, \\
C_n &= P_0 \frac{G_{3n}\left(\omega_n^2 + \lambda_n^2\right) - 2G_{4n}\alpha\lambda_n}{\gamma_n^2\sigma^2\left[\left(\omega_n^2 + \lambda_n^2\right)^2 + 4\alpha^2\lambda_n^2\right]}, \\
D_n &= P_0 \frac{G_{4n}\left(\omega_n^2 + \lambda_n^2\right) - 2G_{3n}\alpha\lambda_n}{\gamma_n^2\sigma^2\left[\left(\omega_n^2 + \lambda_n^2\right)^2 + 4\alpha^2\lambda_n^2\right]}.
\end{aligned}
\tag{2.9}
$$

Constants E_n and F_n result from the zero initial conditions and are equal to:

$$
E_n = \frac{\alpha F_n - \lambda_n\left(A_n + C_n\right)}{\Omega_n}, \qquad F_n = -B_n - D_n. \tag{2.10}
$$

2.2. Case of a concentrated time-varying force

In the next step, we consider vibrations of a plate due to a time-varying force $X_i(t)$ concentrated at point x_i – see Fig. 4. Equation of motion has the form:

$$
EI\left[w^{Xi}(x,t)\right]^{\mathrm{IV}} + c\dot{w}^{Xi}(x,t) + m\ddot{w}^{Xi}(x,t) = X_i(t)\delta\left(x - x_i\right). \tag{2.11}
$$

Let us introduce the dimensionless variables:

$$
\xi = \frac{x_{\mathrm{I}}}{L_{\mathrm{I}}}\xi \in [0,1], \qquad \zeta = \frac{x_{\mathrm{II}}}{L_{\mathrm{II}}}\zeta \in [0,1], \qquad T = \frac{vt}{L_{\mathrm{I}}}T \in [0,1], \tag{2.12}
$$

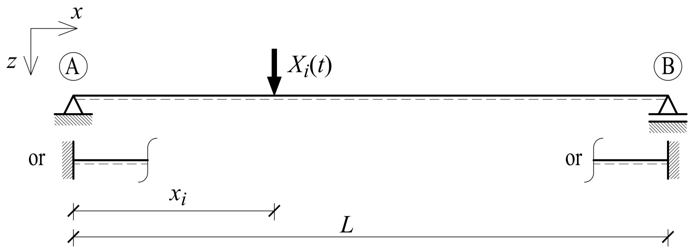

Fig. 4. Single-span beam subjected to a concentrated time-varying force.

where lower index I corresponds to the upper beam while lower index II corresponds to the lower beam. Equation (2.11) for both beams takes the form:

$$\left[w_{\mathrm{I}}^{Xi}(\xi,T)\right]^{\mathrm{IV}}+c_{0,I}\dot{w}_{\mathrm{I}}^{Xi}(\xi,T)+\sigma_{\mathrm{I}}^{2}\ddot{w}_{\mathrm{I}}^{Xi}(\xi,T) = X_{0,\mathrm{I}}\delta(\xi-\xi_i), \tag{2.13}$$

$$\left[w_{\mathrm{II}}^{Xi}(\zeta,T)\right]^{\mathrm{IV}}+c_{0,\mathrm{II}}\dot{w}_{\mathrm{II}}^{Xi}(\zeta,T)+\sigma_{\mathrm{II}}^{2}\ddot{w}_{\mathrm{II}}^{Xi}(\zeta,T) = X_{0,\mathrm{II}}\delta(\zeta-\zeta_i), \tag{2.14}$$

where

$$c_{0,\mathrm{I}}=\frac{c_{\mathrm{I}}vL_{\mathrm{I}}^{3}}{(EI)_{\mathrm{I}}},\qquad \sigma_{\mathrm{I}}^{2}=\frac{m_{\mathrm{I}}v^{2}L_{\mathrm{I}}^{2}}{(EI)_{\mathrm{I}}},\qquad X_{0,\mathrm{I}}=\frac{X_i(T)L_{\mathrm{I}}^{3}}{(EI)_{\mathrm{I}}},$$

$$c_{0,\mathrm{II}}=\frac{c_{\mathrm{II}}vL_{\mathrm{II}}^{4}}{(EI)_{\mathrm{II}}L_{\mathrm{I}}},\qquad \sigma_{\mathrm{II}}^{2}=\frac{m_{\mathrm{II}}v^{2}L_{\mathrm{II}}^{4}}{(EI)_{\mathrm{II}}L_{\mathrm{I}}^{2}},\qquad X_{0,\mathrm{II}}=\frac{X_i(T)L_{\mathrm{II}}^{3}}{(EI)_{\mathrm{II}}}.$$

Vibrations of both beams can be presented in the convolution form:

$$w_{\mathrm{I}}^{Xi}(\xi,T) = \frac{L_{\mathrm{I}}}{v}\int_0^T h_{\mathrm{I},i}(\xi,T-\tau)X_i(\tau)\,\mathrm{d}\tau, \tag{2.15}$$

$$w_{\mathrm{II}}^{Xi}(\zeta,T) = \frac{L_{\mathrm{I}}}{v}\int_0^T h_{\mathrm{II},i}(\zeta,T-\tau)X_i(\tau)\,\mathrm{d}\tau, \tag{2.16}$$

where impulse response functions $h_{\mathrm{I},i}(\xi,T)$ and $h_{\mathrm{II},i}(\zeta,T)$ can be described as:

$$h_{\mathrm{I},i}(\xi,T) = \frac{vL_{\mathrm{I}}^{2}}{\sigma_{\mathrm{I}}^{2}(EI)_{\mathrm{I}}}e^{-\alpha_{\mathrm{I}}T}\sum_{n=1}^{\infty}\frac{\sin\mathit{\Omega}_{\mathrm{I},n}TW_{\mathrm{I},n}(\xi_i)W_{\mathrm{I},n}(\xi)}{\gamma_{\mathrm{I},n}^{2}\mathit{\Omega}_{\mathrm{I},n}}, \tag{2.17}$$

$$h_{\mathrm{II},i}(\zeta,T) = \frac{vL_{\mathrm{II}}^{3}}{\sigma_{\mathrm{II}}^{2}(EI)_{\mathrm{II}}L_{\mathrm{I}}}e^{-\alpha_{\mathrm{II}}T}\sum_{n=1}^{\infty}\frac{\sin\mathit{\Omega}_{\mathrm{II},n}TW_{\mathrm{II},n}(\zeta_i)W_{\mathrm{II},n}(\zeta)}{\gamma_{\mathrm{II},n}^{2}\mathit{\Omega}_{\mathrm{II},n}}. \tag{2.18}$$

3. Vibrations of a double-beam system

By combining solutions for the cases of the load presented above we are able to build a set of k compatibility equations linking displacements of the upper and the lower beam at the locations of intermediate elastic restraints:

$$\left\{\begin{aligned} & w_{\mathrm{I}}^{P}(\xi_1,T)+\sum_{j=1}^{k} w_{\mathrm{I}}^{Xj}(\xi_1,T)+\sum_{j=1}^{k} w_{\mathrm{II}}^{Xj}(\zeta_1,T)=-\frac{X_1(T)}{s_1},\\ & w_{\mathrm{I}}^{P}(\xi_2,T)+\sum_{j=1}^{k} w_{\mathrm{I}}^{Xj}(\xi_2,T)+\sum_{j=1}^{k} w_{\mathrm{II}}^{Xj}(\zeta_2,T)=-\frac{X_2(T)}{s_2},\\ & \vdots\\ & w_{\mathrm{I}}^{P}(\xi_k,T)+\sum_{j=1}^{k} w_{\mathrm{I}}^{Xj}(\xi_k,T)+\sum_{j=1}^{k} w_{\mathrm{II}}^{Xj}(\zeta_k,T)=-\frac{X_k(T)}{s_k}. \end{aligned}\right. \tag{3.1}$$

3.1. Volterra integral equations

After substituting solutions (2.15) and (2.16) into the set of Eqs (3.1), we obtain a set of k Volterra integral equations of the second order:

$$\left\{\begin{aligned} & \frac{L_{\mathrm{I}}}{v}\sum_{j=1}^{k}\int_0^T d_{1j}(T-\tau)X_j(\tau)\,\mathrm{d}\tau+w_{\mathrm{I}}^{P}(\xi_1,T)=-\frac{X_1(T)}{s_1},\\ & \frac{L_{\mathrm{I}}}{v}\sum_{j=1}^{k}\int_0^T d_{2j}(T-\tau)X_j(\tau)\,\mathrm{d}\tau+w_{\mathrm{I}}^{P}(\xi_2,T)=-\frac{X_2(T)}{s_2},\\ & \vdots\\ & \frac{L_{\mathrm{I}}}{v}\sum_{j=1}^{k}\int_0^T d_{kj}(T-\tau)X_j(\tau)\,\mathrm{d}\tau+w_{\mathrm{I}}^{P}(\xi_k,T)=-\frac{X_k(T)}{s_k}, \end{aligned}\right. \tag{3.2}$$

where

$$d_{ij}(T)=h_{\mathrm{I},i}(\xi_j,T)+h_{\mathrm{II},i}(\zeta_j,T). \tag{3.3}$$

Expression $-\frac{X_i(T)}{s_i}$ on the right side of each equation denotes length change of the i elastic restraint at time T.

3.2. Numerical procedure

Because Volterra integral Eqs (3.2) are difficult to solve analytically, we shall apply a simple numerical procedure similar to the one used in [4] to describe vibrations of multi-span beams because of its simplicity and satisfying efficiency. In the first step, the time of force movement along the plate $t = L_{\mathrm{I}}/v$ is divided into N equal time segments. Then we assume collocation points τ_R in the middle of each time segment and values of support reactions $X_j(\tau_r)$ as the unknowns. This allows us to replace a set of integral Eqs (3.2) with algebraic equations by using the midpoint method:

$$\frac{L_{\mathrm{I}}\Delta\tau}{v}\sum_{i=1}^{k}\sum_{r=1}^{R} d_{ij}(T_R-\tau_r)X_i(\tau_r) + w_{\mathrm{I}}^{P}(\xi_i, T_R) = -\frac{X_i(T_R)}{s_i}, \qquad j = 1,\ 2, ..., k, \tag{3.4}$$

where $t_R = R\Delta\tau$; $\tau_r = (r-0.5)\Delta\tau$; $r = 1, 2, ..., R$; $R = 1, 2, ..., N$; $\Delta\tau = L_{\mathrm{I}}/(Nv)$. After solving Eqs (3.4) vibrations of the upper and the lower beam can be described as:

$$w_{\mathrm{I}}(\xi, T_R) = \frac{L_{\mathrm{I}}\Delta\tau}{v}\sum_{i=1}^{k}\sum_{r=1}^{R} h_{\mathrm{I},i}(\xi, T_R-\tau_r)X_i(\tau_r) + w_{\mathrm{I}}^{P}(\xi, T_R), \tag{3.5}$$

$$w_{\mathrm{II}}(\zeta, T_R) = -\frac{L_{\mathrm{I}}\Delta\tau}{v}\sum_{i=1}^{k}\sum_{r=1}^{R} h_{\mathrm{II},i}(\zeta, T_R-\tau_r)X_i(\tau_r). \tag{3.6}$$

4. Numerical example

The presented example is of a three-span double-beam system (see Fig. 5). The upper beam is simply supported while the lower beam is clamped at both ends. The beams have the same length $L = 12$ m, flexural rigidity $EI = 4\cdot 10^6$ N $\cdot$ m^2 and mass density $m = 25$ kg/m and are connected with two elastic restraints of stiffness $s_1 = s_2 = 1\cdot 10^6$ N/m. The system is subjected to a point force of constant magnitude $P = 1000$ N moving on the upper beam with a constant velocity $v = 30$ m/s. Time of force movement alongside the upper beam was divided into $N = 500$ equally long time steps $\Delta\tau = 0.002$. In further calculations we analyze dynamic deflections at sections "a" and "b" situated in the

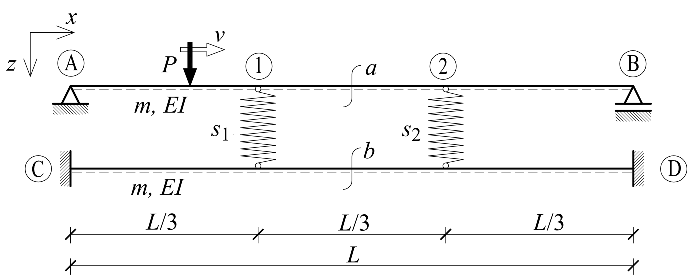

Fig. 5. Double-beam system with two elastic restraints loaded with moving point force.

middle of the upper and the lower beam (Fig. 5), reactions in the "1" and "2" elastic restraint (Fig. 7) and maximum value of dynamic deflection throughout the both beams versus different values of force movement velocity (Fig. 8). Results in Figs 6 and 7 were also compared with results obtained numerically by applying

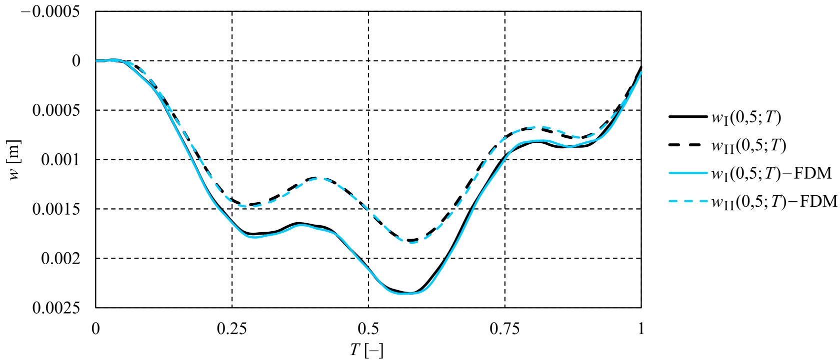

FIG. 6. Dynamic deflection of the middle of the upper and the lower beam.

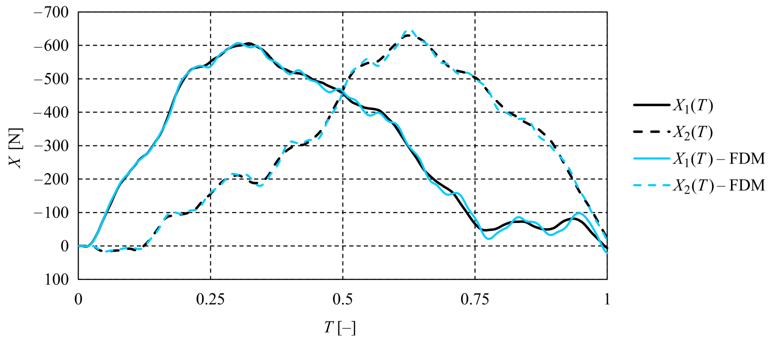

FIG. 7. The reaction in the "1" and "2" elastic restraint.

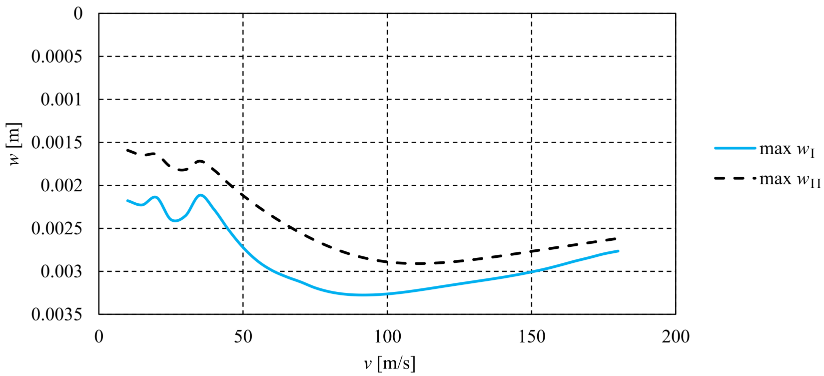

FIG. 8. Maximum dynamic deflection throughout the upper and the lower beam *versus* different force movement velocities.

the finite difference method chosen instead of the finite element method because of its better precision for the moving load cases (in FEA the beam displacement effect is omitted when the load is located between the nodes, which is significant for bending moments and shear forces determination). A very good agreement between the two methods was observed.

5. Conclusion

The proposed method can be applied do describe vibrations of a double Euler-Bernoulli beam system subjected to a moving point force. After appropriate modification this method can be used for different types of moving noninertial load such as moving moment or moving distributed load. By using this method we can avoid spatial discretisation of the structure – we discretise only the time of force movement. The applied numerical procedure allows us to avoid difficulties of solving integral equations analytically and makes this task easy to solve by using simple computer programs. This method can also be used as verification for other numerical methods such as the finite element method or finite difference method. The solution presented for the moving unitary concentrated force treated as a dynamic influence function can be used in the analysis of stochastic vibrations due to the moving load.

References

1. Fryba L., *Vibrations of solids and structures under moving load*, Telford, London, 1999.
2. Abu-Hilal M., *Dynamic response of a double Euler-Bernoulli beam due to a moving constant load*, Journal of Sound and Vibration, **297**(3–5): 477–491, 2006.
3. Oniszczuk Z., *Forced transverse vibrations of an elastically connected simply supported double-beam system*, Journal of Sound and Vibration, **264**(2): 273–286, 2003.
4. Zakęś F., Śniady P., *Application of Volterra integral equations in dynamics of multispan uniform continuous beams subjected to a moving load*, Shock and Vibration, **2016**, article ID 4070627, 12 pages, 2016, http://dx.doi.org/10.1155/2016/4070627.
5. Krylov A.N., *Mathematical collection of papers of the Academy of Sciences*, Vol. 61, St. Petersburg, 1905.
6. Clough R.W., Penzien J., *Dynamics of Structures*, McGraw-Hill, New York, 1993.
7. Weaver W., Timoshenko S., Young D.H., *Vibration Problems in Engineering*, Wiley, New York, 1990.

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