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NATURAL FREQUENCIES OF A CANTILEVER TIMOSHENKO BEAM WITH A TIP MASS (*)

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The aim of the present paper is to deduce the free vibration frequencies of cantilever structures with a tip mass at the free end, by taking into account the rotary inertia and the shear deformation. The analysis is conducted by dividing the beam into rigid bars with elastic constraints, extending a previous work by De Rosa and Franciosi [1]. The proposed method allows us to analyze beams with arbitrarily varying cross-sections, and numerical comparisons with some previous results found in the literature show the good performances of the approach.

Notation

E, G	Young' modulus; shear modulus;
L, L_i	span of the beam; length of the i-th rigid bar;
I, A, ϱ	moment of inertia; area; mass density;
m_i, m_t	i-th mass; beam mass;
M, I_M	mass at the tip; moment of inertia of the mass;
$\overline{J}$	radius of inertia of the mass;
$\hat{k}$	shear factor;
$\mathbf{c}$	vector of the Lagrangian coordinates;
v_1, v_2, v_i	displacements of the rigid bars;
$\Delta\varphi_i, \Delta v_i$	relative rotations, relative displacements;
φ_M	rotation of the mass at the tip;
M_i, T_i	bending moment; shear;
kf, ks	bending stiffness; shear stiffness;
$\mathbf{M}_V, \widetilde{\mathbf{M}}$	mass matrix; matrix of the rotary inertia;
ω, λ	free frequencies; nondimensional parameter;
γ_A, γ_L	nondimensional parameters;
$\overline{Y}, Z$	nondimensional parameters;
t	number of rigid bars.

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1. Introduction

Cantilever Timoshenko beams with concentrated masses at the tip are often used in order to simulate the dynamic behaviour of important structural members, such as moving arms of robots in mechanical engineering, towers, tall buildings, etc..

One of the earlier studies goes back to To [2], who has given the exact solution for a slender beam with constant cross-section. Later on, Gutierrez and Laura [3] generalized the problem by considering beams with varying cross-sections, and solving the problem by means of a combined Rayleigh–Ritz and Schmidt approximate analysis.

The influence of the shear deformation and the rotary inertia has been taken into account by Brüch and Mitchell [4] for a beam with constant cross-section and for a mass whose centroid coincides with the tip of the beam; shortly after, Abramovich and Hamburger [5] extended the analysis to eccentric masses. A transfer matrix approach has been proposed by Liu and Liu [6] in order to examine the dynamic behaviour of a cantilever beam with an elastically flexible constraint. Recently, Farghaly [9] was able to find the governing equations of the problem, by using the above-mentioned Abramovic and Hamburger [5] paper.

If the cross-section is supposed to vary according to a continuous law, then the exact solution is not available. Consequently, Laura, Rossi and Maurizi [8] proposed a FEM-like algorithm, which was illustrated earlier by Przemieniecki [7].

In the present paper a discretization method is employed, which has been already used in the past [1], and it is well tailored to the dynamic analysis of one-dimensional structures. The method is immediately adaptable to every kind of cross-section variation law, and furnishes lower bounds to the true frequencies.

The results presented in [1] are extended by considering an additional Lagrangian coordinate, which represents the rotation of the tip mass. In this way it is possible to calculate the strain energy of the connecting cell between the beam and the mass, as well as the kinetic energy due to the rotary inertia.

The cross-section variation is taken into account directly, writing the cell stiffness in correspondence to the discontinuity.

2. The discretized model

The examined structural system is given in Fig. 1, together with its reduction to a set of rigid bars connected by means of elastic constraints with bending and shear stiffness. Consequently, the structure degrees of freedom can be conveniently assumed to be the displacements at the ends of each rigid bars, and the rotation at the tip of the beam.

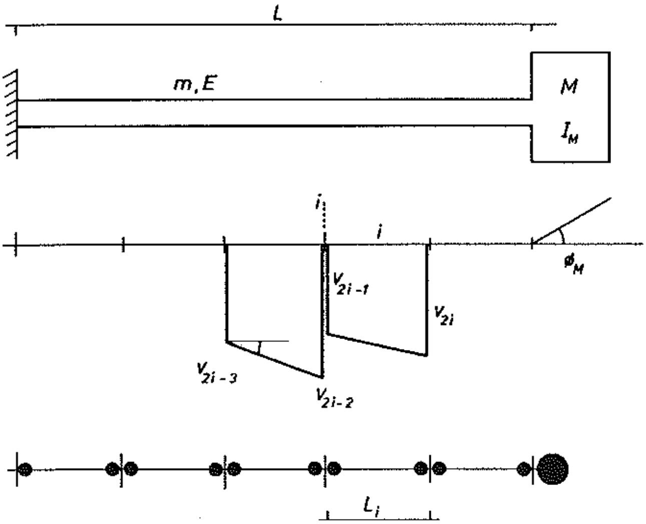

FIG. 1. Structural model.

In matrix notation, the degrees of freedom can be combined into a single vector:

$$\mathbf{c} = [v_1, v_2, v_3, \dots, v_{2t}, \varphi_M]^T \tag{2.1}$$

and therefore the relative rotations of the bars at the elastic constraints can be expressed as:

$$\begin{aligned} \Delta\varphi_1 &= \frac{v_1 - v_2}{L_1}, \\ \Delta\varphi_i &= \frac{v_{2i-2} - v_{2i-3}}{L_{i-1}} - \frac{v_{2i} - v_{2i-1}}{L_i}, \\ \Delta\varphi_{t+1} &= \varphi_M + \frac{v_{2t} - v_{2t-1}}{L_t} \qquad (i = 1, 2, \dots, t). \end{aligned} \tag{2.2}$$

The relative displacements are written as:

$$\begin{aligned} \Delta v_1 &= v_1, \\ \Delta v_i &= v_{2i-1} - v_{2i-2}, \\ \Delta v_{t+1} &= 0 \qquad (i = 1, 2, \dots, t). \end{aligned} \tag{2.3}$$

Consequently, the strain energy of the generic elastic constraint is given by

$$U_i = \frac{1}{2}(M_i \Delta\varphi_i + T_i \Delta v_i), \tag{2.4}$$

where

$$(2.5)\qquad \begin{aligned} M_i &= 2E\left(\frac{I_i I_{i-1}}{I_{i-1}L_i + I_i L_{i-1}}\right)\Delta\varphi_i = kf_i\Delta\varphi_i,\\ T_i &= 2G\hat{k}\left(\frac{A_i A_{i-1}}{A_{i-1}L_i + A_i L_{i-1}}\right)\Delta v_i = ks_i\Delta\varphi_i\,. \end{aligned}$$

Equations (2.2) and (2.3) can be written, using matrix notations, in the form

$$\Delta\varphi = \mathbf{Ac}, \qquad \Delta\mathbf{v} = \mathbf{Bc},$$

and therefore the strain energy of the whole structure is equal to

$$(2.6)\qquad U = \frac{1}{2}\mathbf{c}^T\mathbf{Kc},$$

with

$$(2.7)\qquad \mathbf{K} = \mathbf{A}^T\mathbf{D}_f\mathbf{A} + \mathbf{B}^T\mathbf{D}_s\mathbf{B},$$

$\mathbf{D}_f$ and $\mathbf{D}_s$ are the (diagonal) matrices of the terms kf_i and ks_i, respectively. The kinetic energy T of the structure must also be expressed as a function of the Lagrangian coordinates. From Fig. 1 we have:

$$(2.8)\qquad T = \frac{1}{2}\sum_{i=1}^{2t} m_i\,\dot{v}_i^2 + \frac{1}{2}\sum_{i=1}^{t}\varrho I_i L_i\,\dot{\varphi}_i^2 + \frac{1}{2}I_M\,\dot{\varphi}_M^2\,.$$

The absolute rotations of the rigid bars can also be expressed as functions of the Lagrangian coordinates, by introducing the rectangular matrix $\mathbf{V}$ with $t+1$ rows and $2t+1$ columns:

$$(2.9)\qquad \Phi = \mathbf{Vc}.$$

Henceforth, the kinetic energy becomes:

$$(2.10)\qquad T = \frac{1}{2}\,\dot{\mathbf{c}}^T\,\mathbf{M}_V\,\dot{\mathbf{c}} + \frac{1}{2}\,\dot{\Phi}^T\,\widetilde{\mathbf{M}}\,\dot{\Phi} = \frac{1}{2}\,\dot{\mathbf{c}}^T\left(\mathbf{M}_V + \mathbf{V}^T\widetilde{\mathbf{M}}\mathbf{V}\right)\dot{\mathbf{c}},$$

where the lumped masses at the ends of the bars are grouped into the (diagonal) matrix $\mathbf{M}_V$, and the entry of the $\widetilde{\mathbf{M}}$ matrix read:

$$\begin{aligned} \widetilde{M}_i &= \varrho I_i L_i\,, \qquad i = 1,2,\ldots,t,\\ \widetilde{M}_{t+1} &= \varrho I_M\,. \end{aligned}$$

Finally, the equation of motion can be written as:

$$(2.11)\qquad \mathbf{M}\,\ddot{\mathbf{c}} + \mathbf{Kc} = \mathbf{0},$$

and the free frequencies can be found by solving the eigenvalue problem

$$(2.12)\qquad (\mathbf{K} - \omega^2\mathbf{M})\mathbf{c} = \mathbf{0}.$$

3. Numerical results

As the first example, a beam with constant cross-section is examined, with a mass at the tip, ratio $E/G = 2.6$ and $I_M = \overline{J}^2 M$. The free vibration frequencies and the related nondimensional coefficients

$$\lambda_i = \omega_i \sqrt{\frac{\varrho A L^4}{EI}}$$

are given as functions of the parameters

$$\overline{Y} = \frac{M}{m_t}, \qquad Z = \frac{\overline{J}}{L}, \qquad r^2 = \frac{I}{AL^2}.$$

The shear factor is equal to $\hat{k} = 2/3$ (see Timoshenko [14]). In Fig. 2 the first nondimensional frequency λ_1 is given, for an increasing number of Lagrangian co-ordinates. Nevertheless, it seems sufficient to divide the beam into 20 bars, in order to obtain a good compromise between computational costs and numerical accuracy. Therefore, all the following numerical examples will be performed by dividing the structure into 20 rigid bars.

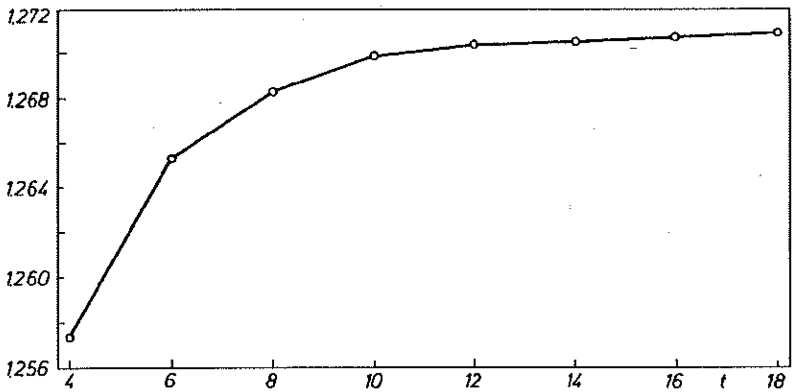

Fig. 2. Convergence curve of the first non-dimensional coefficient for $r = 0.02$, $z = 0.5$, $\overline{Y} = 1$.

The results are given in Table 1 together with some results from the literature. It is worth noting that the first two nondimensional frequencies are approximated with a very small error (0.5%), whereas the error for the other coefficients increases up to 1.3%.

The nondimensional coefficients λ_i for $\hat{k} = 5/6$ are given in Table 2, because the above-mentioned shear factor is also often used for beams with rectangular cross-section (see Cowper [15]). Even in this case, the first eigenvalues agree very well with the results of the literature, whereas the

Table 1. Coefficients λ_i $(i = 1, 2, 3, 4)$ for $r = 0.02$, $\hat{k} = 2/3$.

Mode	1	2	3	4	Z	$\overline{Y}$
Bruch [4]	1.40	5.73	23.64	58.41	0.5	1
Abramovich [5]	1.27	4.53	23.32	58.24		
Farghaly [11]	1.2717	4.5272	23.3163	58.2375		
Author	1.2709	4.524	23.22	57.84		
Bruch [4]	3.50	21.35	57.47	106.93	0	0
Abramovich [5]	3.50	21.35	57.42	106.58		
Author	3.50	21.23	56.84	105.22		

Table 2. Coefficients λ_i $(i = 1, 2, 3, 4)$ for $Z = 0$, $\hat{k} = 5/6$.

Mode	1	2	3	4	r	$\overline{Y}$
Laura [8]	3.50	21.47	58.14	109.02	0.02	0
Author	3.49	21.23	56.84	105.108		
Liu [13]	3.5262	21.152	54.5419		0.04	
Farghaly [10]	3.4636	20.0147	50.5619			
Rossi [12]	3.46	20.01	50.56			
Author	3.469	19.895	50.067			
Laura [8]	1.55	15.93	48.40	95.94	0.02	1
Author	1.5588	15.854	47.820	94.041		
Liu [13]	1.5585	15.6712	45.2083		0.04	
Farghaly [10]	1.5438	15.1038	42.8233			
Rossi [12]	1.54	15.10	42.82			
Author	1.5446	15.045	42.550			

fourth nondimensional frequency λ_4 shows an error which can increase up to 3.5%.

The nondimensional coefficients of the beam in Fig. 3 are given in Table 3, compared with the results of LAURA *et al.* [8]. The cross-section dis-

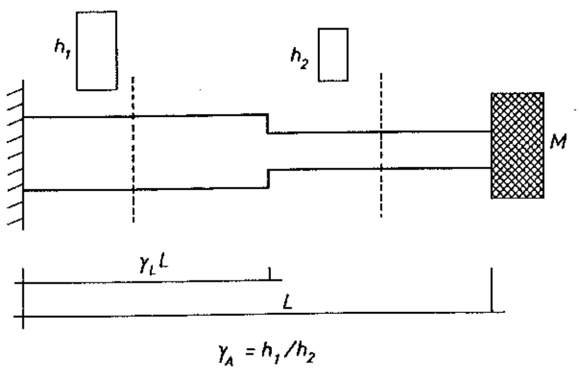

FIG. 3. Stepped beam.

continuity implies greater discrepancies between the frequencies, but the errors can obviously be reduced if the number of Lagrangian coordinates is increased.

Table 3. Coefficients λ_i for $\gamma_L = 2/3$, $\gamma_A = 0.8$, $r = 0.02$.

Mode	1	2	3	4	$\overline{Y}$
Laura [8]	3.82	21.35	55.04	107.50	0
Author	3.75	21.50	53.03	99.20	
Laura [8]	2.26	15.87	46.21	95.44	0.4
Author	2.20	15.53	45.00	88.91	
Laura [8]	1.75	15.17	45.53	94.72	0.8
Author	1.71	14.88	44.34	88.27	
Laura [8]	1.60	15.01	45.37	94.56	1
Author	1.56	14.73	44.20	88.14	

4. Conclusions

A simple model of Timoshenko beams has been presented, in order to calculate the free vibration frequencies of a stocky cantilever beam with a tip mass at the end, taking into account the rotary inertia and the shear deformation. The obtained results seem to be quite satisfactory, at least for the first eigenvalues, and in any case arbitrary precision can be achieved by simply increasing the number of degrees of freedom. The proposed procedure can be easily adapted to beams with arbitrarily varying cross-sections, and, more generally, to all the cases in which analytical solutions are not attainable.

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